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Existence And Multiplicity Results For Sixth-Order Differential Equations *

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Abstract

We employ some known critical point theorems to establish results on the existence of weak solutions for a nonlinear sixth-order ordinary differential equation. One of the results ensures the existence of at least two weak solutions, while another one proves the existence of at least three.

1 Introduction

In this paper, based on two critical point theorems, we prove the existence of at least two and three weak solutions for the following sixth-order ordinary differential equation:

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda f(x,u) \quad a.e. \quad x \in [0,1], \\ u(0) = u(1) = u''(0) = u''^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{1}$$

where $\lambda > 0$, A, B and C are constants and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Caratheodory function. In [3] the authors investigated the existence of at least one nontrivial solution for problem (1). Also, infinitely many solutions for problem (1) have been considered in [6].

Compared to the results obtained in [3] and [6], we provide some new assumptions that ensure the existence of weak solutions for problem (1). More precisely, in one of our main results, we establish the existence of at least two weak solutions for problem (1), while in the other theorem we prove the existence of at least three such solutions.

We refer the interested reader to have an overview on the applications of high order differential equations to [1, 7, 8] and the references therein, and to [10, 11], where non-local conditions are also considered.

The paper consists of four sections. In Section 2, we recall some basic concepts and results that constitute our main tools. The main results and their proofs are given in Section 3. Finally, Section 4 is devoted to some concrete applications.

2 Preliminaries

To ensure the existence of weak solutions for problem (1), we use the following critical point theorems as our main tools. The proofs of these results can be found in [4] and [5], respectively.

Theorem 1 *([4]) Let X be a reflexive real Banach space, $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there are $r \in \mathbb{R}$ and $\overline{u} \in X$, with $0 < \Phi(\overline{u}) < r$, such that*

$$\frac{\sup_{\Phi(u) \le r} \Psi(u)}{r} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})},$$

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and for each $\lambda \in \left(\frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right)$, *the functional* $J_\lambda = \Phi - \lambda\Psi$ *satisfies (PS)-condition and it is unbounded from below. Then, for each* $\lambda \in \left(\frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right)$, *the functional* J_λ *admits at least two non-zero critical points* u_1, u_2 *such that* $J_\lambda(u_1) < 0 < J_\lambda(u_2)$.

Theorem 2 *([5, Theorem 2.6]) Let* X *be a reflexive real Banach space,* $\Phi : X \longrightarrow \mathbb{R}$ *be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^*, $\Psi : X \longrightarrow \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that* $\Phi(0) = \Psi(0) = 0$. *Assume that there exist* $r > 0$ *and* $\overline{v} \in X$, *with* $r < \Phi(\overline{v})$ *such that*

(a) $\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$, *and*

(b) for each $\lambda \in \Lambda_r := \left]\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$ *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

Here, we recall some basic concepts that will be used in what follows. Let

$$X = \left\{u \in H^3(0,1) \cap H_0^1(0,1) : u''(0) = u''(1) = 0\right\},$$

which is endowed with the norm

$$\|u\| = \left(||u'''||_2^2 + ||u''||_2^2 + ||u'||_2^2 + ||u||_2^2\right)^{\frac{1}{2}}, \tag{2}$$

where $||.||_2$ denotes the usual norm in $L^2(0,1)$. It is well known that $||.||$ is induced by the inner product

$$< u, v >= \int_0^1 \big(u'''(x)v'''(x) + u''(x)v''(x) + u'(x)v'(x) + u(x)v(x)\big)dx, \quad \forall u, v \in X.$$

Now, arguing as in [9], we point out some useful Poincare type inequalities.

Proposition 1 ([6]) *For every* $u \in X$, *if* $k = \frac{1}{\pi^2}$, *one has*

$$||u^{(i)}||_2^2 \le k^{j-i}||u^{(j)}||_2^2 \quad i = 0, 1, 2, j = 1, 2, 3 \;\; \textit{with} \;\; i < j.$$

We will introduce a convenient norm, equivalent to $||.||$ that still makes X a Hilbert space. For this reason, for A, B, $C \in \mathbb{R}$, let us define the function $N : X \to \mathbb{R}$ by putting

$$N(u) = ||u'''||_2^2 + A||u''||_2^2 + B||u'||_2^2 + C||u||_2^2$$

for every $u \in X$. Now consider the following set of conditions according to the signs of the constants A, B and C:

(H_1) $A \ge 0$, $B \ge 0$, $C \ge 0$;

(H_2) $A \ge 0$, $B \ge 0$, $C < 0$ and $-Ak - Bk^2 - Ck^3 < 1$;

(H_3) $A \ge 0$, $B < 0$, $C \ge 0$ and $-Ak - Bk^2 < 1$;

(H_4) $A \ge 0$, $B < 0$, $C < 0$ and $-Ak - Bk^2 - Ck^3 < 1$;

(H_5) $A < 0$, $B \ge 0$, $C \ge 0$ and $-Ak < 1$;

(H_6) $A < 0$, $B \geq 0$, $C < 0$ and $\max\{-Ak, -Ak - Bk^2 - Ck^3\} < 1$;

(H_7) $A < 0$, $B < 0$, $C \geq 0$ and $-Ak - Bk^2 < 1$;

(H_8) $A < 0$, $B < 0$, $C < 0$ and $-Ak - Bk^2 - Ck^3 < 1$.

Moreover, fix A, B, $C \in \mathbb{R}$ and consider the following condition:

(H) $\max\{-Ak, -Ak - Bk^2, -Ak - Bk^2 - Ck^3\} < 1$.

We have the following results.

Proposition 2 (**[6]**) *Condition* (H) *holds if and only if one of conditions* $(H_1) - (H_8)$ *holds.*

Proposition 3 (**[6]**) *Assume* (H). *Then, there exits* $m > 0$ *such that*

$$N(u) \geq m||u||^2 \quad \forall u \in X.$$

Proposition 4 (**[6]**) *Assume that* (H) *holds and put*

$$\|u\|_X = \sqrt{N(u)} \qquad \forall u \in X.$$

Then, $\|u\|_X$ *is a norm equivalent to the usual one defined in* (2) *and* $(X, \|u\|_X)$ *is a Hilbert space.*

Clearly, the embedding $(X, \|u\|_X) \hookrightarrow (C^0(0,1), \|u\|_\infty)$ is compact. For a qualitative estimate of the constant of this embedding it is useful to introduce the following number:

$$\delta = \begin{cases} 1 & \text{if } (H_1) \text{ holds,} \\ \min\{1, 1 + Ak + Bk^2 + Ck^3\} & \text{if } (H_2) \text{ or } (H_4) \text{ holds,} \\ \min\{1, 1 + Ak + Bk^2\} & \text{if } (H_3) \text{ holds,} \\ 1 + Ak & \text{if } (H_5) \text{ holds,} \\ \min\{1 + Ak, 1 + Ak + Bk^2\} & \text{if } (H_6) \text{ holds,} \\ 1 + Ak + Bk^2 & \text{if } (H_7) \text{ holds,} \\ 1 + Ak + Bk^2 + Ck^3 & \text{if } (H_8) \text{ holds.} \end{cases} \tag{3}$$

Proposition 5 (**[6]**) *Assume that* (H) *holds. One has*

$$\|u\|_\infty \leq \frac{k}{2\sqrt{\delta}} \|u\|_X \tag{4}$$

for every $u \in X$, *where* δ *is given in* (3).

For the sake of completeness, we recall that for $r \in \mathbb{R}$, $I_\lambda = \Phi - \lambda\Psi$ is said to satisfy the $(PS)^{[r]}$-condition if any sequence $\{u_n\}$ such that

(a_1) $\{I_\lambda(u_n)\}$ is bounded ;

(a_2) $||I'_\lambda(u_n)||_{X^*} \to 0 \quad as \;\; n \to +\infty$;

(a_3) $\Phi(u_n) < r, \quad \forall n \in \mathbb{N}$,

has a convergent subsequence.

As usual, a weak solution to problem (1) is any $u \in X$ such that

$$\int_0^1 \big(u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x)\big)dx = \lambda \int_0^1 f(x, u(x))v(x)dx, \tag{5}$$

for all $v \in X$. In order to clarify the variational structure of problem (1), we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by putting

$$\Phi(u) = \frac{1}{2}\|u\|_X^2, \qquad \Psi(u) = \int_0^1 F(x, u(x))dx, \ \forall u \in X, \tag{6}$$

where $F(x,t) = \int_0^t f(x,s)ds$ for every $(x,t) \in [0,1] \times \mathbb{R}$.

Clearly, Φ is coercive and continuously Gâteaux differentiable and its Gâteaux derivative at point $u \in X$ is defined by

$$\Phi'(u)(v) = \int_0^1 \big(u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x)\big)dx$$

for all $v \in X$. On the other hand Ψ is continuously Gâteaux differentiable and its Gâteaux derivative at point $u \in X$ is defined by

$$\Psi'(u)(v) = \lambda \int_0^1 f(x, u(x))v(x)dx,$$

for all $v \in X$ and $\Phi(0) = \Psi(0) = 0$.

Remark 1 (see [2, Remark 2.1]) *The functional $I_\lambda = \Phi - \lambda\Psi$ defined in (6) satisfies the $(PS)^{[r]}$-condition for every $r \in \mathbb{R}$.*

3 Main Results

The following technical constant will be useful

$$\tau = 4\delta\pi^4\big(96(\frac{12}{5})^5 + 4A(\frac{12}{5})^4 + B\frac{1248}{175} + C\frac{493}{756}\big)^{-1}, \tag{7}$$

where A, B and C are the real numbers involved in problem (1) and such that (H) holds, while δ has been introduced in (3). Now, we present an application of Theorem 1 which we will use to obtain one nontrivial weak solution.

Theorem 3 *Assume that there exist two positive constants d and θ such that*

$$d^2 < \tau\theta^2,$$

(A_1) $F(x,t) \geq 0$, *for every* $(x,t) \in ([0, \frac{5}{12}] \cup [\frac{7}{12}, 1]) \times \mathbb{R}$;

(A_2)

$$\frac{\int_0^1 \max_{|s|\leq\theta} F(x,s)dx}{\theta^2} < \tau\frac{\int_{\frac{5}{12}}^{\frac{7}{12}} F(x,d)dx}{d^2}$$

where the constant τ is given by (7).

Then, for each

$$\lambda \in 2\delta\pi^4\left(\frac{d^2}{\tau\int_{\frac{5}{12}}^{\frac{7}{12}} F(x,d)dx}, \frac{\theta^2}{\int_0^1 \max_{|s|\leq\theta} F(x,s)dx}\right),$$

problem (1) has at least two non-zero weak solutions.

Proof. Our goal is to apply Theorem (1). Consider the Sobolev space X and the operators Φ and Ψ defined in (6). We observe that the regularity assumptions on Φ and Ψ are satisfied and then taking into account Remark (1) by standard computations, for each $\lambda > 0$, I_λ satisfies the $(PS)^{[r]}$-condition. Consider the function

$$w(x) = \begin{cases} dv(x) & \text{if } x \in [0, \frac{5}{12}[, \\ d & \text{if } x \in [\frac{5}{12}, \frac{7}{12}], \\ dv(1-x) & \text{if } x \in]\frac{7}{12}, 1], \end{cases} \tag{8}$$

where $v(x) = (\frac{12}{5})^4x^4 - 2(\frac{12}{5})^3x^3 + \frac{24}{5}x$ for every $x \in [0, \frac{5}{12}]$, a straightforward computation shows that $w \in X$ and, in particular $||w||_X^2 = \frac{4\delta\pi^4}{\tau}d^2$.

Recalling that (H) holds, the positivity of t follows from the positivity of d as seen in the arguments presented in the previous section (see also Proposition 3). Moreover, from (4), since $\|u\|_\infty = 1$, one can even conclude that $0 < \tau < 1$. Therefore

$$\Phi(w) = \frac{\|w\|_X^2}{2} = \frac{2\delta\pi^4}{\tau}d^2, \tag{9}$$

and by (A_1) we have

$$\Psi(w) = \int_0^1 F(x, w)dx \geq \int_{\frac{5}{12}}^{\frac{7}{12}} F(x, d)dx. \tag{10}$$

Moreover, since $0 < d < \theta$ and by (A_2), we obtain that

$$d^2 < \tau\theta^2. \tag{11}$$

Indeed, arguing by contradiction, if we assume that $d^2 > \tau\theta^2$, we have

$$\frac{\int_0^1 \max_{|s|\leq\theta} F(x, s)dx}{\theta^2} \geq \frac{\int_{\frac{5}{12}}^{\frac{7}{12}} F(x, d)dx}{\theta^2} \geq \frac{\tau \int_{\frac{5}{12}}^{\frac{7}{12}} F(x, d)dx}{d^2}$$

which contradicts (A_2). Put $r = 2\delta\pi^4\theta^2$, by (9) and (11) we obtain $0 < \Phi(w) = \frac{2\delta\pi^4}{\tau}d^2 < r$.

Therefore, one has

$$\frac{\Psi(w)}{\Phi(w} \geq \frac{\tau}{2\delta\pi^4}\frac{\int_{\frac{5}{12}}^{\frac{7}{12}} F(x, d)dx}{d^2}.$$

From (4) we have

$$\begin{aligned} \Phi^{-1}(]-\infty, r]) &= \{u \in X;\ \Phi(u) \leq r\} \\ &\subseteq \left\{u \in X;\ \frac{1}{2}\|u\|_X^2 \leq r\right\} \\ &\subseteq \{u \in X;\ |u(x)| \leq \theta \text{ for each } x \in [0, 1]\}, \end{aligned}$$

and it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r])} \int_0^1 F(x, u)dx \leq \int_0^1 \max_{|s|\leq\theta} F(x, s)dx.$$

Therefore, we have

$$\begin{aligned} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \int_0^1 F(x, u(x))dx}{r} \\ &= \frac{\int_0^1 \max_{|s|\leq\theta} F(x, s)dx}{2\delta\pi^4\theta^2}. \end{aligned}$$

All assumptions of Theorem (1) are satisfied and the proof is complete. Then, from (A_2), for each

$$\lambda \in 2\delta\pi^4 \left(\frac{d^2}{\tau \int_{\frac{5}{12}}^{\frac{7}{12}} F(x,d)dx}, \frac{\theta^2}{\int_0^1 \max_{|s|\leq\theta} F(x,s)dx} \right),$$

problem (1) has at least two non-zero weak solutions. ■

Now, we present an application of Theorem 2 which we will use to obtain three nontrivial weak solutions.

Theorem 4 *Assume that there exists two positive constants d and θ such that $\tau\theta^2 < d^2$ with holds (A_1) and (A_2), moreover, there exists two constants $\alpha, \beta > 0$ such that*

(A_3) $|f(x,t)| \leq \alpha + \beta|t|$ *for every* $x \in [0,1]$ *and* $t \in \mathbb{R}$.

Then, for each

$$\lambda \in 2\delta\pi^4 \left(\frac{d^2}{\tau \int_{\frac{5}{12}}^{\frac{7}{12}} F(x,d)dx}, \frac{\theta^2}{\int_0^1 \max_{|s|\leq\theta} F(x,s)dx} \right),$$

problem (1) has at least three non-zero weak solutions.

Proof. From (A_3) we have

$$F(x,t) \leq \alpha t + \frac{\beta}{2}|t|^2 \tag{12}$$

for every $x \in [0,1]$ and $t \in \mathbb{R}$. Taking (12) into account, it follows that, for each $u \in X$,

$$\begin{aligned} I_\lambda(u) = \Phi(u) - \lambda\Psi(u) &= \frac{\|u\|_X^2}{2} - \lambda \int_0^1 F(x,u)dx \\ &\geq \frac{\|u\|_X^2}{2} - \frac{\beta}{8\delta\pi^4}\|u\|_X^2 - \frac{\lambda\alpha}{2\pi^2\sqrt{\delta}}\|u\|_X \\ &\geq (1 - \frac{\beta}{4\delta\pi^4})\frac{\|u\|_X^2}{2} - \frac{\lambda\alpha}{2\pi^2\sqrt{\delta}}\|u\|_X, \end{aligned}$$

with $\frac{\beta}{4\delta\pi^4} < 1$ one has

$$\lim_{\|u\|\to+\infty} I_\lambda = +\infty,$$

which means the functional I_λ is coercive, and the condition Theorem 2 of is verified. ■

4 Applications

In this section, we point out some consequences and applications of the results previously obtained. Now, we point out a result in which the nonlinear term has separable variables. To be precise, let $h : [0,1] \to \mathbb{R}$ be a function such that $h \in L^1([0,1])$, $h(x) \geq 0$ a.e. $x \in [0,1]$, $h \neq 0$ and let $g : \mathbb{R} \to \mathbb{R}$ be nonnegative and continuous function. Consider the following problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda h(x)g(u) \quad \text{a.e. } x \in]0,1[, \\ u(0) = u(1) = u''^{(iv)}(0) = u^{(iv)}(1) = 0. \end{cases} \tag{13}$$

and put $G(t) = \int_0^t g(\xi)d\xi$ for all $t \in \mathbb{R}$,

$$\sigma := \tau . \frac{\int_{\frac{5}{12}}^{\frac{7}{12}} h(x)dx}{||h||_{L^1([0,1])}}, \tag{14}$$

where τ is given by (7).

Taking into account Theorem 3, we have the following result.

Corollary 1 *Assume that there exist two positive constants d and θ such that $d^2 < \sigma\theta^2$ and*

$$\frac{G(\theta)}{\theta^2} < \sigma \frac{G(d)}{d^2},$$

where the constant σ is given by (14). Then, for each $\lambda \in \frac{2\delta\pi^4}{||h||_{L^1([0,1])}} \left(\frac{d^2}{\sigma G(d)}, \frac{\theta^2}{G(\theta)} \right)$, the problem (13) admits two non-zero weak solutions.

A consequence of Corollary 1 is the following result.

Theorem 5 *Assume that*

$$\lim_{t \to 0^+} \frac{g(t)}{t} = +\infty. \tag{15}$$

Then, for each $\lambda \in \frac{2\delta\pi^4}{||h||_{L^1([0,1])}} \left(0, \sup_{\theta>0} \frac{\theta^2}{G(\theta)} \right)$, the problem (13) admits two non-zero weak solutions.

Taking into account Theorem 4, we have the following result.

Theorem 6 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $F(d) > 0$ for some $d > 0$, $F(\xi) \geq 0$ and $\liminf_{\xi \to 0} \frac{f(\xi)}{\xi} = \limsup_{\xi \to +\infty} \frac{f(\xi)}{\xi} = 0$. Then there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ the problem*

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda f(u(t)), \quad a.e. \quad x \in (0,1), \\ u(0) = u(1) = u''^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{16}$$

admits at least three distinct weak solutions in X.

Proof. Fix $\lambda > \lambda^* := \frac{12\delta\pi^4 d^2}{\tau F(d)}$ where τ is given by (7). Since

$$\liminf_{\xi \to 0} \frac{F(\xi)}{\xi^2} = 0,$$

there is a sequence $\{\theta_n\} \subset (0, +\infty)$ such that $\lim_{n\to+\infty} \theta_n = 0$ and

$$\lim_{n \to +\infty} \frac{\max_{|\xi| \leqslant \theta_n} F(\xi)}{\theta_n^2} = 0.$$

Indeed, one has

$$\lim_{n \to +\infty} \frac{\max_{|\xi| \leqslant \theta_n} F(\xi)}{\theta_n^2} = \lim_{n \to +\infty} \frac{\max_{|\xi| \leqslant \theta_n} F(\xi)}{\xi^2} . \frac{\xi^2}{\theta_n^2} = 0.$$

Here, $\tau\theta_n^2 < d^2$ and, from Theorem 4, the conclusion follows. ■

Example 1 *Consider $A = -1$, $B = C = 0$ and the function $g(t) = 4t^3 + 1$ where satisfy (15). Moreover, by (H_5) one has $\delta = \frac{\pi^2-1}{\pi^2}$, therefore due to Theorem 5, for all $\lambda \in]0, \lambda^{**}[$ where*

$$\lambda^{**} = 2\pi^2(\pi^2 - 1) \sup_{\theta>0} \frac{\theta^2}{G(\theta)} = 2\pi^2(\pi^2 - 1) \sup_{\theta>0} \frac{\theta}{\theta^3 + 1} \geq 2\pi^2(\pi^2 - 1) \frac{\theta}{\theta^3 + 1} \,|_{\theta=\frac{1}{2}} = \frac{8\pi^2(\pi^2 - 1)}{9},$$

the problem

$$\begin{cases} -u^{(vi)} - u^{(iv)} = \lambda g(u) \quad a.e. \quad x \in]0,1[, \\ u(0) = u(1) = u''(0) = u''^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{17}$$

admits at least two non-zero and nonnegative solutions.

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References

[1] P. Bates, P. Fife, R. Gardner and C. K. R. J. Jones, The existence of traveling wave solutions of a generalized phase-field model, SIAM J. Math. Anal., 28(1997), 142–149.

[2] G. Bonanno, A critical point theorem via the Ekeland variational principle, Nonlinear Anal. TMA, 75(2012), 2992–3007.

[3] G. Bonanno, P. Candito and D. O. Regan, Existence of nontrivial solutions for sixth-order differential equations, Mathematics, 9(2021), 1–9.

[4] G. Bonanno and G. D. Agui, Two non-zero solutions for elliptic Dirichlet problems, Z. Anal. Anwend., 35(2016), 449–464.

[5] G. Bonanno and S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, Appl. Anal., 89(2010), 1–10.

[6] G. Bonanno and R. Livrea, A sequence of positive solutions for sixth-order ordinary nonlinear differential problems, Electron. J. Qual. Theory Differ. Equ., 20(2021), 1–17.

[7] G. Caginalp and P. Fife, Higher order phase field models and detailed anisotropy, Phys. Rev. B., 34(1986), 4940–4943.

[8] R. A. Gardner and C. K. R. T. Jones, Traveling waves of a perturbed diffusion equation arising in a phase field model, Indiana Univ. Math. J., 39(1990), 1197–1222.

[9] L. A. Peletier, R. K. A. M. Van der Vorst and W. C. Troy, Stationary solutions of a fourth-order nonlinear diffusion equation, Differential Equations, 31(1995), 301–314.

[10] J. R. L. Webb, Positive solutions of some higher order nonlocal boundary value problems, Electron. J. Qual. Theory Differ. Equ., 29(2009), 1–15.

[11] J. R. L. Webb and M. Zima, Multiple positive solutions of resonant and non-resonant non-local fourth-order boundary value problems, Glasg. Math. J., 54(2012), 225–240.