Applied Mathematics E-Notes, 24(2024), 80-88 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Studies On Some Sequence Spaces In Gradual Normed Linear Space*

Hamari Debbarma†, Chiranjib Choudhury‡, Shyamal Debnath§

Received 20 January 2023

Abstract

In this paper, we introduce and investigate some new class of sequences in a gradual normed linear space $(X, \|\cdot\|_G)$ and discuss some properties of these spaces like completeness, solidness, symmetricity, convergence free, sequence algebra, etc. and prove some inclusion relations.

1 Introduction

The idea of fuzzy sets was first introduced by Zadeh [21] in the year 1965 which was an extension of the classical set-theoretical concept. Nowadays it has wide applicability in different branches of science and engineering. The "fuzzy number" plays a crucial role in the study of fuzzy set theory. Fuzzy numbers were the generalization of intervals, not numbers. Even fuzzy numbers do not obey a few algebraic properties of the classical numbers. So the "fuzzy number" is debatable to many authors due to its different behavior. The "fuzzy intervals" is often used by many authors instead of fuzzy number. To overcome the confusion among the researchers, in 2008, Fortin et al. [13] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are mainly known by their respective assignment function which is defined in the interval $(0, 1]$. So in some sense, every real number can be viewed as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and have been used in computation and optimization problems.

In 2011, Sadeqi and Azari [17] first introduced the concept of gradual normed linear space (GNLS). They studied various properties of the space from both the algebraic and topological points of view. Further progress in this direction has occurred due to Ettefagh et al. [11, 12], Choudhury and Debnath [5, 6, 7, 8, 11, 12] and many others. For an extensive study on gradual real numbers [1, 9, 14, 18, 22, 23] can be addressed, where many more references can be found.

In functional analysis, a sequence space is a vector space whose elements are infinite sequences of real or complex numbers. Equivalently, it is a function space whose elements are functions from the natural numbers to the field K of real or complex numbers. The most important sequence spaces are bounded, convergent and null sequence spaces, respectively denoted by ℓ_∞, c, and c_0. A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence α_k of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$. A sequence space E is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where π is a permutation of $\mathbb{N}$. A sequence space E is said to be sequence algebra if $(x_k) \star (y_k) = (x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$. Let $K = \{k_1 < k_2 < k_3 < \dots\} \subset \mathbb{N}$ and $(x_n) \in w$. Then the K-step space of the sequence space E is defined by

$$\lambda_K^E = \{(x_{k_i}) \in w : (x_n) \in E\}.$$

*Mathematics Subject Classifications: 03E72, 40A35, 40A05.

†Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, India
‡Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, India
§Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, Indin

A canonical preimage (y_n) of a sequence $(x_n) \in E$, where K-step space λ_K^E is considered, is defined by

$$y_n = \begin{cases} x_n, & \text{if } n \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A sequence space E is said to be monotone, if it contains all preimages of its step spaces. For extensive study on sequence spaces one may refer to [2, 3, 4, 10, 15, 16, 19, 20].

Motivated by the above works, in this paper, we define some new notions of sequence spaces in GNLS such as space of all, bounded, convergent and null sequences with assignment function

$$\mathcal{A}_{\|x\|_G}(\alpha) = f(\alpha)\|x\| \quad (x \text{ is a element of real normed space}), \forall \alpha \in (0,1],$$

where $f : (0,1] \to \mathbb{R}^+$ is a nonzero function. Then, we investigated that they are complete and prove some inclusion relations. Finally, we define some properties of sequences like solid space, symmetric space, sequence algebra, monotone space, convergence free and prove some theorems about these spaces.

2 Preliminaries

In this section, we present some existing definitions and results which are crucial for the subsequent sections.

Definition 1 ([13]) *A gradual real number $\tilde{r}$ is defined by an assignment function $\mathcal{A}_{\tilde{r}} : (0,1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual real number $\tilde{r}$ is said to be non-negative, if for every $\alpha \in (0,1]$, $\mathcal{A}_{\tilde{r}}(\alpha) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^\star(\mathbb{R})$.*

In [13], the gradual operations between the elements of $G(\mathbb{R})$ was defined as follows:

Definition 2 *Let $\star$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $\mathcal{A}_{\tilde{r}_1}$ and $\mathcal{A}_{\tilde{r}_2}$ respectively. Then, $\tilde{r}_1 \star \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $\mathcal{A}_{\tilde{r}_1 \star \tilde{r}_2}$ given by*

$$\mathcal{A}_{\tilde{r}_1 \star \tilde{r}_2}(\alpha) = \mathcal{A}_{\tilde{r}_1}(\alpha) \star \mathcal{A}_{\tilde{r}_2}(\alpha), \quad \forall \alpha \in (0,1].$$

In particular, the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r}(c \in \mathbb{R})$ are defined by

$$\mathcal{A}_{\tilde{r}_1 + \tilde{r}_2}(\alpha) = \mathcal{A}_{\tilde{r}_1}(\alpha) + \mathcal{A}_{\tilde{r}_2}(\alpha) \quad \text{and} \quad \mathcal{A}_{c\tilde{r}}(\alpha) = c\mathcal{A}_{\tilde{r}}(\alpha), \quad \forall \alpha \in (0,1].$$

For any real number $s \in \mathbb{R}$, the constant gradual real number $\tilde{s}$ is defined by the constant assignment function $\mathcal{A}_{\tilde{s}}(\alpha) = s$ for any $\alpha \in (0,1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are the constant gradual numbers defined by $\mathcal{A}_{\tilde{0}}(\alpha) = 0$ and $\mathcal{A}_{\tilde{1}}(\alpha) = 1$ respectively. It is easy to verify that $G(\mathbb{R})$ with the gradual addition and gradual scalar multiplication forms a real vector space [13].

Definition 3 ([13]) *Let $\tilde{r}, \tilde{s} \in G(\mathbb{R})$. The partial relation "$\leq$"in $G(\mathbb{R})$ defined as $\tilde{r} \leq \tilde{s}$ iff $A_{\tilde{r}}(\alpha) \leq A_{\tilde{s}}(\alpha)$, $\forall \alpha \in (0,1]$.*

Lemma 1 ([17]) *Let $\tilde{r}, \tilde{s}, \tilde{t} \in G(\mathbb{R})$. Then*

(i) if $\tilde{r} \leq \tilde{s}$, then $\tilde{r} - \tilde{t} \leq \tilde{s} - \tilde{t}$;

(ii) if $\tilde{r} \leq \tilde{s}$, and $\tilde{0} \leq \tilde{t}$, then (a) $\tilde{r} \cdot \tilde{t} \leq \tilde{s} \cdot \tilde{t}$ and (b) $\tilde{r}/\tilde{t} \leq \tilde{s}/\tilde{t}$, $\tilde{t} \neq \tilde{0}$;

(iii) $(\tilde{r} \cdot \tilde{s})/\tilde{t} = \tilde{r} \cdot (\tilde{s}/\tilde{t})$, $\tilde{t} \neq \tilde{0}$.

Definition 4 ([17]) *Let X be a real vector space. The function $|| \cdot ||_G : X \to G^\star(\mathbb{R})$ is said to be a gradual norm on X, if for every $\alpha \in (0,1]$, following conditions hold for any $x, y \in X$:*

(i) $\mathcal{A}_{||x||_G}(\alpha) = \mathcal{A}_{\tilde{0}}(\alpha)$ if and only if $x = 0$;

(ii) $\mathcal{A}_{||cx||_G}(\alpha) = |c|\mathcal{A}_{||x||_G}(\alpha)$ *for any* $c \in \mathbb{R}$*;*

(iii) $\mathcal{A}_{||x+y||_G}(\alpha) \leq \mathcal{A}_{||x||_G}(\alpha) + \mathcal{A}_{||y||_G}(\alpha)$.

The pair $(X, ||\cdot||_G)$ *is called a gradual normed linear space (GNLS).*

Definition 5 ([12]) *Let* (x_k) *be a sequence in the GNLS* $(X, ||\cdot||_G)$*. Then,* (x_k) *is said to be gradual bounded if for every* $\alpha \in (0,1]$*, there exists* $B = B(\alpha) > 0$ *such that* $\mathcal{A}_{||x_k||_G} \leq B$ *for all* $k \in \mathbb{N}$.

Definition 6 ([17]) *Let* (x_k) *be a sequence in the GNLS* $(X, ||\cdot||_G)$*. Then,* (x_k) *is said to be gradual convergent to* $x \in X$*, if for every* $\alpha \in (0,1]$ *and* $\varepsilon > 0$*, there exists* $N(= N_\varepsilon(\alpha)) \in \mathbb{N}$ *such that*

$$\mathcal{A}_{||x_k - x||_G}(\alpha) < \varepsilon, \quad \forall k \geq N.$$

Symbolically, $x_k \xrightarrow{||\cdot||_G} x$.

Example 1 ([17]) *Let* $X = \mathbb{R}$ *and for* $x \in \mathbb{R}, \alpha \in (0,1]$*, define* $||\cdot||_G$ *by*

$$\mathcal{A}_{||x||_G}(\alpha) = e^{\alpha}|x|.$$

Then, $||\cdot||_G$ *is a gradual norm on* $\mathbb{R}$*, and* $(\mathbb{R}, ||\cdot||_G)$ *is a GNLS.*

Definition 7 ([17]) *Let* (x_k) *be a sequence in the GNLS* $(X, \|\cdot\|_G)$*. Then,* (x_k) *is said to be gradually Cauchy if for every* $\alpha \in (0,1]$ *and* $\varepsilon > 0$*, there exists* $N(= N_\varepsilon(\alpha)) \in \mathbb{N}$ *such that*

$$\mathcal{A}_{\|x_k - x_j\|_G}(\alpha) < \varepsilon, \quad \forall k, j \geq N.$$

The pair $(X, \|\cdot\|_G)$ *is said to be complete if every Cauchy sequence in* $(X, \|\cdot\|_G)$ *is convergent in* $(X, \|\cdot\|_G)$*.*

3 Main Results

In this section, we present the main results. Throughout the paper, w^G, ℓ_∞^G, c^G and c_0^G denotes collection of all, bounded, convergent and null sequences $x = (x_k)$ with gradual terms, respectively and defined as follows:

$w^G =$ Class of all sequences of gradual real numbers.

$\ell_\infty^G = \{(x_k) \in w^G : \exists B = B(\alpha) > 0, \mathcal{A}_{||x_k||_G}(\alpha) < B \ \forall k \in \mathbb{N}, \alpha \in (0,1]\}$.

$c^G = \{(x_k) \in w^G : \exists x_0 \in G(\mathbb{R}), \lim_{k\to\infty} \mathcal{A}_{||x_k - x_0||_G}(\alpha) = 0 \ \forall k \in \mathbb{N}, \alpha \in (0,1]\}$.

$c_0^G = \{(x_k) \in w^G : \lim_{k\to\infty} \mathcal{A}_{||x_k||_G}(\alpha) = 0 \ \forall k \in \mathbb{N}, \alpha \in (0,1]\}$.

Theorem 1 *The classes of sequences* ℓ_∞^G, c^G *and* c_0^G *are GNLS with the gradual norm*

$$\|x\|_G = \|(x_k)\|_G = \sup_{k\in\mathbb{N}, \alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right|.$$

Proof. *Let* a, b *be scalars and* $(x_k), (y_k) \in \ell_\infty^G$*. Then*

$$\sup_{k\in\mathbb{N}, \alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{N}, \alpha\in(0,1]} \left|\mathcal{A}_{\|y_k\|_G}(\alpha)\right| < \infty. \tag{1}$$

Now,

$$\begin{aligned} \|ax + by\|_G &= \sup_{k\in\mathbb{N}, \alpha\in(0,1]} \left|\mathcal{A}_{\|ax_k + by_k\|_G}(\alpha)\right| \\ &\leq |a| \sup_{k\in\mathbb{N}, \alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| + |b| \sup_{k\in\mathbb{N}, \alpha\in(0,1]} \left|\mathcal{A}_{\|y_k\|_G}(\alpha)\right| \\ &< \infty, \quad \text{by (1)}. \end{aligned}$$

Hence, ℓ_∞^G is a linear space. Similarly it can be shown that c^G and c_0^G are linear spaces.

Next for $x = (x_k) = \theta$, we have $\|x\|_G = \|(x_k)\|_G = 0$. Conversely, let $\|(x_k)\|_G = 0$. Then

$$\|(x_k)\|_G = \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| = 0.$$

Then $x_k = 0,\ \forall k \in \mathbb{N}$. It follows that $x = \theta$,

$$\begin{aligned}
\|cx\|_G &= \|(cx_k)\|_G = \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|cx_k\|_G}(\alpha)\right| \\
&= \sup_{k\in\mathbb{N},\alpha\in(0,1]} |c| \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| = |c| \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| \\
&= |c|\|x\|_G,
\end{aligned}$$

$$\begin{aligned}
\|x+y\|_G &= \|(x_k+y_k)\|_G = \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|x_k+y_k\|_G}(\alpha)\right| \\
&\le \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| + \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|y_k\|_G}(\alpha)\right| \\
&= \|(x_k)\|_G + \|(y_k)\|_G = \|x\|_G + \|y\|_G.
\end{aligned}$$

Hence, $\|\cdot\|_G$ is a gradual norm on ℓ_∞^G. Similarly it can be shown that $\|\cdot\|_G$ is a gradual norm on c^G and c_0^G. This completes the proof. ■

Theorem 2 *The sequence spaces $\ell_\infty^G, c^G, c_0^G$ are complete under the gradual norm*

$$\|x\|_G = \|(x_k)\|_G = \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right|.$$

Proof. Let (x^n) be a gradual Cauchy sequence in ℓ_∞^G, where $x^n = (x_i^n) = (x_1^n, x_2^n, \dots) \in \ell_\infty^G$, for each $n \in \mathbb{N}$. Then,

$$\|x^n - x^m\|_G = \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|x_k^n - x_k^m\|_G}(\alpha)\right| \to 0 \ \text{ as } n, m \to \infty.$$

Hence, for a given $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that

$$\|x^n - x^m\|_G = \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|x_k^n - x_k^m\|_G}(\alpha)\right| < \varepsilon, \quad \forall\ n, m \ge n_0.$$

Therefore, $|\mathcal{A}_{\|x_k^n - x_k^m\|_G}(\alpha)| < \varepsilon$, $\forall\ n, m \ge n_0$, $\alpha \in (0,1]$, $k \in \mathbb{N}$. Then (x_k^m) is a gradually Cauchy sequence in $\mathbb{R}$. So (x_k^m) is a gradually convergent in $\mathbb{R}$ (since $\mathbb{R}$ is complete w.r.t. gradual norm).

Let, $\lim\limits_{m\to\infty} x_k^m = x_k$ (say) for $k \in \mathbb{N}$. Now

$$\|x^n - x\|_G = \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|x_k^n - x_k\|_G}(\alpha)\right| \to 0 \ \text{ as } n \to \infty.$$

Therefore, $x^n \to x$ as $n \to \infty$. Finally we will prove that $x \in \ell_\infty^G$.

$$\begin{aligned}
\mathcal{A}_{\|x_k\|_G}(\alpha) &= \mathcal{A}_{\|x_k - x_k^n + x_k^n\|_G}(\alpha), \quad \forall \alpha \in (0,1],\ k \in \mathbb{N} \\
&\le \mathcal{A}_{\|x_k - x_k^n\|_G}(\alpha) + \mathcal{A}_{\|x_k^n\|_G}(\alpha), \quad \forall \alpha \in (0,1],\ k \in \mathbb{N} \\
&< \varepsilon + \mathcal{A}_{\|x_k^n\|_G}(\alpha), \quad \forall \varepsilon > 0,\ k \in \mathbb{N}.
\end{aligned}$$

It follows that $x \in \ell_\infty^G$. Hence, ℓ_∞^G is complete space. Similarly it can be shown that the spaces c^G and c_0^G are also complete. ■

Remark 1 *$c_0^G \subset c^G \subset \ell_\infty^G$ and the inclusion are proper. It follows from the next example.*

Example 2 *(1) Take $f(\alpha) = 1$. Consider the sequence (x_k) in $\mathbb{R}$, defined by $x_k = 1 + \frac{1}{k}$, for $k \in \mathbb{N}$. Then*

$$\lim_{k\to\infty} \mathcal{A}_{\|x_k\|_G}(\alpha) = \lim_{k\to\infty} |x_k|, \quad \forall \alpha \in (0,1] = \lim_{k\to\infty} \left|1 + \frac{1}{k}\right| = 1.$$

So $(x_k) \in c^G$. But $(x_k) \notin c_0^G$. Hence, $c_0^G \subset c^G$ is proper.

(2) Take $f(\alpha) = 1$. Consider the sequence (x_k), defined by $x_k = (-1)^k$, for $k \in \mathbb{N}$. Then

$$\sup_{k\in\mathbb{N}, \alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| < \infty.$$

So $(x_k) \in \ell_\infty^G$. But $(x_k) \notin c^G$. Hence, $c^G \subset \ell_\infty^G$ is proper.

Definition 8 *A subset E^G of w^G is said to be solid or normal if $(x_k) \in E^G$ implies $(y_k) \in E^G$, for all sequences (y_k) such that*

$$\left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| \geq \left|\mathcal{A}_{\|y_k\|_G}(\alpha)\right|, \quad \forall k \in \mathbb{N}, \ \alpha \in (0,1].$$

Theorem 3 *The sequence spaces ℓ_∞^G, c_0^G are solid but c^G is not solid.*

Proof. Let $(x_k) \in \ell_\infty^G$. Then $\exists\, M = M(\alpha) > 0$ such that

$$\|(x_k)\|_G = \sup_{k\in\mathbb{N}, \alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| < M.$$

Let $\left|\mathcal{A}_{\|y_k\|_G}(\alpha)\right| \leq \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right|$. Then we have

$$\|(y_k)\|_G = \sup_{k\in\mathbb{N}, \alpha\in(0,1]} \left|\mathcal{A}_{\|y_k\|_G}(\alpha)\right| \leq \sup_{k\in\mathbb{N}, \alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| < M.$$

Thus $(y_k) \in \ell_\infty^G$. Hence, ℓ_∞^G is a solid.

Let, $(x_k) \in c_0^G$. Then we have $\mathcal{A}_{\|x_k\|_G}(\alpha) \to 0$ as $k \to \infty$. Let, (y_k) be such that $\left|\mathcal{A}_{\|y_k\|_G}(\alpha)\right| \leq \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right|$. Then,

$$\left|\mathcal{A}_{\|y_k\|_G}(\alpha)\right| \to 0 \ \text{ as } k \to \infty.$$

Therefore, $(y_k) \in c_0^G$. Hence, c_0^G is solid. But c^G is not solid. Example 3 illustrates the fact. ■

Example 3 *Take $f(\alpha) = e^\alpha$, $\forall \alpha \in (0,1]$. Suppose $x_k = 1$ and $y_k = (-1)^k$. Then $(x_k) \in c^G$ because $\mathcal{A}_{\|x_k - \tilde{1}\|_G}(\alpha) = e^\alpha |x_k - 1| \to 0$ as $k \to \infty$, $\forall \alpha \in (0,1]$. Also, $\|x_k\|_G = \|y_k\|_G$, because*

$$\mathcal{A}_{\|x_k\|_G}(\alpha) = \mathcal{A}_{\|y_k\|_G}(\alpha), \quad \forall \alpha \in (0,1] \text{ and } k \in \mathbb{N}.$$

But $(y_k) \notin c^G$, since $\mathcal{A}_{\|y_k - \tilde{1}\|_G}(\alpha) = e^\alpha |y_k - 1|$ does not tends to 0 as $k \to \infty$, $\forall \alpha \in (0,1]$. So, c^G is not solid.

Definition 9 *A sequence space E^G is said to be symmetric if $(x_k) \in E^G$ implies $(x_{\pi(k)}) \in E^G$, where π is a permutation of $\mathbb{N}$.*

Remark 2 *If all the rearrangement of the terms of the sequence (x_k) belongs to E^G, then we say that the sequence space E^G is symmetric.*

Theorem 4 *The classes of sequences $\ell_\infty^G, c^G, c_0^G$ are symmetric.*

Proof. Let, $(x_k) \in \ell_\infty^G$. Then

$$\sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right| < \infty.$$

Let (y_k) be the rearrangement of (x_k), then clearly,

$$\sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|y_k\|_G}(\alpha)\right| < \infty.$$

Hence, $(y_k) \in \ell_\infty^G$. Thus ℓ_∞^G is symmetric.

Let, $(x_k) \in c^G$, then $\lim\limits_{k\to\infty} \mathcal{A}_{\|x_k\|_G}(\alpha)$ exists and let it be x. For a given $\varepsilon > 0$, $\exists N(= N_\varepsilon(\alpha)) \in \mathbb{N}$ such that $\mathcal{A}_{\|x_k-x\|_G}(\alpha) < \varepsilon$, $\forall k \geq N$. Let, (y_k) be the rearrangement of (x_k), i.e., $x_k = y_{n_k}$ for some $n_k \in \mathbb{N}$. Consider the set $\{y_{n_1}, y_{n_2}, y_{n_3}, \ldots, y_{n_{k_0}}\}$. Let, $M_0 = \max\{n_1, n_2, n_3, \ldots, n_{k_0}\}$ then we have $\mathcal{A}_{\|y_k-x\|_G}(\alpha) < \varepsilon$, $\forall k \geq M_0$. This shows that $(y_k) \in c^G$. Since (x_k) is arbitrary chosen, so c^G is symmetric.

Suppose, $x = 0$ then we can easily establish that c_0^G is symmetric. ■

Definition 10 *Let, E^G be a sequence space. Then E^G is said to be a sequence algebra if there is defined an operation $\star$ on E^G such that $(x_k), (y_k) \in E^G$ implies $(x_k) \star (y_k) \in E^G$.*

Theorem 5 *The classes of sequences $c_0^G, c^G, \ell_\infty^G$ are sequence algebra with respect to the term wise gradual multiplication of sequences and term wise gradual addition of sequences*

$$\mathcal{A}_{\|x_k+y_k\|_G}(\alpha) = \mathcal{A}_{\|x_k\|_G} + \mathcal{A}_{\|y_k\|_G}(\alpha) \quad \textit{and} \quad \mathcal{A}_{\|x_k\cdot y_k\|_G}(\alpha) = \mathcal{A}_{\|x_k\|_G} \cdot \mathcal{A}_{\|y_k\|_G}(\alpha)$$

$\forall\alpha \in (0,1]$ and $k \in \mathbb{N}$.

Proof. We proof that c_0^G is a sequence algebra. For the space c^G and ℓ_∞^G, the result can be proved similarly. Let $(x_k), (y_k) \in c_0^G$. Then

$$\lim_{k\to\infty} \mathcal{A}_{\|x_k\|_G}(\alpha) = 0 \ \text{ and } \ \lim_{k\to\infty} \mathcal{A}_{\|y_k\|_G}(\alpha) = 0, \quad \forall\alpha \in (0,1],\ k \in \mathbb{N}.$$

This shows that

$$\lim_{k\to\infty} \mathcal{A}_{\|x_k\cdot y_k\|_G}(\alpha) = 0, \quad \forall\alpha \in (0,1],\ k \in \mathbb{N}.$$

Thus $(x_k \cdot y_k) \in c_0^G$. Hence, c_0^G is a sequence algebra. ■

Definition 11 *A subset E^G of w^G is said to be convergence free if $(x_k) \in E^G$ and $\mathcal{A}_{\|x_k\|_G}(\alpha) \to 0$ implies $\mathcal{A}_{\|y_k\|_G}(\alpha) \to 0$ together with $(y_k) \in E^G$, $\forall\alpha \in (0,1]$.*

Theorem 6 *The class of sequences $\ell_\infty^G, c^G, c_0^G$ are not convergence free.*

Proof. Let $(x_k) \in \ell_\infty^G$ defined by $\forall\alpha \in (0,1]$ and $k \in \mathbb{N}$

$$\mathcal{A}_{\|x_k\|_G}(\alpha) = \begin{cases} 1, & \text{if } k \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

Now construct (y_k) by

$$\mathcal{A}_{\|y_k\|_G}(\alpha) = \begin{cases} k, & \text{if } k \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus $(y_k) \notin \ell_\infty^G$. Similarly for the others. ■

Definition 12 *Let, $K = \{k_1 < k_2 < k_3 < \cdots < k_n < \ldots\} \subset \mathbb{N}$ and $(x_n) \in w^G$. Then the K-step space of the sequence space E^G is defined by*

$$\lambda_K^{E^G} = \{(x_{k_i}) \in w^G : (x_n) \in E^G\}.$$

Definition 13 *A canonical pre-image* (y_n) *of a sequence* $(x_n) \in E^G$*, where* K*-step space* $\lambda_K^{E^G}$ *is considered, is defined by* $\forall\alpha \in (0,1]$,

$$\mathcal{A}_{\|y_n\|_G}(\alpha) = \begin{cases} \mathcal{A}_{\|x_n\|_G}(\alpha), & \text{if } n \in K, \\ \mathcal{A}_{\tilde{0}}(\alpha), & \text{otherwise.} \end{cases}$$

Definition 14 *A sequence space* E^G *is said to be monotone if it contains all pre-images of its step space.*

Theorem 7 *A sequence space* E^G *is solid implies* E^G *is monotone.*

Proof. Let E^G be a solid space, then we have $(x_n) \in E^G$ implies $(y_n) \in E^G$ for all sequence (y_n) such that

$$\left|\mathcal{A}_{\|y_n\|_G}(\alpha)\right| \leq \left|\mathcal{A}_{\|x_n\|_G}(\alpha)\right|, \ \forall\ n \in \mathbb{N}, \alpha \in (0,1].$$

Consider the canonical pre-image of E^G with respect to K. Then, (y_n) is the canonical pre-image. This implies

$$\mathcal{A}_{\|y_n\|_G}(\alpha) = \begin{cases} \mathcal{A}_{\|x_n\|_G}(\alpha), & \text{if } n \in K, \\ \mathcal{A}_{\tilde{0}}(\alpha), & \text{otherwise.} \end{cases}$$

This follows that $\left|\mathcal{A}_{\|y_n\|_G}(\alpha)\right| \leq \left|\mathcal{A}_{\|x_n\|_G}(\alpha)\right|, \ \forall\alpha \in (0,1], n \in \mathbb{N}$. Hence, $(y_n) \in E^G$. Since K is arbitrary subset of $\mathbb{N}$, E^G contains all the canonical pre-image of E^G. Hence E^G is monotone. ■

Theorem 8 *The class of sequences* c_0^G *and* ℓ_∞^G *are monotone but* c^G *is not monotone.*

Proof. Since the class of sequences ℓ_∞^G and c_0^G are solid, so are monotone by Theorem 7. The class of sequence c^G is not monotone follows from the following example: ■

Example 4 *Consider the sequence* $(x_n) \in c^G$ *defined by* $\mathcal{A}_{\|x_n\|_G}(\alpha) = \mathcal{A}_{\tilde{1}}(\alpha)$*,* $\forall\alpha \in (0,1]$ *and* $n \in \mathbb{N}$*. Consider the canonical pre-image of* (x_n) *i.e.,* (y_n) *defined by*

$$\mathcal{A}_{\|y_n\|_G}(\alpha) = \begin{cases} \mathcal{A}_{\|x_n\|_G}(\alpha), & \text{if } n \text{ is odd}, \\ \mathcal{A}_{\tilde{0}}(\alpha), & \text{otherwise.} \end{cases}$$

Then, (y_n) *is not a gradual convergent sequence. Thus* $(y_n) \notin c^G$*. It follows that* c^G *is not monotone.*

Conclusion

In this paper, we have investigated a few fundamental properties of space of bounded, convergent and null sequences in gradual normed linear spaces which are connected by the relation

$$c_0^G \subset c^G \subset \ell_\infty^G.$$

We also prove that, ℓ_∞^G, c^G and c_0^G are Banach spaces with respect to the gradual norm

$$\|x\|_G = \|(x_k)\|_G = \sup_{k\in\mathbb{N},\alpha\in(0,1]} \left|\mathcal{A}_{\|x_k\|_G}(\alpha)\right|.$$

Finally, we have introduced the concept of solid space, symmetric space, sequence algebra, convergence free and monotone space in the gradual normed linear spaces and established some theorem for the first time.

Summability theory and sequence spaces have wide applications in various branches of mathematics particularly, in mathematical analysis. Research in this direction based on gradual normed linear spaces has not yet gained much ground and it is still in its infant stage. In the future, this work can be extended over difference sequences and several analytical properties can be investigated.

Additionally, the following questions can be asked: Let $E^G \subset w^G$ be a sequence space. If E^G is monotone then is it solid? Is E^G separable for $E = \ell_\infty, c$ and c_0?

Acknowledgment. The authors thank the anonymous referees for their constructive suggestions to improve the quality of the paper. The first author is grateful to the **Tripura University** for their non-NET Ph.D. fellowship funding under the **UGC MERGED** scheme during the preparation of this paper. The second author is grateful to the **University Grants Commission, India** for their fellowships funding under the **UGC-SRF** scheme **(F.No.16-6(DEC. 2018)/2019(NET/CSIR))** during the preparation of this paper.

References

[1] F. Aiche and D. Dubois, Possibility and gradual number approaches to ranking methods for random fuzzy intervals, Commun. Comput. Inf. Sci., 299(2012), 9–18.

[2] M. Altinok and M. Küçükaslan, A-statistical convergence and A-statistical monotonicity, Appl. Math. E-Notes, 13(2013), 249–260.

[3] M. Başarir, On some new sequence spaces and related matrix transformations, Indian J. Pure Appl. Math., 26(1995), 1003–1010.

[4] M. Başarir and M. Mursaleen, Some sequence spaces of fuzzy numbers generated by infinite matrices, J. Fuzzy Math., 11(2003), 757–764.

[5] C. Choudhury and S. Debnath, On $\mathcal{I}$-convergence of sequences in gradual normed linear spaces, Facta Univ. Ser. Math. Inform., 36(2021), 595–604.

[6] C. Choudhury and S. Debnath, On $\mathcal{I}$-statistical convergence of sequences in gradual normed linear spaces, Mat. Vesn., 74(2022), 218–228.

[7] C. Choudhury and S. Debnath, On lacunary statistical convergence of sequences in gradual normed linear spaces, Ann. Univ. Craiova Math. Comput. Sci. Ser., 49(2022), 110–119.

[8] C. Choudhury and S. Debnath, Further results on $\mathcal{I}$ and $\mathcal{I}^*$-convergence of sequences in gradual normed linear spaces, Jordan J. Math. Stat., 15(2022), 967–982.

[9] D. Dubois and H. Prade, Gradual elements in a fuzzy set, Soft Comput., 12(2007), 165–175.

[10] M. Et and M. Başarir, On some new generalized difference sequence spaces, Period. Math. Hungar., 35(1997), 169–175.

[11] M. Ettefagh, F. Y. Azari and S. Etemad, On some topological properties in gradual normed spaces, Facta Univ. Ser. Math. Inform., 35(2020), 549–559.

[12] M. Ettefagh, S. Etemad, and F. Y. Azari, Some properties of sequences in gradual normed spaces, Asian-Eur. J. Math., 13(2020), 2050085.

[13] J. Fortin, D. Dubois and H. Fargier, Gradual numbers and their application to fuzzy interval analysis, IEEE Trans. Fuzzy Syst., 16(2008), 388–402.

[14] L. Lietard and D. Rocacher, Conditions with aggregates evaluated using gradual numbers, Control Cybernet., 38(2009), 395–417.

[15] I. J. Maddox, Elements of Functional Analysis, Cambridgen Univ. Press, 1970.

[16] M. Mursaleen and M. Başarir, On some new sequence spaces of fuzzy numbers, Indian J. Pure Appl. Math., 34(2003), 1351–1357.

[17] I. Sadeqi and F. Y. Azari, Gradual normed linear space, Iran. J. Fuzzy Syst., 8(2011), 131–139.

[18] E. A. Stock, Gradual Numbers and Fuzzy Optimization, Ph.D. Thesis, University of Colorado Denver, America, 2010.

[19] B. C. Tripathy and B. Hazarika, Paranorm $\mathcal{I}$-convergent sequence spaces, Math. Slovaca, 59(2009), 485–494.

[20] B. C. Tripathy and B. Hazarika, Some $\mathcal{I}$-convergent sequence spaces defined by Orlicz functions, Acta Math. Sin. Engl. Ser., 27(2011), 149–154.

[21] L. A. Zadeh, Fuzzy sets, Inf. Control, 8(1965), 338–353.

[22] C. Zhou, Gradual metric spaces, Appl. Math. Sci., 9(2015), 689–701.

[23] C. Zhou and J. Li, New fuzzy measure based on gradual numbers, Int. J. Math. Anal., 9(2015), 101–110.