Applied Mathematics E-Notes, 24(2024), 486-493 © ISSN 1607-2510
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Coincidence Points For Hybrid Pair Of Mappings Via Digraphs In Vector Metric Spaces*

Sushanta Kumar Mohanta†

Received 24 October 2023

Abstract

In this paper, we establish some coincidence point results for a hybrid pair of mappings satisfying a generalized contraction type condition involving a digraph in vector metric spaces. As some consequences of this study, we obtain several fixed point results for multi valued mappings in vector metric spaces.

1 Introduction

It is well known that the Banach contraction theorem [4] is a very useful and simple tool in modern analysis, and it has many applications in applied mathematics and sciences. Several authors successfully extended this famous theorem in different directions (see [8, 14, 15, 16]). In recent investigations, the study of fixed point theory combining a graph is a new development in the domain of contractive type single valued and multi valued theory. In 2005, Echenique [11] studied fixed point theory by using graphs. Later on, Espinola and Kirk [12] applied fixed point results in graph theory. Afterwards, combining fixed point theory and graph theory, a series of articles (see [1, 5, 6] and references therein) have been dedicated to the improvement of fixed point theory. In 2009, Çevic et al. [9] introduced the concept of vector metric spaces as a generalization of metric spaces, where the metric is Riesz space valued and studied some properties of such spaces. Motivated by the works in [2, 3, 9, 17], we will prove some coincidence point results for a hybrid pair of mappings satisfying a generalized contraction type condition involving a digraph in vector metric spaces. Moreover, we apply our main result to derive some new coincidence point results including fixed points of multi valued mappings in vector metric spaces. Our results extend and unify several existing results in the literature. Finally, we provide an example to analyze and illustrate our main result.

2 Some Basic Concepts

In this section, we recall some basic facts about Riesz spaces mostly of which can be found in [3, 9].

A partially ordered set $(X, \preceq)$ is called a lattice if each pair of elements $x, y \in X$ has a supremum and an infimum.
A real vector space E with an order relation $\preceq$ on E that is compatible with the algebraic structure of E in the sense that satisfies properties:

(1) $x \preceq y$ implies $x + z \preceq y + z$ for each $z \in E,\ x, y \in E$;

(2) $x \preceq y$ implies $tx \preceq ty$ for each $t > 0$, $x, y \in E$

is called an ordered vector space.

*Mathematics Subject Classifications: 54H25, 47H10.

†Department of Mathematics, West Bengal State University, Barasat, 24 Parganas (North), Kolkata-700126, West Bengal, India

An ordered vector space that is also a lattice is called a Riesz space or vector lattice. Throughout this paper, we take θ as the zero vector of the vector space E. Let E be a Riesz space with the positive cone $E_+ = \{x \in E : \theta \preceq x\}$. If (a_n) is a decreasing sequence in E such that $\inf a_n = a$, we write $a_n \downarrow a$. The Riesz space E is said to be Archimedean if $\frac{1}{n}a \downarrow \theta$ holds for every $a \in E_+$. A sequence (b_n) in E is called order convergent or o-convergent to b if there exists a sequence (a_n) in E satisfying $a_n \downarrow \theta$ and $\mid b_n - b \mid \preceq a_n$ for all n, and written $b_n \xrightarrow{o} b$ or $o - \lim b_n = b$, where $\mid a \mid = \sup\{a, -a\}$ for any $a \in E$. Moreover, (b_n) is called order-Cauchy or o-Cauchy if there exists a sequence (a_n) in E such that $a_n \downarrow \theta$ and $\mid b_n - b_{n+p} \mid \preceq a_n$ holds for all n and p. E is called o-complete if every o-Cauchy sequence is o-convergent. For other notations and facts about Riesz spaces, we refer to [2].

Definition 1 *[9] Let X be a nonempty set and E be a Riesz space. The function $d : X \times X \to E$ is said to be a vector metric (or E-metric) if it satisfies the following properties:*

(v_1) $d(x, y) = \theta$ *if and only if* $x = y$,

(v_2) $d(x, y) \preceq d(x, z) + d(y, z)$ *for all* $x, y, z \in X$.

The triple (X, d, E) is said to be a vector metric space.

It is easy to observe that vector metric spaces generalize metric spaces.

In a vector metric space (X, d, E), the following assertions hold:

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $\mid d(x, z) - d(y, z) \mid \preceq d(x, y)$ for all $x, y, z \in X$;

(iv) $\mid d(x, z) - d(y, w) \mid \preceq d(x, y) + d(z, w)$ for all $x, y, z, w \in X$.

Example 1 *[9] A Riesz space E is a vector metric space with $d : E \times E \to E$ defined by*

$$d(x, y) = \mid x - y \mid .$$

This vector metric is said to be absolute valued metric on E.

Example 2 *[9] It is well known that $\mathbb{R}^2$ is a Riesz space with coordinate wise ordering defined by*

$$(x_1, y_1) \preceq (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

Again, $\mathbb{R}^2$ is a Riesz space with lexicographical ordering defined by

$$(x_1, y_1) \preceq (x_2, y_2) \text{ if and only if } x_1 < x_2 \text{ or } x_1 = x_2,\ y_1 \leq y_2.$$

It is worth noting that $\mathbb{R}^2$ is Archimedean with coordinate wise ordering but not with lexicographical ordering.

Then, $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$d((x_1, y_1), (x_2, y_2)) = (\alpha \mid x_1 - x_2 \mid, \beta \mid y_1 - y_2 \mid)$$

is a vector metric, where α, β are positive real numbers.

Example 3 *[9] Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be defined by*

$$d(x, y) = (\alpha \mid x - y \mid, \beta \mid x - y \mid)$$

where α, $\beta \geq 0$ and $\alpha + \beta > 0$. Then d is a vector metric with coordinate wise or lexicographical ordering.

Definition 2 *[9] Let (X, d, E) be a vector metric space and let (x_n) be a sequence in X. Then*

(i) (x_n) vectorial converges or E-converges to a point $x \in X$, written $x_n \xrightarrow{d,E} x$, if there is a sequence (a_n) in E satisfying $a_n \downarrow \theta$ and $d(x_n, x) \preceq a_n$ for all n.

(ii) (x_n) is called vectorially Cauchy or E-Cauchy, if there is a sequence (a_n) in E such that $a_n \downarrow \theta$ and $d(x_n, x_{n+p}) \preceq a_n$ holds for all n and p.

(iii) (X, d, E) is said to be E-complete if each E-Cauchy sequence in X E-converges to a point in X.

Let X be a nonempty set and $P(X)$ be the set of all nonempty subsets of X. A point $x \in X$ is called a fixed point of the multi valued mapping $T : X \to P(X)$ if $x \in Tx$.

Definition 3 *Let $g : X \to X$ and $T : X \to P(X)$ be two mappings. If $y = gx \in Tx$ for some x in X, then x is called a coincidence point of g and T and y is called a point of coincidence of g and T.*

We now assign a digraph on a vector metric space (X, d, E). We assume that G is a digraph with the set of vertices $V(G) = X$ and the set $E(G)$ of its edges contains all the loops, i.e., $\Delta \subseteq E(G)$ where $\Delta = \{(x, x) : x \in X\}$. We also assume that G has no parallel edges. So we can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the graph obtained from G by reversing the direction of edges, that is, $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, such as [7, 10, 13].

Definition 4 *Let (X, d, E) be a vector metric space and let $G = (V(G), E(G))$ be a digraph. Then the mapping $g : X \to X$ is called edge preserving if*

$$x, y \in X, \ (x, y) \in E(\tilde{G}) \Rightarrow (gx, gy) \in E(\tilde{G}).$$

Definition 5 *Let (X, d, E) be a vector metric space with a digraph $G = (V(G), E(G))$. Then the mapping $T : X \to P(X)$ is called edge preserving if*

$$x, y \in X, \ (x, y) \in E(\tilde{G}) \Rightarrow (z_1, z_2) \in E(\tilde{G}) \text{ for all } z_1 \in Tx, \ z_2 \in Ty.$$

Definition 6 *Let (X, d, E) be a vector metric space with a digraph $G = (V(G), E(G))$ and let $g : X \to X$, $T : X \to P(X)$ be two mappings. Then T is called edge preserving w.r.t. g if*

$$x, y \in X, \ (gx, gy) \in E(\tilde{G}) \Rightarrow (z_1, z_2) \in E(\tilde{G}) \text{ for all } z_1 \in Tx, \ z_2 \in Ty.$$

3 Main Results

In this section, we assume that (X, d, E) is a vector metric space and G is a reflexive digraph such that $V(G) = X$ and G has no parallel edges.

Before presenting our main result, we state a property of the graph G, call it property $(*)$.

Property $(*)$**:** If (gx_k) is a sequence in X such that $gx_k \xrightarrow{d,E} x$ and $(gx_k, gx_{k+1}) \in E(\tilde{G})$ for all $k \geq 1$, then there exists a subsequence (gx_{k_i}) of (gx_k) such that $(gx_{k_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

Taking $g = I$, the above property reduces to property $(\acute{*})$:

Property $(\acute{*})$**:** If (x_k) is a sequence in X such that $x_k \xrightarrow{d,E} x$ and $(x_k, x_{k+1}) \in E(\tilde{G})$ for all $k \geq 1$, then there exists a subsequence (x_{k_i}) of (x_k) such that $(x_{k_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

We begin with the following definitions.

Definition 7 *Let (X, d, E) be a vector metric space endowed with a digraph G. The mapping $T : X \to P(X)$ is called a G-contraction if for x, $y \in X$ with $(x, y) \in E(\tilde{G})$ and any $u \in Tx$, there exists $v \in Ty$ such that*

$$d(u, v) \preceq kd(x, y)$$

where $k \in [0, 1)$ is a constant.

Definition 8 *Let (X, d, E) be a vector metric space endowed with a digraph G. The mappings $T : X \to P(X)$ and $g : X \to X$ are called (T, g)-G-contraction if for x, $y \in X$ with $(gx, gy) \in E(\tilde{G})$ and any $u \in Tx$, there exists $v \in Ty$ such that*

$$d(u, v) \preceq kd(gx, gy)$$

where $k \in [0, 1)$ is a constant.

It is to be noted that (T, g)-G-contraction reduces to (T, g)-contraction if we take $G = G_0$, where G_0 is the complete graph $(X, X \times X)$.

Theorem 1 *Let E be Archimedean and let (X, d, E) be a vector metric space with a digraph $G = (V(G), E(G))$. Let $T : X \to P(X)$, $g : X \to X$ be (T, g)-G-contraction, $T(X) \subseteq g(X)$ and $g(X)$ is an E-complete subspace of X. Assume that T is edge preserving w.r.t. g and the graph G has the property $(*)$.*
If there exists $x_0 \in X$ such that $(gx_0, z) \in E(\tilde{G})$ for all $z \in Tx_0$, then T and g have a point of coincidence in $g(X)$.

Proof. Suppose there exists $x_0 \in X$ such that $(gx_0, z) \in E(\tilde{G})$ for all $z \in Tx_0$. As Tx_0 is nonempty and contained in $g(X)$, there exists $x_1 \in X$ such that $gx_1 \in Tx_0$ and $(gx_0, gx_1) \in E(\tilde{G})$. By using (T, g)-G-contraction property, there exists $gx_2 \in Tx_1$ such that

$$d(gx_1, gx_2) \preceq kd(gx_0, gx_1).$$

Since T is edge preserving w.r.t. g, it follows that $(z_1, z_2) \in E(\tilde{G})$ for all $z_1 \in Tx_0$, $z_2 \in Tx_1$. This ensures that $(gx_1, gx_2) \in E(\tilde{G})$. Again, using (T, g)-G-contraction property, there exists $gx_3 \in Tx_2$ such that

$$d(gx_2, gx_3) \preceq kd(gx_1, gx_2) \preceq k^2 d(gx_0, gx_1).$$

Continuing this process, we can construct a sequence (gx_n) in $g(X)$ such that $gx_n \in Tx_{n-1}$, $n = 1, 2, \cdots$ with $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for $n = 0, 1, 2, \cdots$ and

$$d(gx_n, gx_{n+1}) \preceq kd(gx_{n-1}, gx_n) \preceq \cdots \preceq k^n d(gx_0, gx_1)$$

for all $n \in \mathbb{N}$.

We shall show that (gx_n) is an E-Cauchy sequence in $g(X)$.

For any $n \in \mathbb{N}$ and $p = 1, 2, 3, \cdots$, we have

$$\begin{aligned} d(gx_n, gx_{n+p}) &\preceq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{n+p-1}, gx_{n+p}) \\ &\preceq [k^n + k^{n+1} + \cdots + k^{n+p-1}] d(gx_0, gx_1) \\ &= k^n \frac{1-k^p}{1-k} d(gx_0, gx_1) \\ &\preceq \frac{k^n}{1-k} d(gx_0, gx_1) \\ &\preceq \frac{1}{n} \frac{1}{1-k} d(gx_0, gx_1) \\ &= \frac{1}{n} b, \end{aligned}$$

where $b = \frac{1}{1-k} d(gx_0, gx_1) \in E_+$.

Since E is Archimedean, we have $\frac{1}{n} b \downarrow \theta$. This proves that the sequence (gx_n) is E-Cauchy in $g(X)$. As $g(X)$ is E-complete, there exists $u (= gt) \in g(X)$ for some $t \in X$ such that $gx_n \xrightarrow{d,E} u$. So, there exists a sequence (a_n) in E such that $a_n \downarrow \theta$ and $d(gx_n, u) \preceq a_n$ for all n.

By using property $(*)$ of the graph G, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gt) \in E(\tilde{G})$ for all $i \geq 1$. Now, $gx_{n_i+1} \in Tx_{n_i}$ for any $i \in \mathbb{N}$. So, by applying (T, g)-G-contraction property, there exists $z \in Tt$ such that

$$d(gx_{n_i+1}, z) \preceq kd(gx_{n_i}, gt)$$

for any $i \in \mathbb{N}$.

Now,

$$\begin{aligned} d(z, u) &\preceq d(gx_{n_i+1}, z) + d(gx_{n_i+1}, u) \\ &\preceq kd(gx_{n_i}, u) + a_{n_i+1} \\ &\preceq (k+1) a_{n_i} \downarrow \theta. \end{aligned}$$

This implies that $d(z, u) = \theta$ and hence $z = u = gt \in Tt$. This proves that u is a point of coincidence of T and g in $g(X)$. ■

Corollary 1 *Let E be Archimedean and let (X, d, E) be an E-complete vector metric space with a digraph $G = (V(G), E(G))$. Assume that $T : X \to P(X)$ is a G-contraction which is edge preserving and the graph G has the property $(\acute{*})$.*
If there exists $x_0 \in X$ such that $(x_0, z) \in E(\tilde{G})$ for all $z \in Tx_0$, then T has a fixed point in X.

Proof. The proof follows from Theorem 1 by taking $g = I$, the identity map on X. ■

Corollary 2 *Let E be Archimedean and let (X, d, E) be a vector metric space. Let $T : X \to P(X)$, $g : X \to X$ be (T, g)-contraction. If $T(X) \subseteq g(X)$ and $g(X)$ is an E-complete subspace of X, then T and g have a point of coincidence in $g(X)$.*

Proof. The proof follows from Theorem 1 by taking $G = G_0$. ■

Corollary 3 *Let E be Archimedean and let (X, d, E) be an E-complete vector metric space. Suppose $T : X \to P(X)$ is a mapping which satisfies the following condition: For $x, y \in X$ and any $u \in Tx$, there exists $v \in Ty$ such that*

$$d(u, v) \preceq kd(x, y)$$

where $k \in [0, 1)$ is a constant. Then T has a fixed point in X.

Proof. The proof follows from Theorem 1 by taking $g = I$ and $G = G_0$. ■

We now give an application of Theorem 1. For this purpose, we need the following definitions.

Let (X, d, E) be a vector metric space and ρ be a binary relation over X. Denote $S = \rho \cup \rho^{-1}$. Then

$$x, y \in X, \ xSy \Leftrightarrow x\rho y \text{ or } y\rho x.$$

Definition 9 *We say that (X, d, E, S) is regular if the following condition holds:*

If the sequence (x_n) in X and the point $x \in X$ are such that $x_n S x_{n+1}$ for all $n \geq 1$ and $x_n \xrightarrow{d,E} x$, then there exists a subsequence (x_{n_i}) of (x_n) such that $x_{n_i} S x$ for all $i \geq 1$.

Definition 10 *Let (X, d, E) be a vector metric space and ρ be a binary relation over X. Then the mapping $g : X \to X$ is called comparative if g maps comparable elements into comparable elements, that is,*

$$x, y \in X, \ xSy \Rightarrow (gx)\, S\, (gy).$$

Similarly, for $T : X \to P(X), \ g : X \to X$, we call T is comparative w.r.t. g if

$$x, y \in X, \ (gx)S(gy) \Rightarrow z_1 S z_2 \text{ for all } z_1 \in Tx, \ z_2 \in Ty.$$

Theorem 2 *Let E be Archimedean and (X, d, E) a vector metric space endowed with a binary relation ρ over X. Let the mappings $T : X \to P(X), \ g : X \to X$ satisfy the following condition:*
For $x, y \in X$ with $(gx)S(gy)$ and any $u \in Tx$, there exists $v \in Ty$ such that

$$d(u, v) \preceq kd(gx, gy)$$

where $k \in [0, 1)$ is a constant. Suppose that T is comparative w.r.t. g, $T(X) \subseteq g(X)$ and $g(X)$ is an E-complete subspace of X. Suppose also that the following conditions hold:

(i) (X, d, E, S) is regular;

(ii) there exists $x_0 \in X$ such that $(gx_0)Sz$ for all $z \in Tx_0$.

Then T and g have a point of coincidence in $g(X)$.

Proof. We consider the digraph $G = (V(G), E(G))$ where $V(G) = X, \ E(G) = \{(x, y) \in X \times X : xSy\} \cup \Delta$. Then it is easy to observe that all the hypotheses of Theorem 1 hold true under this digraph. Now applying Theorem 1, we obtain the desired conclusion. ■

Remark 1 *As a special case of Theorem 2, we may obtain several important fixed point and coincidence point results in vector metric spaces.*

Now we furnish an example to justify the validity of our main result.

Example 4 *Let $E=\mathbb{R}^2$ with component wise ordering and let $X=\{(x,0)\in\mathbb{R}^2:0\leq x\leq 1\}\cup\{(0,y)\in\mathbb{R}^2:0\leq y\leq 1\}$. Define $d:X\times X\to E$ by*

$$d((x,0),(y,0))=(\mid x-y\mid,\frac{2}{3}\mid x-y\mid),$$

$$d((0,x),(0,y))=(\frac{4}{3}\mid x-y\mid,\mid x-y\mid),$$

$$d((x,0),(0,y))=d((0,y),(x,0))=(x+\frac{4}{3}y,\frac{2}{3}x+y).$$

Then (X,d,E) is a vector metric space. Let G be a digraph such that $V(G)=X$ and $E(G)=\Delta\cup\{((\frac{1}{2},0),(0,\frac{1}{2}))\}$. Let $g:X\to X$ be defined by

$$gx=\frac{x}{2} \text{ for all } x\in X$$

and $T:X\to P(X)$ be defined by

$$T(x,0)=\{(\frac{x}{2},0)\},$$

$$T(0,y)=\{(\frac{y}{2},0)\}.$$

Then, $T(X)\subseteq g(X)$, where $g(X)=\{(\frac{x}{2},0)\in\mathbb{R}^2:0\leq x\leq 1\}\cup\{(0,\frac{y}{2})\in\mathbb{R}^2:0\leq y\leq 1\}$ and $g(X)$ is an E-complete subspace of (X,d,E). It is easy to observe that (T,g)-G-contraction property holds trivially.

On the other hand, T is edge preserving w.r.t. g and for $x_0=(1,0)$, $(gx_0,z)\in E(\tilde{G})$ for all $z\in Tx_0$. To check the property $(*)$, we note that any sequence (gx_n) in $g(X)$ with $(gx_n,gx_{n+1})\in E(\tilde{G})$ and $gx_n\xrightarrow{d,E}u$ must be a constant sequence. In fact, if $gx_n=a$ for all n, where a is a fixed element of $g(X)$, then u must be a and hence the property $(*)$ holds true. Thus, all the conditions of Theorem 1 are fulfilled and we conclude that T and g have a point of coincidence $(\frac{1}{2},0)$ in $g(X)$. It is valuable to note that $(\frac{1}{n},0)$ for any $n(\geq 2)\in\mathbb{N}$ is a point of coincidence of T and g in $g(X)$. Thus, T and g have infinitely many points of coincidence in $g(X)$.

Acknowledgment. The author would like to thank the referees for their valuable comments.

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