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Unique Solvability Conditions For The Absolute Value Matrix Equation*

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Abstract

In this paper, we discuss some conditions for the unique solvability and unsolvability of the absolute value matrix equation (AVME) $AX - B|X| = D$. Furthermore, we examine some examples in which the existing conditions fail to determine the unique solvability of the AVME. These conditions are subject to future revisions. Finally, we raise some new challenging problems related to the AVME.

1 Introduction

The absolute value matrix equation (AVME) with unknown matrix X is defined as

$$AX - B|X| = D \tag{1}$$

where $A, B, D \in \mathbb{R}^{n\times n}$ are given matrices, and the goal is to determine the matrix $X \in \mathbb{R}^{n\times n}$. When dealing with vectors and matrices, we interpret the inequalities, and absolute values ($|.|$) entrywise.

The generalized absolute value equations (GAVE) in the unknown vector x are defined as

$$Ax - B|x| = d \tag{2}$$

with $A, B \in \mathbb{R}^{n\times n}$ and $d \in \mathbb{R}^n$ are given and $x \in \mathbb{R}^n$ is to be determined. Clearly, the AVME is a generalization form of the GAVE. We use to denote $\sigma_{max}(A)$ for maximum singular value and $\sigma_{min}(A)$ for minimum singular value of a matrix A. The spectral radius of a matrix A is denoted by $\rho(A)$. The zero matrix is denoted by $\mathbf{0}$ and identity matrix of order n is represented by $I_{n\times n}$. The Kronecker product and vec operator are denoted by '$\otimes$' and 'vec' respectively.

The GAVE has many applications in complementarity problems, convex quadratic programming and linear programming. A theorem of the alternatives for the GAVE (2) was presented by Rohn [15]. For theoretical study and numerical methods for GAVE, see [1, 2, 3, 7, 8, 9, 10, 12, 13, 14, 19, 20] and references cited therein. AVME represents a generalized form of the GAVE. Dehghan et al. [4] first considered the AVME (1) and provided the condition $\sigma_{min}(A) > \sigma_{max}(|B|)$ for the unique solution of (1) and they also provided a matrix multi-splitting Picard-iterative method for the solution of AVME (1). Their proposed method is convergent under the condition $n\sigma_{max}(|B|) < \sigma_{min}(A)$. Xie [22] gave the condition $\sigma_{min}(A) > \sigma_{max}(B)$ for the unique solution of (1) and provided new convergence condition $n\sigma_{max}(B) < \sigma_{min}(A)$ for matrix multi-splitting Picard-iterative scheme. In [6], the authors provided a new condition to check the unique solvability of the AVME (1). The condition [6, Theorem 0.2] is superior to that of Dehghan et al. [4, Theorem 2.1] and Xie [22, Theorem 1]. Tang et al. [18] further discussed the unique solvability of the AVME and provided convergence conditions for the Picard-type scheme. The AVME has a number of applications in the field of interval matrix equations [11, 16] and robust control [17].

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In contrast to the GAVE (2), the AVME (1) has received comparatively less attention in research. To the best of our knowledge, the unique solvability conditions of the AVME (1) are addressed in [4, 5, 6, 18, 22]. The main objective of this article is to explain the criteria determining the unique solvability and unsolvability conditions of the AVME (1).

2 Unique Solvability Conditions for AVME

Based on the spectral radius conditions, for the AVME and GAVE, we recall the following results.

Theorem 1 ([5]) *The AVME (1) has exactly one solution for any $D \in \mathbb{R}^{n\times n}$, if the condition $\rho(|A^{-1}||B|) < 1$ is satisfied for invertible matrix A.*

Theorem 2 ([21]) *If 0 is not an eigenvalue of A and $\rho(A^{-1}B\bar{D}) < 1$ for any diagonal matrix $\bar{D} \in [-I_{n\times n}, I_{n\times n}]$, then the GAVE (2) has exactly one solution for any $d \in \mathbb{R}^{n\times n}$.*

Based on Theorem 2, Tang et al. [18] presented the following result regarding the unique solution of the AVME (1).

Theorem 3 ([18]) *The AVME (1) has exactly one solution for any $D \in \mathbb{R}^{n\times n}$, if 0 is not an eigenvalue of A and the condition $\rho((I_{n\times n} \otimes A^{-1}B)\bar{\Lambda}) < 1$ for any diagonal matrix $\bar{\Lambda} \in [-I_{n^2\times n^2}, I_{n^2\times n^2}]$.*

We now provide the following result for the AVME (1) in accordance with Theorem 2.

Theorem 4 *If 0 is not an eigenvalue of A and $\rho(A^{-1}B\bar{D}) < 1$ for any diagonal matrix $\bar{D} \in [-I_{n\times n}, I_{n\times n}]$, then the AVME (1) has a unique solution for any $D \in \mathbb{R}^{n\times n}$.*

Proof. The proof is simple and can be followed by Theorem 0.2 of [6]. Let $D = (d_1, ..., d_n)$ and $X = (x_1, ..., x_n)$, where d_j and x_j are the jth column of the matrices D and X, respectively. Then, AVME (1) can be rewritten as the following form:

$$Ax_j - B|x_j| = d_j \tag{3}$$

where $j = 1, ..., n$. By Theorem 2, if condition $\rho(A^{-1}B\bar{D}) < 1$ for any diagonal matrix $\bar{D} \in [-I_{n\times n}, I_{n\times n}]$, then Equ. (3) has a unique solution for each d_j, where $j = 1, ..., n$. Hence, we can compute each x_j, individually. ■

Remark 1 *The condition $\rho(A^{-1}B\bar{D}) < 1$ is slightly superior than the condition $\rho(|A^{-1}||B|) < 1$. Since*

$$A^{-1}B\bar{D} \leq |A^{-1}B\bar{D}| \leq |A^{-1}B||\bar{D}| \leq |A^{-1}B| \leq |A^{-1}||B|$$

and this leads to $\rho(A^{-1}B\bar{D}) \leq \rho(|A^{-1}||B|)$. To check the condition of Theorem 3, size of the matrix grows up to $n^2 \times n^2$, while the condition of Theorem 4 work with the initial size $n \times n$. That is why, for a given matrix of order n, checking the condition of Theorem 4 requires less memory space compared to the condition of Theorem 3.

Based on Theorem 3, we have the following result.

Theorem 5 *If 0 is not an eigenvalue of A and $\sigma_{max}(A^{-1}B) < 1$, then the AVME (1) has exactly one solution for any $D \in \mathbb{R}^{n\times n}$.*

Proof. By using the Kronecker product property, we have

$$\begin{aligned}\rho((I_{n\times n} \otimes A^{-1}B)\bar{\Lambda}) &\leq \sigma_{max}((I_{n\times n} \otimes A^{-1}B)\bar{\Lambda}) \leq \sigma_{max}(I_{n\times n} \otimes A^{-1}B)\sigma_{max}(\bar{\Lambda}) \\ &\leq \sigma_{max}(I_{n\times n} \otimes A^{-1}B) = \sigma_{max}(I_{n\times n})\sigma_{max}(A^{-1}B) = \sigma_{max}(A^{-1}B).\end{aligned}$$

So, when $\sigma_{max}(A^{-1}B) < 1$, then with the help of Theorem 3, the AVME (1) has a unique solution for each $D \in \mathbb{R}^{n\times n}$. ■

Remark 2 *Theorem 5 is the main result in [6], but our approach is different than [6]. We applied the Kronecker product property and Theorem 3 to prove Theorem 5. Whereas, in [6], first the AVME (1) is written in the equivalent form of the GAVE and then the unique solvability result of GAVE utilized for the AVME.*

Based on singular value conditions, we have the following results.

Theorem 6 ([4, 6, 22]) *The AVME (1) has a unique solution for any $D \in \mathbb{R}^{n\times n}$ if either of the below conditions is satisfied:*

(i) $\sigma_{min}(A) > \sigma_{max}(|B|)$.

(ii) $\sigma_{min}(A) > \sigma_{max}(B)$.

(iii) $\sigma_{max}(A^{-1}B) < 1$ *or* $\sigma_{min}(B^{-1}A) > 1$, *where matrices A and B are invertible.*

Remark 3 *The condition $\sigma_{max}(A^{-1}B) < 1$ is slightly superior than the condition $\sigma_{max}(B) < \sigma_{min}(A)$ [6], and the condition $\sigma_{max}(B) < \sigma_{min}(A)$ is slightly superior than the condition $\sigma_{max}(|B|) < \sigma_{min}(A)$ [22]. In general, there is no relation between the singular value and spectral radius of a matrix, so there are some cases when the conditions of Theorem 6 are satisfied and the condition of Theorem 1 is not, and conversely. Moreover, the conditions given in Theorem 1 and Theorem 6 are sufficient but not necessary. So sometimes, given conditions are not satisfied, still the AVME (1) has a unique solution; see Example (1).*

Example 1 *Consider the AVME (1), where matrices A and B are given by*

$$A = \begin{pmatrix} 1.8 & 0 \\ 0 & 1.7 \end{pmatrix},\ B = \begin{pmatrix} -1.68 & 0.46 \\ -0.82 & -1.55 \end{pmatrix} \ \text{ and } D = \begin{pmatrix} 2.1 & -2.08 \\ 0.37 & 14.64 \end{pmatrix}.$$

Here, the AVME (1) has a unique solution

$$X = \begin{pmatrix} 2.1 & -2.08 \\ 0.37 & 14.64 \end{pmatrix},$$

but conditions of Theorem 1 and Theorem 6 are not valid to judge the unique solvability of the AVME (1). Since

$$\sigma_{min}(A) = 1.7, \quad \sigma_{max}(B) = 1.9286, \quad \sigma_{max}(|B|) = 2.2683$$

and

$$\sigma_{min}(B^{-1}A) = 0.9029 \ngtr 1,\ \ \sigma_{max}(A^{-1}B) = 1.1075 \nless 1,\ \ \rho(|A^{-1}||B|) = 1.2738 \nless 1.$$

In Example (1), conditions of Theorem 1 and Theorem 6 failed to detect the unique solvability of the AVME (1). There is always a possibility to derive superior conditions concerning the conditions of Theorem 1 and Theorem 6.

Rohn established the following result for the GAVE (2).

Theorem 7 ([15]) *The GAVE (2) has a unique solution for any $d \in \mathbb{R}^n$ if either of the following conditions is satisfied:*

(i) The inequality $|Ax| \leq |B||x|$ has only the trivial solution $x = \mathbf{0}$.

(ii) For any $\lambda \in [0, 1]$, the condition $|Ax| = \lambda|B||x|$ does not hold, where $x \neq \mathbf{0}$.

We extend the condition of Theorem 7 for the AVME (1) as follows:

Theorem 8 *The AVME (1) has a unique solution for any $D \in \mathbb{R}^{n\times n}$ if either of the below conditions is satisfied:*

(i) There exists only trivial solution $X = \mathbf{0}$ of the inequality $|AX| \leq |B||X|$.

(ii) The condition $|AX| = \lambda|B||X|$ with $X \neq \mathbf{0}$ does not hold for each $\lambda \in [0,1]$.

Proof. By taking $T = I_{n\times n} \otimes B$, $S = I_{n\times n} \otimes A$, $x = vec(X)$ and $d = vec(D)$, AVME (1) can be equivalently expressed in the GAVE form as $Sx - T|x| = d$. So, we apply the Theorem 7 for the equation $Sx - T|x| = d$. By simple calculation, we have $|Sx| \leq |T||x|$. Then

$$\begin{aligned} & |(I_{n\times n} \otimes A)vec(X)| \leq (|I_{n\times n} \otimes B|)vec(|X|) = (|I_{n\times n}| \otimes |B|)vec(|X|) \\ \Longrightarrow \quad & vec(|AXI_{n\times n}|) \leq vec(|B||X||I_{n\times n}|) \\ \Longrightarrow \quad & |AX| \leq |B||X|. \end{aligned}$$

Similarly, we have that $|Sx| \neq \lambda|T||x| \implies |AX| \neq \lambda|B||X|$. So, based on the above calculation, both results of Theorem 8 hold. ■

Now consider the following example:

Example 2 *Let $A = 2I_{n\times n}$ and $B = I_{n\times n}$. Then*

(i) $|AX| \leq |B||X|$ or $2|X| \leq |X|$ has only trivial solution $X = \mathbf{0}$.

(ii) $|AX| = \lambda|B||X|$ or $2|X| = \lambda|X|$ satisfied only for $\lambda = 2$, i.e., $\lambda \notin [0,1]$.

So, from Theorem 8 it implies that the AVME (1) possesses a unique solution for each $D \in \mathbb{R}^{n\times n}$.

Now, we establish the following results.

Theorem 9 *The AVME (1) has a unique solution for any $D \in \mathbb{R}^{n\times n}$ if and only if matrix $(I_{n\times n} \otimes A) - (I_{n\times n} \otimes B)\bar{D}$ is nonsingular for each $\bar{D} \in [-I_{n^2\times n^2}, I_{n^2\times n^2}]$.*

Proof. Utilizing the properties of the Kronecker product and the vec operator, the AVME $AX - B|X| = D$ can be written as

$$(I_{n\times n} \otimes A)vec(X) - (I_{n\times n} \otimes B)vec(|x|) = vec(D).$$

So, according to Theorem 3.2 of [21], the AVME (1) has a unique solution if and only if $(I_{n\times n} \otimes A) - (I_{n\times n} \otimes B)\bar{D}$ is nonsingular for each $\bar{D} \in [-I_{n^2\times n^2}, I_{n^2\times n^2}]$. ■

Theorem 10 *The AVME $\alpha AX - B|X| = D$ has a unique solution for each $D \in \mathbb{R}^{n\times n}$ and for each $|\alpha| \geq 1$ if and only if the AVME $AX - B|X| = D$ has a unique solution for each $D \in \mathbb{R}^{n\times n}$.*

Proof. The AVME $AX - B|X| = D$ has a unique solution for each $D \in \mathbb{R}^{n\times n}$ if and only if matrix $(I_{n\times n} \otimes A) - (I_{n\times n} \otimes B)\bar{D}$ is nonsingular for each $\bar{D} \in [-I_{n^2\times n^2}, I_{n^2\times n^2}]$. The matrix $(I_{n\times n} \otimes A) - (I_{n\times n} \otimes B)\bar{D}$ is nonsingular for each $\bar{D} \in [-I_{n^2\times n^2}, I_{n^2\times n^2}]$ if and only if $(I_{n\times n} \otimes \alpha A) - (I_{n\times n} \otimes B)\bar{D}$ is nonsingular for each $\bar{D} \in [-I_{n^2\times n^2}, I_{n^2\times n^2}]$ and for each $|\alpha| \geq 1$. This completes the proof. ■

Here arises an open question: "Is the unique solvability of AVME preserved under matrix transposition? i.e., if AVME (1) can be uniquely solvable for each $D \in \mathbb{R}^{n\times n}$, then $A^T X - B^T|X| = D$ is also uniquely solvable for each $D \in \mathbb{R}^{n\times n}$." This needs further investigation.

In the following result, we have an unsolvability condition for the AVME (1).

Theorem 11 *If 0 is not an eigenvalue of B, $\mathbf{0} \leq B^{-1}D \neq \mathbf{0}$ and $\sigma_{max}(B^{-1})\sigma_{max}(A) < 1$, then the AVME (1) has no solution.*

Proof. Let us assume that, a non-zero solution X of the AVME (1) exists. The AVME (1) can be written as

$$B^{-1}AX - |X| = B^{-1}D \quad \Longrightarrow \quad |X| \leq B^{-1}AX.$$

Now taking the 2-norm and using the property

$$||B^{-1}||_2||A||_2 = \sigma_{max}(B^{-1})\sigma_{max}(A) < 1,$$

we have

$$||X||_2 \leq ||B^{-1}AX||_2 \leq ||B^{-1}||_2||A||_2||X||_2 < ||X||_2.$$

This contradicts $||X||_2 < ||X||_2$. ∎

In support of Theorem 11, we have the following example.

Example 3 *Consider the AVME (1), where matrices A, B and D are given by*

$$A = \begin{pmatrix} -2 & 3 \\ -4 & 3 \end{pmatrix}, \ B = \begin{pmatrix} -4 & 5 \\ 6 & 5 \end{pmatrix}, \ \textit{and } D = \begin{pmatrix} -1 & 2 \\ 6 & 4 \end{pmatrix}.$$

Clearly, $\sigma_{max}(B^{-1})\sigma_{max}(A) = 0.1563 \times 6.0850 = 0.95109 < 1$. *Further,*

$$B^{-1}D = \begin{pmatrix} 0.7 & 0.2 \\ 0.36 & 0.56 \end{pmatrix} \geq \mathbf{0}.$$

Hence, the AVME (1) does not possess a unique solution for the given D.

In the literature, several conditions which are used to detect the unique solvability of the GAVE (2) are also used to determined the unique solvability of the AVME (1) (see, e.g., Theorem 1, Theorem 4 and Theorem 6). For example, the condition $\rho(|A^{-1}||B|) < 1$ is used to check the unique solvability of the GAVE (2) as well as AVME (1).

A natural question arises here whether the unique solvability of $Ax - B|x| = d$ implies the unique solvability of $AX - B|X| = D$. This needs further investigation.

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