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Some Results On Continuity And Boundedness Of Linear Operators On Neutrosophic Fuzzy n-Normed Spaces*

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Abstract

In this research, our primary objective is to elucidate various aspects of continuity in the context of linear operators defined on neutrosophic fuzzy n-normed spaces (NFnNS). Specifically, we investigate weak, strong, and sequential continuity in this setting. We present an illustrative example that underscores the fact that linear operators that are sequentially continuous may not necessarily exhibit strong continuity when applied to NFnNS. Furthermore, we introduce the concepts of weak and strong boundedness for operators in these spaces and explore their interrelationships with continuity.

1 Introduction

In our day-to-day experiences, we often encounter complex and multifaceted classes of objects or phenomena that do not conform to the binary classification of traditional, crisp sets. For instance, consider the classes of beautiful women, intelligent students, or tall individuals; each of these categories embodies a gradation of characteristics that cannot be neatly confined to simple "yes" or "no" categories. To address the inherent fuzziness and variability in these classes, Lotfi Zadeh introduced the groundbreaking concept of fuzzy sets in his seminal work in 1965 [25]. Fuzzy sets enable us to incorporate the concept of membership degrees, allowing for a more nuanced representation of reality. Atanassov [1] initially noted that Zadeh's concept of fuzzy sets lacked adequacy for addressing certain issues. As a result, he extended this notion by incorporating a non-membership function alongside the membership function, naming it the intuitionistic fuzzy set. Furthermore, these sets find applications in defining intuitionistic topological spaces and intuitionistic normed spaces. To gain a comprehensive understanding of these spaces, we refer to the following references [6, 16, 18].

Over the years, Zadeh's initial concept of fuzzy sets has grown and evolved, leading to the development of various fuzzy analogues of classical mathematical concepts. One of the most prominent extensions is the field of fuzzy topology, which has demonstrated its versatility and applicability in various domains, including quantum physics [7]. Fuzzy topology provides a framework for understanding the structure and properties of spaces where the boundnaries between open and closed sets are not sharply defined.

In 1984, during the examination of fuzzy topological spaces, Katsaras [8] introduced the ideas of fuzzy semi-norm and fuzzy norm, and conducted an analysis of several characteristics associated with fuzzy semi-normed and fuzzy normed spaces. This work laid the foundation for exploring mathematical structures and properties in spaces defined by fuzzy norms, culminating in Xiao and Zhu's development of fuzzy norms for linear operators and their examination of the properties of spaces endowed with these fuzzy norms [22]. Subsequently, Bag and Samanta delved into the study of strong and weak boundedness for fuzzy bounded linear operators, shedding light on their relationships with fuzzy continuity [2, 3]. For a more in-depth understanding of these topics, interested readers are encouraged to explore additional sources such as [4, 5].

The notion of neutrosophic sets was first introduced by Smarandache in his pioneering work [20, 21]. The classic neutrosophic set is defined by its three elements: truth, indeterminacy, and falseness values. Neutrosophic sets serve as a mathematical framework to manage issues related to imprecision, uncertainty,

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inconsistency, and conflicting data within belief systems. They address challenges that were present in both fuzzy sets and intuitionistic fuzzy sets. Consequently, neutrosophic sets have gained significant traction in various scientific and engineering disciplines.

Recent developments in this field by Kirişci and Şimşek introduced neutrosophic normed spaces (NNS) and explored statistical convergence in these spaces [13]. This research has opened up new possibilities for studying mathematical structures in spaces characterized by the presence of neutrosophic information. For a more extensive exploration of NNS, readers are encouraged to delve into sources like [9, 10, 11, 12, 14, 15, 19].

In this current study, our focus lies squarely on NNS. We aim to introduce and analyze specific concepts related to continuity and boundedness for operators within NFnNS. Through our investigation, we seek to unravel compelling relationships and connections among these fundamental mathematical concepts, thereby advancing our understanding of these multifaceted spaces and their potential applications in various fields.

2 Preliminaries

In this section, we give significant existing conceptions and results which are crucial for our findings.

A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ is considered a continuous t-norm if it adheres to the following criteria, as stated in [18]:

(i) $\circ$ is commutative and associative,

(ii) $\circ$ is continuous,

(iii) For all $q \in [0,1]$, it satisfies the condition $q \circ 1 = q$,

(iv) Whenever $q \leq p$ and $r \leq s$, with q, p, r, s belonging to the interval $[0,1]$, it holds that $q \circ r \leq p \circ s$.

A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is considered a continuous t-conorm if it adheres to the following criteria, as stated in [18]:

(i) $\diamond$ is commutative and associative,

(ii) $\diamond$ is continuous,

(iii) For all $q \in [0,1]$, it satisfies the condition $q \diamond 0 = q$,

(iv) Whenever $q \leq p$ and $r \leq s$, with q, p, r, s belonging to the interval $[0,1]$, it holds that $q \diamond r \leq p \diamond s$.

A neutrosophic fuzzy n normed spaces, denoted as NFnNS, is represented by a six-tuple $(X, \mathcal{G}, \mathcal{B}, \mathcal{W}, \circ, \diamond)$ where X is a vector space, $\circ$ is a t-norm, $\diamond$ is a t-conorm, and $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{W}$ are single valued fuzzy sets on $X^n \times (0, \infty)$. In this context, $\mathcal{G}$ stands for the membership function, $\mathcal{B}$ for the indeterminacy function, and $\mathcal{W}$ for the non-membership function. To qualify as an NFnNS, the following conditions must hold for every $(w_1, w_2, ..., w_n) \in X^n$, with $s,\, t \geq 0$ and $\varsigma \neq 0$:

(i) $0 \leq \mathcal{G}(w_1, w_2, ..., w_n, t) + \mathcal{B}(w_1, w_2, ..., w_n, t) + \mathcal{W}(w_1, w_2, ..., w_n, t) \leq 3$;

(ii) $\mathcal{G}(w_1, w_2, ..., w_n, t) = 1$ iff $w_1, w_2, ..., w_n$ are linearly dependent;

(iii) $\mathcal{G}(\varsigma w_1, w_2, ..., w_n, t) = \mathcal{G}\left(w_1, w_2, ..., w_n, \frac{t}{|\varsigma|}\right)$ for each $\varsigma \neq 0$;

(iv) $\mathcal{G}(w_1, w_2, ..., w_n, s) \circ \mathcal{G}(w_1, w_2, ..., w_n', t) \leq \mathcal{G}(w_1, w_2, ..., w_n + w_n', s + t)$;

(v) $\mathcal{G}(w_1, w_2, ..., w_n, t)$ is continuous on $(0, \infty)$;

(vi) $\lim_{t\to\infty} \mathcal{G}(w_1, w_2, ..., w_n, t) = 1$;

(vii) $\mathcal{B}(w_1, w_2, ..., w_n, t) = 0$ iff $w_1, w_2, ..., w_n$ are linearly dependent;

(viii) $\mathcal{B}(\varsigma w_1, w_2, ..., w_n, t) = \mathcal{B}\left(w_1, w_2, ..., w_n, \frac{t}{|\varsigma|}\right)$ for each $\varsigma \neq 0$;

(ix) $\mathcal{B}(w_1, w_2, ..., w_n, s) \diamond \mathcal{B}(w_1, w_2, ..., w_n', t) \geq \mathcal{G}(w_1, w_2, ..., w_n + w_n', s+t)$;

(x) $\mathcal{B}(w_1, w_2, ..., w_n, t)$ is continuous on $(0, \infty)$;

(xi) $\lim_{t\to\infty} \mathcal{B}(w_1, w_2, ..., w_n, t) = 0$;

(xii) $\mathcal{W}(w_1, w_2, ..., w_n, t) = 0$ iff $w_1, w_2, ..., w_n$ are linearly dependent;

(xiii) $\mathcal{W}(\varsigma w_1, w_2, ..., w_n, t) = \mathcal{W}\left(w_1, w_2, ..., w_n, \frac{t}{|\varsigma|}\right)$ for each $\varsigma \neq 0$;

(xiv) $\mathcal{W}(w_1, w_2, ..., w_n, s) \diamond \mathcal{W}(w_1, w_2, ..., w_n', t) \geq \mathcal{W}(w_1, w_2, ..., w_n + w_n', s+t)$;

(xvi) $\mathcal{W}(w_1, w_2, ..., w_n, t)$ is continuous on $(0, \infty)$;

(xvii) $\lim_{t\to\infty} \mathcal{W}(w_1, w_2, ..., w_n, t) = 0$.

In this case, we define $N_n(\mathcal{G}, \mathcal{B}, \mathcal{W})$ as a neutrosophic n-norm on X^n. Yaying [23] and Yaying et al. [24] put forward new types of continuity in cone metric space and asymmetric metric spaces. Now, we provide an overview of the concept of continuous, sequentially continuous, and strongly continuous mappings in NNS. (see in detail, [17]).

Definition 1 ([17]) *Let $(P, N_P, \circ, \diamond)$ and $(Q, N_Q, \circ, \diamond)$ be two NNS. The mapping $T : (P, N_P, \circ, \diamond) \to (Q, N_Q, \circ, \diamond)$ is said to be continuous at $w_0 \in P$ if for all $w \in P$, for each $0 < \epsilon < 1$ and $t > 0$, there exists $0 < \delta < 1$ and $s > 0$, such that*

$$\mathcal{G}_Q(T(w) - T(w_0), t) > (1-\epsilon), \mathcal{B}_Q(T(w) - T(w_0), t) < \epsilon, \mathcal{W}_Q(T(w) - T(w_0), t) < \epsilon,$$

whenever

$$\mathcal{G}_P(w - w_0, s) > (1-\delta), \mathcal{B}_P(w - w_0, s) < \delta, \mathcal{W}_P(w - w_0, s) < \delta,$$

respectively. In other words:

$$\begin{aligned} \mathcal{G}_P(w - w_0, s) &> (1-\delta) \Rightarrow \mathcal{G}_Q(T(w) - T(w_0), t) > (1-\epsilon), \\ \mathcal{B}_P(w - w_0, s) &< \delta \Rightarrow \mathcal{B}_Q(T(w) - T(w_0), t) < \epsilon, \\ \mathcal{W}_P(w - w_0, s) &< \delta \Rightarrow \mathcal{W}_Q(T(w) - T(w_0), t) < \epsilon, \end{aligned}$$

T is continuous on P if it is continuous at every point in P.

Definition 2 ([17]) *The mapping $T : (P, N_P, \circ, \diamond) \to (Q, N_Q, \circ, \diamond)$ is called sequentially continuous at $w_0 \in P$, any sequence (w_n) in P satisfying $w_n \to w_0$ leads to $T(w_n) \to T(w_0)$. In other words:*

$$\begin{aligned} \lim_{n\to\infty} \mathcal{G}_P(w_n - w_0, t) &= 1 \Rightarrow \lim_{n\to\infty} \mathcal{G}_Q(T(w_n) - T(w_0), t) = 1, \\ \lim_{n\to\infty} \mathcal{B}_P(w_n - w_0, t) &= 0 \Rightarrow \lim_{n\to\infty} \mathcal{B}_Q(T(w_n) - T(w_0), t) = 0, \\ \lim_{n\to\infty} \mathcal{W}_P(w_n - w_0, t) &= 0 \Rightarrow \lim_{n\to\infty} \mathcal{W}_Q(T(w_n) - T(w_0), t) = 0, \end{aligned}$$

where $t > 0$. We call T is sequentially continuous on P when T is sequentially continuous at each point of P.

Definition 3 ([17]) *The mapping $T : (P, N_P, \circ, \diamond) \to (Q, N_Q, \circ, \diamond)$ is called strongly continuous at $w_0 \in P$ if for each $t > 0$, $\exists s > 0$ such that $\forall w \in U$,*

$$\begin{aligned} \mathcal{G}_P(w - w_0, s) &\leq \mathcal{G}_Q(T(w) - T(w_0), t), \\ \mathcal{B}_P(w - w_0, s) &\geq \mathcal{B}_Q(T(w) - T(w_0), t), \\ \mathcal{W}_P(w - w_0, s) &\geq \mathcal{W}_Q(T(w) - T(w_0), t), \end{aligned}$$

we say T is strongly continuous on P when it is strongly continuous at every point in P.

3 Main Results

Let, $U = (X, \mathcal{G}_1, \mathcal{B}_1, \mathcal{W}_1, \circ_1, \diamond_1)$ and $V = (Y, \mathcal{G}_2, \mathcal{B}_2, \mathcal{W}_2, \circ_2, \diamond_2)$ be two NFnNS, where X and Y are linear spaces over $\mathbb{R}$.

Definition 4 *A mapping $T : U \to V$ is called to be neutrosophic fuzzy continuous (nf-continuous) at $w_0 \in U$ if for every $\epsilon > 0$, $\eta > 0$ $(0 < \eta < 1)$, $u_1, u_2, ..., u_{n-1} \in U$ and $T(u_1), T(u_2), ..., T(u_{n-1}) \in V$, there exist $\delta = \delta(\eta, \epsilon, w_0) > 0$ and $\xi = \xi(\eta, \epsilon, w_0) \in (0, 1)$ such that $\forall w \in U$ we have*

$$\begin{cases} \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta) > \xi, \\ \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta) < 1 - \xi, \\ \mathcal{W}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta) < 1 - \xi. \end{cases}$$

$$\Rightarrow \begin{cases} \mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) > \eta, \\ \mathcal{B}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) < 1 - \eta, \\ \mathcal{W}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) < 1 - \eta. \end{cases}$$

$T : U \to V$ is said to be nf-continuous on U if T is nf-continuous at each point of U.

Definition 5 *A map $T : U \to V$ is called to be strongly nf-continuous at $w_0 \in U$ if for each $\epsilon > 0$, $u_1, u_2, ..., u_{n-1} \in U$ and $T(u_1), T(u_2), ..., T(u_{n-1}) \in V$, $\exists \delta > 0$ such that $\forall w \in U$*

$$\begin{aligned} \mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) &\geq \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta), \\ \mathcal{B}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) &\leq \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta), \\ \mathcal{W}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) &\leq \mathcal{W}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta). \end{aligned}$$

$T : U \to V$ is said to be strongly nf-continuous on U if T is strongly nf-continuous at each point of U.

Definition 6 *A map $T : U \to V$ is said to be weakly nf-continuous at $w_0 \in U$ if, for each $\epsilon > 0, \eta \in (0, 1)$, $u_1, u_2, ..., u_{n-1} \in U$ and $T(u_1), T(u_2), ..., T(u_{n-1}) \in V$, $\exists \delta = (\eta, \epsilon) > 0$ such that $\forall w \in U$*

$$\begin{cases} \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta) \geq \eta \\ \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta) \leq 1 - \eta \\ \mathcal{W}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta) \leq 1 - \eta. \end{cases}$$

$$\Rightarrow \begin{cases} \mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) \geq \eta, \\ \mathcal{B}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) \leq 1 - \eta, \\ \mathcal{W}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) \leq 1 - \eta. \end{cases}$$

We say $T : U \to V$ weakly nf-continuous on U if T is weakly nf-continuous at each point of U.

Definition 7 *A map $T : U \to V$ is said to be sequentially nf-continuous at $w_0 \in U$ if for any sequence (w_n) with $w_n \to w_0$ implies $T(w_n) \to T(w_0)$ i.e, for all $r > 0$, $u_1, u_2, ..., u_{n-1} \in U$ and $T(u_1), T(u_2), ..., T(u_{n-1}) \in V$,*

$$\begin{cases} \lim\limits_{n\to\infty} \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w_n - w_0, r) = 1, \\ \lim\limits_{n\to\infty} \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w_n - w_0, r) = 0, \\ \lim\limits_{n\to\infty} \mathcal{W}_1(u_1, u_2, ..., u_{n-1}, w_n - w_0, r) = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \lim_{n\to\infty} \mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w_0\right), r\right) = 1 \\ \lim_{n\to\infty} \mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w_0\right), r\right) = 0, \\ \lim_{n\to\infty} \mathcal{W}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w_0\right), r\right) = 0. \end{cases}$$

$T : U \to V$ is said to be sequentially nf-continuous on U if T is sequentially nf-continuous at each point of U.

Theorem 1 *If a map $T : U \to V$ is strongly nf-continuous then it is sequentially nf-continuous.*

Proof. Consider a mapping $T : U \to V$ that is strongly nf-continuous. Our goal is to demonstrate that T is also sequentially nf-continuous. To do so, let's take an arbitrary point $w_0 \in U$. Since $T : U \to V$ is strongly nf-continuous so for each $\epsilon > 0$, $\exists \delta > 0$ such that $\forall w \in U$,

$$\begin{aligned} &\mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w\right) - T\left(w_0\right), \epsilon\right) \geq \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w - w_0, \delta\right), \\ &\mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w\right) - T\left(w_0\right), \epsilon\right) \leq \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w - w_0, \delta\right), \\ &\mathcal{W}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w\right) - T\left(w_0\right), \epsilon\right) \leq \mathcal{W}_1\left(u_1, u_2, ..., u_{n-1}, w - w_0, \delta\right). \end{aligned} \tag{1}$$

Let (w_n) be any sequence in U such that $w_n \to w_0$ w.r.t $N_1\left(\mathcal{G}_1,\mathcal{B}_1,\mathcal{Y}_1\right)$. Then

$$\lim_{n\to\infty} \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta\right) = 1 \tag{2}$$

and

$$\lim_{n\to\infty} \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta\right) = \lim_{n\to\infty} \mathcal{Y}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta\right) = 0. \tag{3}$$

Now, by (1),

$$\begin{aligned} &\mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w_0\right), \epsilon\right) \geq \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta\right), \\ &\mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w_0\right), \epsilon\right) \leq \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta\right), \\ &\mathcal{W}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w_0\right), \epsilon\right) \leq \mathcal{W}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta\right), \end{aligned}$$

and therefore,

$$\lim_{n\to\infty} \mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w_0\right), \epsilon\right) \geq \lim_{n\to\infty} \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta\right) = 1$$

by (2) and (3). This gives

$$\lim_{n\to\infty} G_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w_0\right), \epsilon\right) = 1.$$

Further,

$$\lim_{n\to\infty} \mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w_0\right), \epsilon\right) \leq \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta\right) = 0$$

and

$$\lim_{n\to\infty} \mathcal{W}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w_0\right), \epsilon\right) \leq \mathcal{W}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta\right) = 0.$$

This demonstrates that the sequence $T\left(w_n\right)$ converges to $T\left(w_0\right)$ w.r.t $N_2\left(\mathcal{G}_2,\mathcal{B}_2,\mathcal{Y}_2\right)$, thereby establishing T as sequentially nf-continuous. ■

The converse of the previous result is generally not valid, as illustrated by the following example.

Example 1 *Let $(X, \|\cdot\|_n)$ be a n-normed space. Define the t-norm, t-conorm, $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_2, \&\mathcal{W}_1, \mathcal{W}_2$ by $s \circ t = \min\{s,t\}$, $s \diamond t = \max\{s,t\}$ for $s,t \in [0,1]$;*

$$\mathcal{G}_1(w_1, w_2, ..., w_n, \delta) = \frac{\delta}{\delta + \|w_1, w_2, ..., w_n\|}, \quad \mathcal{B}_1(w_1, w_2, ..., w_n, \delta) = \frac{\|w_1, w_2, ..., w_n\|}{\delta + \|w_1, w_2, ..., w_n\|},$$

$$\mathcal{W}_1(w_1, w_2, ..., w_n, \delta) = \frac{\|(w_1, w_2, ..., w_n)\|}{\delta};$$

$$\mathcal{G}_2(w_1, w_2, ..., w_n, \epsilon) = \frac{\epsilon}{\epsilon + \alpha\|w_1, w_2, ..., w_n\|}, \quad \mathcal{B}_2(w_1, w_2, ..., w_n, \epsilon) = \frac{\alpha\|w_1, w_2, ..., w_n\|}{\epsilon + \alpha\|w_1, w_2, ..., w_n\|_2},$$

$$\mathcal{W}_2(w_1, w_2, ..., w_n, \epsilon) = \frac{\alpha\|w_1, w_2, ..., w_n\|}{\epsilon},$$

where $\epsilon > 0$, $\exists\delta > 0$, $\alpha > 0$, and $w = (w_1, w_2, ..., w_n) \in X^n$. Then $U = (X^n, \mathcal{G}_1, \mathcal{B}_1, \mathcal{Y}_1, \circ, \diamond)$, $V = (X^n, \mathcal{G}_2, \mathcal{B}_2, \mathcal{Y}_2, \circ, \diamond)$ are NFnNS. Establish a map $T : U \to V$ by $T(w) = \frac{w^4}{1+w^2}$ where $w = (w_1, w_2, ..., w_n) \in X^n$. We first demonstrate that T is sequentially nf-continuous. Let $w_0 \in U$ and (w_n) be any sequence in U such that $(w_n) \to w_0$ w.r.t $N_1(\mathcal{G}_1, \mathcal{B}_1, \mathcal{Y}_1)$. For any $r > 0$, we have

$$\lim_{n\to\infty} \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta) = 1$$

and

$$\lim_{n\to\infty} \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta) = \lim_{n\to\infty} \mathcal{W}_1(u_1, u_2, ..., u_{n-1}, w_n - w_0, \delta) = 0.$$

Then

$$\lim_{n\to\infty} \frac{\delta}{\delta + \|u_1, u_2, ..., u_{n-1}, w_n - w_0\|} = 1$$

and

$$\lim_{n\to\infty} \frac{\|u_1, u_2, ..., u_{n-1}, w_n - w_0\|}{\delta + \|u_1, u_2, ..., u_{n-1}, w_n - w_0\|} = \lim_{n\to\infty} \frac{\|u_1, u_2, ..., u_{n-1}, w_n - u_0\|}{\delta} = 0.$$

So we obtain

$$\lim_{n\to\infty} \|u_1, u_2, ..., u_{n-1}, w_n - u_0\| = 0. \tag{4}$$

Now consider

$$\begin{aligned}
&\mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_n) - T(w_0), \epsilon)\\
= &\frac{\epsilon}{\epsilon + \alpha\|T(u_1), T(u_2), ..., T(u_{n-1}), T(w_n) - T(w_0)\|}\\
= &\frac{\epsilon}{\epsilon + \alpha\left\|\frac{w_n^4}{1+w_n^2} - \frac{w_0^4}{1+w_0^2}\right\|} = \frac{\epsilon\left\|1 + w_n^2\right\|\left\|1 + w_0^2\right\|}{\epsilon\|1 + w_n^2\|\|1 + w_0^2\| + \alpha\|(w_n^4(1 + w_0^2) - w_0^4(1 + w_n^2))\|}\\
= &\frac{\epsilon\left\|1 + w_n^2\right\|\left\|1 + w_0^2\right\|}{\epsilon\|1 + w_n^2\|\|1 + w_0^2\| + \alpha\|w_n^4 + w_n^4 w_0^2 - w_0^4 - w_0^4 w_n^2\|}\\
= &\frac{\epsilon\left\|1 + w_n^2\right\|\left\|1 + w_0^2\right\|}{\epsilon\|1 + w_n^2\|\|1 + w_0^2\| + \alpha\|(w_n - w_0)(w_n + w)(w_n^2 + w_0^2) + w_n^2 w_0^2(w_n^2 - w_0^2)\|}\\
= &\frac{\epsilon\left\|1 + w_n^2\right\|\left\|1 + w_0^2\right\|}{\epsilon\|1 + w_n^2\|\|1 + w_0^2\| + \alpha\|w_n - w_0\|\|(w_n + w_0)(w_n^2 + w_0^2) + w_n^2 w_0^2(w_n + w_0)\|}
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} \mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_n) - T(w_0), \epsilon) = 1$$

by (4). Further,

$$\begin{aligned}
&\mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), \ldots, T\left(u_{n-1}\right), T\left(w_n\right)-T\left(w_0\right), \epsilon\right) \\
= & \frac{\alpha\left\|\frac{w_n^4}{1+w_n^2}-\frac{w_0^4}{1+w_0^2}\right\|}{\epsilon+\alpha\left\|\frac{w_n^4}{1+w_n^2}-\frac{w_0^4}{1+w_0^2}\right\|} \\
= & \frac{\alpha\left\|w_n^4\left(1+w_0^2\right)-w_0^4\left(1+w_n^2\right)\right\|}{\epsilon\left\|1+w_n^2\right\|\left\|1+w_0^2\right\|+\alpha\left\|w_n^4\left(1+w_0^2\right)-w_0^4\left(1+w_n^2\right)\right\|} \\
= & \frac{\alpha\left\|w_n^4-w_0^4+w_n^4 w_0^2-w_0^4 w_n^2\right\|}{\epsilon\left\|1+w_n^2\right\|\left\|1+w_0^2\right\|+\alpha\left\|w_n^4-w_0^4+w_n^4 w_0^2-w_0^4 w_n^2\right\|} \\
= & \frac{\alpha\left\|\left(w_n-w_0\right)\left(w_n+w_0\right)\left(w_n^2+w_0^2\right)+w_n^2 w_0^2\left(w_n^2-w_0^2\right)\right\|}{\epsilon\left\|1+w_n^2\right\|\left\|1+w_0^2\right\|+\alpha\left\|\left(w_n-w_0\right)\left(w_n+w_0\right)\left(w_n^2+w_0^2\right)+w_n^2 w_0^2\left(w_n^2-w_0^2\right)\right\|} \\
= & \frac{\alpha\left\|w_n-w_0\right\|\left\|\left(w_n+w_0\right)\left(w_n^2+w_0^2\right)+w_n^2 w_0^2\left(w_n+w_0\right)\right\|}{\epsilon\left\|1+w_n^2\right\|\left\|1+w_0^2\right\|+\alpha\left\|w_n-w_0\right\|\left\|\left(w_n+w_0\right)\left(w_n^2+w_0^2\right)+w_n^2 w_0^2\left(w_n+w_0\right)\right\|}.
\end{aligned}$$

and therefore

$$\lim_{n \rightarrow \infty} \mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), \ldots, T\left(u_{n-1}\right), T\left(w_n\right)-T\left(w_0\right), \epsilon\right)=0.$$

by (4). Similarly, we have

$$\lim_{n \rightarrow \infty} \mathcal{W}_2\left(T\left(u_1\right), T\left(u_2\right), \ldots, T\left(u_{n-1}\right), T\left(w_n\right)-T\left(w_0\right), \epsilon\right)=0,$$

and hence $T\left(w_n\right) \rightarrow T\left(w_0\right)$ w.r.t $N_2\left(\mathcal{G}_2, \mathcal{B}_2, \mathcal{Y}_2\right)$. This demonstrates that T exhibits sequential nf-continuity on U. However, we assert that T does not possess strong neutrosophic continuity on U. Assume that T is strongly continuous on U. Let $\epsilon>0$ be given and $w_0 \in X^n$. Since T is strongly nf-continuous, there exists $\delta>0$ such that $\forall w \in X^n$, $u_1, u_2, \ldots, u_{n-1} \in U$ and $T\left(u_1\right), T\left(u_2\right), \ldots, T\left(u_{n-1}\right) \in V$,

$$\begin{aligned}
&\mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), \ldots, T\left(u_{n-1}\right), T(w)-T\left(w_0\right), \epsilon\right) \geq \mathcal{G}_1\left(u_1, u_2, \ldots, u_{n-1}, w-w_0, \delta\right) \\
&\Rightarrow \frac{\epsilon\left\|1+w^2\right\|\left\|1+w_0^2\right\|}{\epsilon\left\|1+w^2\right\|\left\|1+w_0^2\right\|+\alpha\left\|w-w_0\right\|\left\|\left(w+w_0\right)\left(w^2+w_0^2\right)+w^2 w_0^2\left(w+w_0\right)\right\|} \geq \frac{\delta}{\delta+\left\|w-w_0\right\|},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), \ldots, T\left(u_{n-1}\right), T(w)-T\left(w_0\right), \epsilon\right) \leq \mathcal{B}_1\left(u_1, u_2, \ldots, u_{n-1}, w-w_0, \delta\right) \\
&\Rightarrow \frac{\alpha\left\|w-w_0\right\|\left\|\left(w+w_0\right)\left(w^2+w_0^2\right)+w^2 w_0^2\left(w+w_0\right)\right\|}{\epsilon\left\|1+w^2\right\|\left\|1+w_0^2\right\|+\alpha\left\|w-w_0\right\|\left\|\left(w+w_0\right)\left(w^2+w_0^2\right)+w^2 w_0^2\left(w+w_0\right)\right\|} \leq \frac{\left\|w-w_0\right\|}{\delta+\left\|w-w_0\right\|},
\end{aligned}$$

$$\begin{aligned}
&\alpha \delta\left\|w-w_0\right\|\left\|w+w_0\right\|\left\|w^2+w_0^2+w^2 w_0^2\right\|+\alpha\left\|w-w_0\right\|^2\left\|w+w_0\right\|\left\|w^2+w_0^2+w^2 w_0^2\right\| \\
&\leq \epsilon\left\|1+w^2\right\|\left\|1+w_0^2\right\|\left\|w-w_0\right\|+\alpha\left\|w-w_0\right\|^2\left\|w+w_0\right\|\left\|w^2+w_0^2+w^2 w_0^2\right\| \\
&\Rightarrow \alpha \delta\left\|w-w_0\right\|\left\|w+w_0\right\|\left\|w^2+w_0^2+w^2 w_0^2\right\| \leq \epsilon\left\|1+w^2\right\|\left\|1+w_0^2\right\|\left\|w-w_0\right\| \\
&\Rightarrow \delta \leq \frac{\epsilon\left\|1+w^2\right\|\left\|1+w_0^2\right\|\left\|w-w_0\right\|}{\alpha\left\|w-w_0\right\|\left\|w+w_0\right\|\left\|w^2+w_0^2+w^2 w_0^2\right\|}=\frac{\epsilon\left\|1+w^2\right\|\left\|1+w_0^2\right\|}{\alpha\left\|w+w_0\right\|\left\|w^2+w_0^2+w^2 w_0^2\right\|}.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{W}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right)-T\left(w_0\right), \epsilon\right) \leq \mathcal{W}_1\left(u_1, u_2, ..., u_{n-1}, w_n-w_0, \delta\right) \\
&\Rightarrow \tfrac{\alpha\left\|T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right)-T\left(w_0\right)\right\|}{\epsilon} \leq \tfrac{\left\|T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), w_n-w_0\right\|}{\delta} \\
&\Rightarrow \alpha\left\|\tfrac{w^4}{1+w^2}-\tfrac{w_0^4}{1+w_0^2}\right\| \leq \tfrac{\epsilon}{\delta}\left\|w-w_0\right\| \\
&\Rightarrow \tfrac{\alpha\left\|w^4\left(1+w_0^2\right)-w_0^4\left(1+w^2\right)\right\|}{\left\|1+w^2\right\|\left\|1+w_0^2\right\|} \leq \tfrac{\epsilon}{\delta}\left\|w-w_0\right\| \\
&\Rightarrow \delta\alpha\left\|w^4-w_0^4+w^4 w_0^2-w_0^4 w^2\right\| \leq \epsilon\left\|w-w_0\right\|\left\|1+w_0^2\right\|\left\|1+w^2\right\| \\
&\Rightarrow \delta\alpha\left\|\left(w^2-w_0^2\right)\left(w^2+w_0^2\right)+w^2 w_0^2\left(w^2-w_0^2\right)\right\| \leq \epsilon\left\|w-w_0\right\|\left\|1+w_0^2\right\|\left\|1+w^2\right\| \\
&\Rightarrow \delta\alpha\left\|w-w_0\right\|\left\|w+w_0\right\|\left\|w^2+w_0^2+w^2 w_0^2\right\| \leq \epsilon\left\|w-w_0\right\|\left\|1+w_0^2\right\|\left\|1+w^2\right\| \\
&\Rightarrow \delta \leq \tfrac{\epsilon}{\alpha} \tfrac{\left\|1+w_0^2\right\|\left\|1+w^2\right\|}{\left\|w+w_0\right\|\left\|w^2+w_0^2+w^2 w_0^2\right\|}.
\end{aligned}$$

Hence, in all cases

$$\Rightarrow \delta \leq \frac{\epsilon}{\alpha} \frac{\left\|1+w_0^2\right\|\left\|1+w^2\right\|}{\left\|w+w_0\right\|\left\|w^2+w_0^2+w^2 w_0^2\right\|}.$$

Let,

$$\delta^*=\inf_{\substack{w \\ w \neq w_0}} \frac{\left\|1+w_0^2\right\|\left\|1+w^2\right\|}{\left\|w+w_0\right\|\left\|w^2+w_0^2+w^2 w_0^2\right\|}.$$

Then $\delta=\frac{\epsilon}{\alpha} \delta^*$*. But* $\delta^*=0$ *which is not possible. As a result,* T *is not strongly nf-continuous on* U*.*

Theorem 2 *A map* $T: U \rightarrow V$ *is nf-continuous iff* T *is sequentially nf-continuous on* U*.*

Proof. Suppose $T: U \rightarrow V$ is nf-continuous on U. We shall prove that T is sequentially nf-continuous. Let $w_0 \in U$ be any element and $w=\left(w_n\right)$ be any sequence in U converging to w_0 w.r.t $N_1\left(\mathcal{G}_1, \mathcal{B}_1, \mathcal{Y}_1\right)$ i.e. $N_1\left(\mathcal{G}_1, \mathcal{B}_1, \mathcal{Y}_1\right)-\lim_{n \rightarrow \infty} w_n=w_0$. Let $\epsilon>0$ and $0<\eta<1$. Since, $T: U \rightarrow V$ is nf-continuous at w_0, there exists $\delta=\delta(\eta, \epsilon)>0$ and $\xi=\xi(\eta, \epsilon)>0$ such that for all $w \in X^n$ satisfying

$$\mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w-w_0, \delta\right)>\xi$$

and

$$\mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w-w_0, \delta\right)<1-\xi,\ \mathcal{Y}_1\left(u_1, u_2, ..., u_{n-1}, w-w_0, \delta\right)<1-\xi.$$

We have

$$\begin{aligned}
&\mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w)-T\left(w_0\right), \epsilon\right)>\eta, \\
&\mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w)-T\left(w_0\right), \epsilon\right)<1-\eta, \\
&\mathcal{Y}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w)-T\left(w_0\right), \epsilon\right)<1-\eta.
\end{aligned} \tag{5}$$

Since $N_1\left(\mathcal{G}_1, \mathcal{B}_1, \mathcal{Y}_1\right)-\lim_{n \rightarrow \infty} w_n=w_0$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we have

$$\mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w_n-w_0, \delta\right)>\xi$$

and

$$\mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w_n-w_0, \delta\right)<1-\xi,\ \mathcal{Y}_1\left(u_1, u_2, ..., u_{n-1}, w_n-w_0, \delta\right)<1-\xi.$$

So by (5), we have for all $n \geq n_1$,

$$\begin{aligned}
&\mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right)-T\left(w_0\right), \epsilon\right)>\eta, \\
&\mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right)-T\left(w_0\right), \epsilon\right)<1-\eta, \\
&\mathcal{Y}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right)-T\left(w_0\right), \epsilon\right)<1-\eta.
\end{aligned}$$

This denotes that $T(w_n) \to T(w_0)$ w.r.t $N_2(\mathcal{G}_2,\mathcal{B}_2,\mathcal{Y}_2)$ and therefore T is sequentially nf-continuous on U as w_0 was selected arbitrary. This demonstrates that $T(w_n) \to T(w_0)$ w.r.t $N_2(\mathcal{G}_2,\mathcal{B}_2,\mathcal{Y}_2)$ and so T is sequentially nf-continuous on U as w_0 was selected arbitrary.

Conversely, assume that $T : U \to V$ is sequentially nf-continuous on U. We have to demonstrate that T is nf-continuous on U. Suppose that T is not nf-continuous on U. Then $\exists w_0 \in U$ such that T is not nf-continuous at w_0. Then $\exists \epsilon > 0$ and $\eta > 0$ such that for any $\delta > 0$ and $0 < \xi < 1$ there exists $w' \in X^n$ such that

$$\mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w_0 - w', \delta) > \xi$$

and

$$\mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w_0 - w', \delta) < 1 - \xi,\ \mathcal{Y}_1(u_1, u_2, ..., u_{n-1}, w_0 - w', \delta) < 1 - \xi.$$

We have

$$\begin{aligned} &\mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_0) - T(w'), \epsilon) \leq \eta,\\ &\mathcal{B}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_0) - T(w'), \epsilon) \geq 1-\eta,\\ &\mathcal{Y}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_0) - T(w'), \epsilon) \geq 1-\eta. \end{aligned} \tag{6}$$

If we take $\xi = 1 - \frac{1}{n+1}$ and $\delta = \frac{1}{n+1}$, $k = 1, 2, 3, \ldots$, then we have a sequence (w'_n) such that

$$\begin{aligned} &\mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \tfrac{1}{n+1}\right) > 1 - \tfrac{1}{n+1},\\ &\mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \tfrac{1}{n+1}\right) < 1 - \tfrac{1}{n+1},\\ &\mathcal{Y}_1\left(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \tfrac{1}{n+1}\right) < 1 - \tfrac{1}{n+1}. \end{aligned} \tag{7}$$

However,

$$\begin{aligned} &\mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_0) - T(w'), \epsilon) \leq \eta,\\ &\mathcal{B}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_0) - T(w'), \epsilon) \geq 1-\eta,\\ &\mathcal{Y}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_0) - T(w'), \epsilon) \geq 1-\eta. \end{aligned}$$

Further, for $\delta > 0$, we can select $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $\frac{1}{n+1} < \delta$. Now,

$$\begin{aligned} &\mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \delta) > \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \tfrac{1}{n+1}\right) > 1 - \tfrac{1}{n+1},\\ &\mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \delta) \leq \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \tfrac{1}{n+1}\right) < \tfrac{1}{n+1},\\ &\mathcal{Y}_1(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \delta) \leq \mathcal{Y}_1\left(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \tfrac{1}{n+1}\right) < \tfrac{1}{n+1}, \text{ by using (7)} \end{aligned}$$

will imply

$$\begin{aligned} &\lim_{n\to\infty} \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \delta) = 1,\\ &\lim_{n\to\infty} \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \delta) = 0,\\ &\lim_{n\to\infty} \mathcal{Y}_1(u_1, u_2, ..., u_{n-1}, w_0 - w'_n, \delta) = 0. \end{aligned}$$

Thus show that $(w'_n) \to w_0$ w.r.t $N_1(\mathcal{G}_1,\mathcal{B}_1,\mathcal{Y}_1)$.

Now by (6),

$$\begin{cases} \mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_0) - T(w'_n), \epsilon) \leq \eta,\\ \mathcal{B}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_0) - T(w'_n), \epsilon) \geq 1-\eta,\\ \mathcal{Y}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w_0) - T(w'_n), \epsilon) \geq 1-\eta. \end{cases}$$

$$\Rightarrow \begin{cases} \lim_{n\to\infty} \mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_0\right) - T\left(w_n'\right), \epsilon\right) \neq 1, \\ \lim_{n\to\infty} \mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_0\right) - T\left(w_n'\right), \epsilon\right) \neq 0, \\ \lim_{n\to\infty} \mathcal{Y}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_0\right) - T\left(w_n'\right), \epsilon\right) \neq 0. \end{cases}$$

So $T\left(w_n'\right) \nrightarrow T\left(w_0\right)$ w.r.t $N_2\left(\mathcal{G}_2, \mathcal{B}_2, \mathcal{Y}_2\right)$. This demonstrates the lack of sequential continuity of T as $\left(w_n'\right) \rightarrow w_0$ w.r.t $N_1\left(\mathcal{G}_1, \mathcal{B}_1, \mathcal{Y}_1\right)$. Consequently, we encounter a contradiction, confirming that T exhibits neutrosophic continuity in U. ■

We will now establish the concepts of neutrosophic weak and strong boundedness for a linear operator and explore their pertinent associations.

Definition 8 *A linear operator $T : U \rightarrow V$ is called to be strongly neutrosophic bounded on U iff $\exists M > 0$ such that for all $w \in U$ and $\eta > 0$*

$$\begin{aligned} \mathcal{G}_2\left(T\left(w_1, w_2, ..., w_n\right), \eta\right) &\geq \mathcal{G}_1\left(w_1, w_2, ..., w_n, \frac{\eta}{M}\right), \\ \mathcal{B}_2\left(T\left(w_1, w_2, ..., w_n\right), \eta\right) &\leq \mathcal{B}_2\left(w_1, w_2, ..., w_n, \frac{\eta}{M}\right), \\ \mathcal{Y}_2\left(T\left(w_1, w_2, ..., w_n\right), \eta\right) &\leq \mathcal{Y}_2\left(w_1, w_2, ..., w_n, \frac{\eta}{M}\right). \end{aligned}$$

Example 2 *Let $(X, \|\cdot\|_n)$ be a n-normed linear space. Define $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{Y}_1, \mathcal{Y}_2$ as follows:*

$$\begin{aligned} \mathcal{G}_1\left(w_1, w_2, ..., w_n, \eta\right) &= \begin{cases} \frac{\eta}{\eta+\alpha_1\|w_1,w_2,...,w_n\|} & \text{if } \eta > 0, \\ 0 & \text{if } \eta \leq 0, \end{cases} \\ \mathcal{B}_1\left(w_1, w_2, ..., w_n, \eta\right) &= \begin{cases} \frac{\alpha_1\|w_1,w_2,...,w_n\|}{\eta+\alpha_1\|w_1,w_2,...,w_n\|} & \text{if } \eta > 0, \\ 0 & \text{if } \eta \leq 0, \end{cases} \\ \mathcal{Y}_1\left(w_1, w_2, ..., w_n, \eta\right) &= \begin{cases} \frac{\alpha_1\|w_1,w_2,...,w_n\|}{\eta} & \text{if } \eta > 0, \\ 0 & \text{if } \eta \leq 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_2\left(w_1, w_2, ..., w_n, \eta\right) &= \begin{cases} \frac{\eta}{\eta+\alpha_2\|w_1,w_2,...,w_n\|} & \text{if } \eta > 0, \\ 0 & \text{if } \eta \leq 0, \end{cases} \\ \mathcal{B}_2\left(w_1, w_2, ..., w_n, \eta\right) &= \begin{cases} \frac{\alpha_2\|w_1,w_2,...,w_n\|}{\eta+\alpha_2\|w_1,w_2,...,w_n\|} & \text{if } \eta > 0, \\ 0 & \text{if } \eta \leq 0, \end{cases} \\ \mathcal{Y}_2\left(w_1, w_2, ..., w_n, \eta\right) &= \begin{cases} \frac{\alpha_2\|w_1,w_2,...,w_n\|}{\eta} & \text{if } \eta > 0, \\ 0 & \text{if } \eta \leq 0. \end{cases} \end{aligned}$$

If $\eta > 0$ and $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{Y}_1, \mathcal{Y}_2$ are defined to be zero of $\eta \leq 0$, where α_1 and α_2 are fixed positive real numbers and $\alpha_1 > \alpha_2$. It is evident that the structures $(X, \mathcal{G}_1, \mathcal{B}_1, \mathcal{Y}_1, \circ, \diamond)$ and $(X, \mathcal{G}_2, \mathcal{B}_2, \mathcal{Y}_2, \circ, \diamond)$ become $NnNLS$. We define an operator $T : (X, \mathcal{G}_1) \rightarrow (X, \mathcal{G}_2)$ by $T(w) = lw$, where $w = (w_1, w_2, ..., w_n) \in X^n$, where $l \neq 0 \in \mathbb{R}$ is fixed, then it is easy to see that T is a linear operator. It's worth noting that we can select M such that $M \geq |l|$. In this context, we have

$$\mathcal{G}_2\left(T\left(w_1, w_2, ..., w_n\right), \eta\right) \geq \mathcal{G}_1\left(w_1, w_2, ..., w_n, \frac{\eta}{M}\right), \forall \left(w_1, w_2, ..., w_n\right) \in X^n, \forall \eta \in \mathbb{R}.$$

Since, $w=(w_1,w_2,...,w_n)\in U,\ M\geq|l|$ *we have,* $\alpha_1 M\geq\alpha_2|l|$ *since* $(\alpha_1>\alpha_2>0)$

$$\begin{aligned}
&\Rightarrow\quad \alpha_1 M\left\|w_1,w_2,...,w_n\right\|\geq\alpha_2|l|\left\|w_1,w_2,...,w_n\right\|\\
&\Rightarrow\quad \eta+\alpha_1 M\left\|w_1,w_2,...,w_n\right\|\geq\eta+\alpha_2|l|\left\|w_1,w_2,...,w_n\right\|,\ \forall\eta>0\\
&\Rightarrow\quad \frac{1}{\eta+\alpha_2|l|\left\|w_1,w_2,...,w_n\right\|}\geq\frac{1}{\eta+\alpha_1 M\left\|w_1,w_2,...,w_n\right\|}\\
&\Rightarrow\quad \frac{\eta}{\eta+\alpha_2\left\|lw_1,w_2,...,w_n\right\|}\geq\frac{\frac{\eta}{M}}{\frac{\eta}{M}+\alpha_1\left\|w_1,w_2,...,w_n\right\|}.
\end{aligned}$$

$$\mathcal{G}_2\left(T\left(w_1,w_2,...,w_n\right),\eta\right)\geq\mathcal{G}_1\left(w_1,w_2,...,w_n,\frac{\eta}{M}\right),\ \forall\left(w_1,w_2,...,w_n\right)\in X^n,\forall\eta\in\mathbb{R}.$$

Further,

$$\begin{aligned}
&\quad\ \ \alpha_2\left|l\right|\leq\alpha_1 M\\
&\Rightarrow\quad \alpha_2|l|\eta\leq\alpha_1 M\eta\\
&\Rightarrow\quad \alpha_2|l|\eta+\alpha_1\alpha_2 M|l|\left\|w_1,w_2,...,w_n\right\|\leq\alpha_1 M\eta+\alpha_1\alpha_2 M|l|\left\|w_1,w_2,...,w_n\right\|\\
&\Rightarrow\quad \frac{\alpha_2|l|}{\eta+\alpha_2|l|\left\|w_1,w_2,...,w_n\right\|}\leq\frac{\alpha_1 M}{\eta+\alpha_1 M\left\|w_1,w_2,...,w_n\right\|}\\
&\Rightarrow\quad \frac{\alpha_2\left\|lw_1,w_2,...,w_n\right\|}{\eta+\alpha_2\left\|lw_1,w_2,...,w_n\right\|}\leq\frac{\alpha_1\left\|w_1,w_2,...,w_n\right\|}{\frac{\eta}{M}+\alpha_1\left\|w_1,w_2,...,w_n\right\|}.
\end{aligned}$$

$$\mathcal{B}_2\left(T\left(w_1,w_2,...,w_n\right),\eta\right)\leq\mathcal{B}_1\left(w_1,w_2,...,w_n,\frac{\eta}{M}\right),\ \forall\left(w_1,w_2,...,w_n\right)\in X^n,\forall\eta\in\mathbb{R}.$$

Similarly,

$$\mathcal{Y}_2\left(T\left(w_1,w_2,...,w_n\right),\eta\right)\leq\mathcal{Y}_1\left(w_1,w_2,...,w_n,\frac{\eta}{M}\right),\ \forall\left(w_1,w_2,...,w_n\right)\in X^n,\forall\eta\in\mathbb{R}.$$

This demonstrates the strong neutrosophic boundedness of the operator T.

Definition 9 *A linear operator* $T:U\to V$ *is called to be weakly neutrosophic bounded on* U *if for any* η, $0<\eta<1$, $\exists M_\eta>0$ *such that* $\forall w\in U$ *and* $\xi>0$,

$$\mathcal{G}_1\left(w_1,w_2,...,w_n,\frac{\xi}{M_\eta}\right)\geq\eta,\ \ \mathcal{B}_1\left(w_1,w_2,...,w_n,\frac{\xi}{M_\eta}\right)\leq 1-\eta,\ \ \mathcal{Y}_1\left(w_1,w_2,...,w_n,\frac{\xi}{M_\eta}\right)\leq 1-\eta.$$

Then

$$\mathcal{G}_2(T\left(w_1,w_2,...,w_n\right),\xi)\geq\eta,\ \ \mathcal{B}_2(T\left(w_1,w_2,...,w_n\right),\xi)\leq 1-\eta\ \textit{and}\ \ \mathcal{Y}_2(T\left(w_1,w_2,...,w_n\right),\xi)\leq 1-\eta.$$

Example 3 *Consider* $(X,\|\cdot\|_n)$ *as a n-normed space. We define* $s\circ t$ *as the minimum of* s *and* t*, and* $s\diamond t$ *as the maximum of* s *and* t *for any* s*,* t *in the interval* $[0,1]$*;*

$$\begin{aligned}
\mathcal{G}_1\left(w_1,w_2,...,w_n,\xi\right)&=\begin{cases}\frac{\xi^2-(\|w_1,w_2,...,w_n\|)^2}{\xi^2+(\|w_1,w_2,...,w_n\|)^2} & \textit{if } \xi>\left\|w_1,w_2,...,w_n\right\|,\\ 0 & \textit{if } \xi\leq\left\|w_1,w_2,...,w_n\right\|,\end{cases}\\
\mathcal{B}_1\left(w_1,w_2,...,w_n,\xi\right)&=\begin{cases}\frac{2(\|w_1,w_2,...,w_n\|)^2}{\xi^2+(\|w_1,w_2,...,w_n\|)^2} & \textit{if } \xi>\left\|w_1,w_2,...,w_n\right\|,\\ 0 & \textit{if } \xi\leq\left\|w_1,w_2,...,w_n\right\|,\end{cases}\\
\mathcal{Y}_1\left(w_1,w_2,...,w_n,\xi\right)&=\begin{cases}\frac{2(\|w_1,w_2,...,w_n\|)^2}{\xi^2} & \textit{if } \xi>\left\|w_1,w_2,...,w_n\right\|,\\ 0 & \textit{if } \xi\leq\left\|w_1,w_2,...,w_n\right\|,\end{cases}
\end{aligned}$$

and

$$\mathcal{G}_2(w_1,w_2,...,w_n,\xi) = \begin{cases} \frac{\xi}{\xi+\|w_1,w_2,...,w_n\|} & \text{if } \xi>0, \forall w_1,w_2,...,w_n\in X, \\ 0 & \text{if } \xi\leq 0, \forall w_1,w_2,...,w_n\in X, \end{cases}$$

$$\mathcal{B}_2(w_1,w_2,...,w_n,\xi) = \begin{cases} \frac{\|w_1,w_2,...,w_n\|}{\xi+\|w_1,w_2,...,w_n\|} & \text{if } \xi>0, \forall w_1,w_2,...,w_n\in X, \\ 0 & \text{if } \xi\leq 0, \forall w_1,w_2,...,w_n\in X, \end{cases}$$

$$\mathcal{Y}_2(w_1,w_2,...,w_n,\xi) = \begin{cases} \frac{\|w_1,w_2,...,w_n\|}{\xi} & \text{if } \xi>0, \forall w_1,w_2,...,w_n\in X, \\ 0 & \text{if } \xi\leq 0, \forall w_1,w_2,...,w_n\in X, \end{cases}$$

If $\xi>0$ and $\mathcal{G}_1,\mathcal{G}_2,\mathcal{B}_1,\mathcal{B}_2,\mathcal{Y}_1$ and $\mathcal{Y}_2$ are said to be zero for $\xi\leq 0$. It is evident that $U=(X,\mathcal{G}_1,\mathcal{B}_1,\mathcal{Y}_1,\circ,\diamond)$ and $V=(X,\mathcal{G}_2,\mathcal{B}_2,\mathcal{Y}_2,\circ,\diamond)$ both qualify as NnNLS.
Establish an operator $T:U\to V$ by $T(w)=w$ where $w=(w_1,w_2,...,w_n)\in X^n$. If we select $M_\eta=\frac{1}{1-\eta}$, for all $\eta\in(0,1)$, then for $\xi>\|w_1,w_2,...,w_n\|$ we obtain

$$\mathcal{G}_1\left(w_1,w_2,...,w_n,\frac{\xi}{M_\eta}\right)\geq\eta\Rightarrow\frac{\xi^2(1-\eta)^2-(\|w_1,w_2,...,w_n\|)^2}{\xi^2(1-\eta)^2+(\|w_1,w_2,...,w_n\|)^2}\geq\eta.$$

$$\begin{aligned}
&\xi^2(1-\eta)^2-(\|w_1,w_2,...,w_n\|)^2\geq\eta\xi^2(1-\eta)^2+\eta(\|w_1,w_2,...,w_n\|)^2\\
&\Rightarrow\xi^2(1-\eta)^2-\eta\xi^2(1-\eta)^2\geq(\|w_1,w_2,...,w_n\|)^2+\eta(\|w_1,w_2,...,w_n\|)^2\\
&\Rightarrow\xi^2(1-\eta)^2(1-\eta)\geq(1+\eta)(\|w_1,w_2,...,w_n\|)^2\\
&\Rightarrow\xi^2(1-\eta)^3\geq(1+\eta)(\|w_1,w_2,...,w_n\|)^2\\
&\Rightarrow\tfrac{\xi^2(1-\eta)^3}{(1+\eta)}\geq(\|w_1,w_2,...,w_n\|)^2\\
&\Rightarrow(\|w_1,w_2,...,w_n\|)^2\leq\tfrac{\xi^2(1-\eta)^3}{(1+\eta)}\\
&\Rightarrow\|w_1,w_2,...,w_n\|\leq\tfrac{\xi(1-\eta)^{\frac{3}{2}}}{(1+\eta)^{\frac{1}{2}}}\\
&\Rightarrow\|w_1,w_2,...,w_n\|\leq\tfrac{\xi(1-\eta)(1-\eta)^{\frac{1}{2}}}{(1+\eta)^{\frac{1}{2}}}\\
&\Rightarrow\xi+\|w_1,w_2,...,w_n\|\leq\tfrac{\xi(1-\eta)(1-\eta)^{\frac{1}{2}}}{(1+\eta)^{\frac{1}{2}}}+\xi\\
&\Rightarrow\xi+\|w_1,w_2,...,w_n\|\leq\tfrac{\xi(1-\eta)(1-\eta)^{\frac{1}{2}}+\xi(1+\eta)^{\frac{1}{2}}}{(1+\eta)^{\frac{1}{2}}}\\
&\Rightarrow\tfrac{\xi+\|w_1,w_2,...,w_n\|}{\xi}\leq\tfrac{(1-\eta)(1-\eta)^{\frac{1}{2}}+(1+\eta)^{\frac{1}{2}}}{(1+\eta)^{\frac{1}{2}}}
\end{aligned}$$

$$\Rightarrow\frac{\xi}{\xi+\|w_1,w_2,...,w_n\|}\geq\frac{(1+\eta)^{\frac{1}{2}}}{(1-\eta)(1-\eta)^{\frac{1}{2}}+(1+\eta)^{\frac{1}{2}}}. \tag{8}$$

Now,

$$\begin{aligned}
&\tfrac{(1+\eta)^{\frac{1}{2}}}{(1-\eta)(1-\eta)^{\frac{1}{2}}+(1+\eta)^{\frac{1}{2}}}\geq\eta\\
&\Rightarrow(1+\eta)^{\frac{1}{2}}\geq\eta(1-\eta)(1-\eta)^{\frac{1}{2}}+\eta(1+\eta)^{\frac{1}{2}}\\
&\Rightarrow(1+\eta)^{\frac{1}{2}}-\eta(1+\eta)^{\frac{1}{2}}\geq\eta(1-\eta)(1-\eta)^{\frac{1}{2}}\\
&\Rightarrow(1-\eta)(1+\eta)^{\frac{1}{2}}\geq\eta(1-\eta)(1-\eta)^{\frac{1}{2}}\\
&\Rightarrow(1+\eta)^{\frac{1}{2}}\geq\eta(1-\eta)^{\frac{1}{2}}\ \textit{(squaring both sides)}\\
&\Rightarrow(1+\eta)\geq\eta^2(1-\eta)\Rightarrow1+\eta\geq\eta^2-\eta^3\Rightarrow1+\eta+\eta^3\geq\eta^2.
\end{aligned}$$

This holds for all $\eta\in(0,1)$ by (8), we get,

$$\mathcal{G}_2(T(u_1),T(u_2),...,T(u_{n-1}),T(w),\xi)\geq\eta$$

if $\xi > \|w_1, w_2, ..., w_n\|$. *Since*

$$\xi \leq \|w_1, w_2, ..., w_n\|, \frac{\xi^2 - (\|w_1, w_2, ..., w_n\|)^2}{\xi^2 + (\|w_1, w_2, ..., w_n\|)^2} = 0,$$

it yields that

$$\mathcal{G}_1\left(w_1, w_2, ..., w_n, \frac{\xi}{M_\eta}\right) \geq \eta \Rightarrow \mathcal{G}_2\left(T\left(w_1, w_2, ..., w_n\right), \xi\right) \geq \eta, \quad \forall \eta \in (0,1).$$

For all cases, we get,

$$\mathcal{G}_1\left(w_1, w_2, ..., w_n, \frac{\xi}{M_\eta}\right) \geq \eta \Rightarrow \mathcal{G}_2\left(T\left(w_1, w_2, ..., w_n\right), \xi\right) \geq \eta, \quad \forall \eta \in (0,1).$$

Now

$$\mathcal{B}_1\left(w_1, w_2, ..., w_n, \frac{\xi}{M_\eta}\right) \leq 1 - \eta \Rightarrow \mathcal{B}_2\left(T\left(w_1, w_2, ..., w_n\right), \xi\right) \leq 1 - \eta, \quad \forall \eta \in (0,1).$$

$$\begin{aligned}
& \mathcal{B}_1\left(w_1, w_2, ..., w_n, \frac{\xi}{M_\eta}\right) \leq 1 - \alpha \\
\Rightarrow \quad & \frac{2\|w_1, w_2, ..., w_n\|^2}{\xi^2(1-\alpha)^2 + \|w_1, w_2, ..., w_n\|^2} \leq 1 - \alpha \\
\Rightarrow \quad & 2\|w_1, w_2, ..., w_n\|^2 \leq (1-\alpha)\left(\xi^2(1-\alpha)^2 + \|w_1, w_2, ..., w_n\|^2\right) \\
\Rightarrow \quad & 2\|w_1, w_2, ..., w_n\|^2 \leq (1-\alpha)\left(\xi^2(1-\alpha)^2 + (1-\alpha)\|w_1, w_2, ..., w_n\|^2\right) \\
\Rightarrow \quad & 2\|w_1, w_2, ..., w_n\|^2 - (1-\alpha)\|w_1, w_2, ..., w_n\|^2 \leq (1-\alpha)^3\xi^2 \\
\Rightarrow \quad & 2\|w_1, w_2, ..., w_n\|^2 - \|w_1, w_2, ..., w_n\|^2 + \alpha\|w_1, w_2, ..., w_n\|^2 \leq (1-\alpha)^3\xi^2 \\
\Rightarrow \quad & \|w_1, w_2, ..., w_n\|^2 + \alpha\|w_1, w_2, ..., w_n\|^2 \leq (1-\alpha)^3\xi^2 \\
\Rightarrow \quad & (1+\alpha)\|w_1, w_2, ..., w_n\|^2 \leq (1-\alpha)^3\xi^2 \\
\Rightarrow \quad & \|w_1, w_2, ..., w_n\|^2 \leq \frac{(1-\alpha)^3\xi^2}{(1+\alpha)} \\
\Rightarrow \quad & \|w_1, w_2, ..., w_n\| \leq \frac{(1-\alpha)^{\frac{3}{2}}\xi}{(1+\alpha)^{\frac{1}{2}}} \\
\Rightarrow \quad & \|w_1, w_2, ..., w_n\| \leq \frac{(1-\alpha)(1-\alpha)^{\frac{1}{2}}\xi}{(1+\alpha)^{\frac{1}{2}}} \\
\Rightarrow \quad & \xi + \|w_1, w_2, ..., w_n\| \leq \frac{(1-\alpha)(1-\alpha)^{\frac{1}{2}}\xi}{(1+\alpha)^{\frac{1}{2}}} + \xi \\
\Rightarrow \quad & \xi + \|w_1, w_2, ..., w_n\| \leq \frac{(1-\alpha)(1-\alpha)^{\frac{1}{2}}\xi + \xi(1+\alpha)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}} \\
\Rightarrow \quad & \xi + \|w_1, w_2, ..., w_n\| \leq \frac{\xi\left[(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}\right]}{(1+\alpha)^{\frac{1}{2}}} \\
\Rightarrow \quad & \xi + \|w_1, w_2, ..., w_n\| \leq \frac{\|w_1, w_2, ..., w_n\|\left[(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}\right]}{(1+\alpha)^{\frac{1}{2}}} \\
\Rightarrow \quad & \frac{\xi + \|w_1, w_2, ..., w_n\|}{\|w_1, w_2, ..., w_n\|} \leq \frac{(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}} \\
\Rightarrow \quad & \frac{\|w_1, w_2, ..., w_n\|}{\xi + \|w_1, w_2, ..., w_n\|} \geq \frac{(1+\alpha)^{\frac{1}{2}}}{(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{(1+\alpha)^{\frac{1}{2}}}{(1-\alpha)(1-\alpha)^{\frac{1}{2}}+(1+\alpha)^{\frac{1}{2}}} \leq (1-\alpha)\\
\Rightarrow\quad & (1+\alpha)^{\frac{1}{2}} \leq (1-\alpha)(1-\alpha)(1-\alpha)^{\frac{1}{2}}+(1-\alpha)(1+\alpha)^{\frac{1}{2}}\\
\Rightarrow\quad & (1+\alpha)^{\frac{1}{2}}-(1-\alpha)(1+\alpha)^{\frac{1}{2}} \leq (1-\alpha)^2(1-\alpha)^{\frac{1}{2}}\\
\Rightarrow\quad & (1+\alpha)^{\frac{1}{2}}-(1+\alpha)^{\frac{1}{2}}+\alpha(1+\alpha)^{\frac{1}{2}} \leq (1-\alpha)^2(1-\alpha)^{\frac{1}{2}}\\
\Rightarrow\quad & \alpha(1+\alpha)^{\frac{1}{2}} \leq (1-\alpha)^2(1-\alpha)^{\frac{1}{2}} \ (\textit{squaring both sides})
\end{aligned}$$

$$\begin{aligned}
& \alpha^2(1+\alpha) \leq (1-\alpha)^4(1-\alpha)\\
\Rightarrow\quad & \alpha^2+\alpha^3 \leq (1-\alpha)^4-\alpha(1-\alpha)^4\\
\Rightarrow\quad & \alpha^2+\alpha^3 \leq \left[(1-\alpha)^2\right]^2-\alpha\left[(1-\alpha)^2\right]^2\\
\Rightarrow\quad & \alpha^2+\alpha^3 \leq \left[1-2\alpha+\alpha^2\right]^2-\alpha\left[1-2\alpha+\alpha^2\right]^2\\
\Rightarrow\quad & \alpha^2+\alpha^3 \leq \left[1+4\alpha^2+\alpha^4-4\alpha-4\alpha^3+2\alpha^2\right]-\alpha\left[1+4\alpha^2+\alpha^4-4\alpha-4\alpha^3+2\alpha^2\right]\\
\Rightarrow\quad & \alpha^2+\alpha^3 \leq \left[1+4\alpha^2+\alpha^4-4\alpha-4\alpha^3+2\alpha^2\right]-\alpha-4\alpha^3-\alpha^5+4\alpha^2+4\alpha^4-2\alpha^3\\
\Rightarrow\quad & \alpha^2+\alpha^3 \leq 1+10\alpha^2+5\alpha^4-5\alpha-10\alpha^3-\alpha^5\\
\Rightarrow\quad & \alpha^3+10\alpha^3+5\alpha+\alpha^5 \leq 1+10\alpha^2+5\alpha^4-\alpha^2.
\end{aligned}$$

This holds for all $\eta \in (0,1)$ *we obtain,* $\mathcal{B}_2\left(T\left(w_1,w_2,...,w_n\right),\xi\right) \leq 1-\eta$ *if* $\xi < \|w_1,w_2,...,w_n\|$. *Since*

$$\xi \geq \|w_1,w_2,...,w_n\|, \frac{\xi^2}{\xi^2+(\|w_1,w_2,...,w_n\|)^2}=0,$$

it means that

$$\mathcal{B}_1\left(w_1,w_2,...,w_n,\frac{\xi}{M_\eta}\right) \leq 1-\eta \Rightarrow \mathcal{B}_2\left(T\left(w_1,w_2,...,w_n\right),\xi\right) \leq 1-\eta,\ \forall\eta \in (0,1).$$

Similarly,

$$\mathcal{Y}_1\left(w_1,w_2,...,w_n,\frac{\xi}{M_\eta}\right) \leq 1-\eta \Rightarrow \mathcal{Y}_2\left(T\left(w_1,w_2,...,w_n\right),\xi\right) \leq 1-\eta, \forall\eta \in (0,1).$$

This shows that T *is weakly neutrosophic bounded.*

Theorem 3 *If a linear operator* $T : U \to V$ *is strongly neutrosophic bounded on* U*, then it is weakly neutrosophic bounded on* U*.*

Proof. Suppose that $T : U \to V$ is strongly neutrosophic bounded on U. So, there exists $\exists M > 0$ such that for all $w = (w_1,w_2,...,w_n) \in U$ and $\eta > 0$

$$\begin{aligned}
\mathcal{G}_2(T(w_1,w_2,...,w_n),\eta) &\geq \mathcal{G}_1\left(w_1,w_2,...,w_n,\tfrac{\eta}{M}\right),\\
\mathcal{B}_2\left(T\left(w_1,w_2,...,w_n\right),\eta\right) &\leq \mathcal{B}_1\left(w_1,w_2,...,w_n,\tfrac{\eta}{M}\right),\\
\mathcal{Y}_2(T\left(w_1,w_2,...,w_n\right),\eta) &\leq \mathcal{Y}_1\left(w_1,w_2,...,w_n,\tfrac{\eta}{M}\right).
\end{aligned} \tag{9}$$

Let $0 < \xi < 1$. Then $\exists M_\xi (= M > 0)$ such that

$$\mathcal{G}_1\left(w_1,w_2,...,w_n,\frac{\eta}{M_\xi}\right) \geq \xi,\ \ \mathcal{B}_1\left(w_1,w_2,...,w_n,\frac{\eta}{M_\xi}\right) \leq 1-\xi \ \text{ and } \ \mathcal{Y}_1\left(w_1,w_2,...,w_n,\frac{\eta}{M_\xi}\right) \leq 1-\xi.$$

Then

$$\begin{aligned}
\mathcal{G}_2(T(w_1, w_2, ..., w_n), \eta) &\geq \mathcal{G}_1\left(w_1, w_2, ..., w_n, \frac{\eta}{M_\xi}\right) \geq \xi, \\
\mathcal{B}_2(T(w_1, w_2, ..., w_n), \eta) &\leq \mathcal{B}_1\left(w_1, w_2, ..., w_n, \frac{\eta}{M_\xi}\right) \leq 1-\xi, \\
\mathcal{Y}_2(T(w_1, w_2, ..., w_n), \eta) &\leq \mathcal{Y}_1\left(w_1, w_2, ..., w_n, \frac{\eta}{M_\xi}\right) \leq 1-\xi. \quad \text{(by using (9))}.
\end{aligned}$$

Since this is valid for all $w \in U$ and $\eta > 0$, it follows that $T : U \to V$ exhibits weak neutrosophic boundedness. ■

Theorem 4 *A linear operator $T : U \to V$ is strongly nf-continuous everywhere on U if T is strongly nf-continuous at a point $w_0 \in U$.*

Proof. Let $w_0 \in U$ be a point in U such that $T : U \to V$ is strongly nf-continuous at w_0. We denote that T is strongly nf-continuous everywhere in U. Since T is strongly nf-continuous at w_0, we see that for each $\epsilon > 0$, $u_1, u_2, ..., u_{n-1} \in U$ and $T(u_1), T(u_2), ..., T(u_{n-1}) \in V$, $\exists \delta > 0$ such that $\forall w \in U$

$$\begin{aligned}
&\mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) \geq \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta), \\
&\mathcal{B}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) \leq \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta), \\
&\mathcal{W}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(w_0), \epsilon) \leq \mathcal{W}_1(u_1, u_2, ..., u_{n-1}, w - w_0, \delta).
\end{aligned} \tag{10}$$

Suppose $\sigma \in U$ is any element of U. Then $w + w_0 - \sigma$ is also an element of U, and therefore by replacing w by $w + w_0 - \sigma$ in (10), we have

$$\mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w + w_0 - \sigma) - T(w_0), \epsilon) \geq \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w + w_0 - \sigma - w_0, \delta)$$

and

$$\mathcal{B}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w + w_0 - \sigma) - T(w_0), \epsilon) \leq \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w + w_0 - \sigma - w_0, \delta).$$

Then

$$\mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w + w_0 - \sigma) - T(w_0), \epsilon) \geq \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w - \sigma, \delta)$$

and

$$\mathcal{B}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w + w_0 - \sigma) - T(w_0), \epsilon) \leq \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w + w_0 - \sigma - w_0, \delta).$$

That is,

$$\mathcal{G}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(\sigma), \epsilon) \geq \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w - \sigma, \delta)$$

and

$$\mathcal{B}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(\sigma), \epsilon) \leq \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, w - \sigma, \delta).$$

Similarly,

$$\mathcal{W}_2(T(u_1), T(u_2), ..., T(u_{n-1}), T(w) - T(\sigma), \epsilon) \leq \mathcal{W}_1(u_1, u_2, ..., u_{n-1}, w - \sigma, \delta).$$

Since, $\sigma \in U$ was arbitrarily selected, we see that $T : U \to V$ is strongly nf-continuous. ■

Theorem 5 *A linear map $T : U \to V$ is strongly nf-continuous iff T is strongly neutrosophic bounded.*

Proof. Suppose that $T: U \to V$ is strongly nf-continuous on U, then T is strongly nf-continuous at $\theta \in U$ where θ denote the zero element of U. So for $\epsilon = 1$, $\exists \delta > 0$ such that for all $w \in U$

$$\begin{aligned}
\mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w\right) - T\left(\theta\right), 1\right) \geq \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w - \theta, \delta\right), \\
\mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w\right) - T\left(\theta\right), 1\right) \leq \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w - \theta, \delta\right), \\
\mathcal{W}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w\right) - T\left(\theta\right), 1\right) \leq \mathcal{W}_1\left(u_1, u_2, ..., u_{n-1}, w - \theta, \delta\right).
\end{aligned}$$

Case 1. Let $w \neq \theta$ and $\eta > 0$. Take $\sigma = \frac{w}{\eta}$. Then

$$\begin{aligned}
\mathcal{G}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w), \eta) &= \mathcal{G}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(\eta\sigma), \eta) \\
&= \mathcal{G}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), \eta T(\sigma), \eta) \\
&= \mathcal{G}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(\sigma), 1) \\
&\geq \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, \sigma, \delta) \\
&= \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, \frac{w}{\eta}, \delta\right) \\
&= \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w, \frac{\eta}{\frac{1}{\delta}}\right) \\
&= \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w, \frac{\eta}{M}\right),
\end{aligned}$$

where $M = \frac{1}{\delta}$, i.e

$$\mathcal{G}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w), \eta) \geq \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w, \frac{\eta}{M}\right)$$

and

$$\begin{aligned}
\mathcal{B}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w), \eta) &= \mathcal{B}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(\eta\sigma), \eta) \\
&= \mathcal{B}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), \eta T(\sigma), \eta) \\
&= \mathcal{B}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\sigma), 1) \\
&\leq \mathcal{B}_1(u_1, u_2, ..., u_{n-1}, \sigma, \delta) \\
&= \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, \frac{w}{\eta}, \delta\right) \\
&= \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w, \frac{\eta}{\frac{1}{\delta}}\right) \\
&= \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w, \frac{\eta}{M}\right),
\end{aligned}$$

where $M = \frac{1}{\delta}$, i.e

$$\mathcal{B}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w), \eta) \leq \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w, \frac{\eta}{M}\right).$$

Similarly,

$$\mathcal{W}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w), \eta) \leq \mathcal{W}_1\left(u_1, u_2, ..., u_{n-1}, w, \frac{\eta}{M}\right).$$

Case 2. If $w = \theta$ and $\eta > 0$, then

$$T(\theta) = \theta,\ \mathcal{G}_2(\theta, \eta) = \mathcal{G}_1\left(\theta, \frac{\eta}{M}\right) = 1,\ \mathcal{B}_2(\theta, \eta) = \mathcal{B}_1\left(\theta, \frac{\eta}{M}\right) = 0 \text{ and } \mathcal{W}_2(\theta, \eta) = \mathcal{W}_1\left(\theta, \frac{\eta}{M}\right) = 0.$$

Hence, in both instances, T is strongly bounded in the neutrosophic sense. Conversely, suppose that T is strongly neutrosophic bounded so $\exists M > 0$ such that $\forall w \in U$ and $\eta > 0$

$$\begin{aligned} \mathcal{G}_2\left(T\left(w_1, w_2, ..., w_n\right), \eta\right) &\geq \mathcal{G}_1\left(w_1, w_2, ..., w_n, \frac{\eta}{M}\right), \\ \mathcal{B}_2\left(T\left(w_1, w_2, ..., w_n\right), \eta\right) &\leq \mathcal{B}_2\left(w_1, w_2, ..., w_n, \frac{\eta}{M}\right), \\ \mathcal{Y}_2\left(T\left(w_1, w_2, ..., w_n\right), \eta\right) &\leq \mathcal{Y}_2\left(w_1, w_2, ..., w_n, \frac{\eta}{M}\right). \end{aligned}$$

Let $\epsilon > 0$. Then we have

$$\mathcal{G}_2(T(w), \epsilon) \geq \mathcal{B}_1\left(w, \frac{\epsilon}{M}\right), \mathcal{B}_2(T(w), \epsilon) \leq \mathcal{B}_1\left(w, \frac{\epsilon}{M}\right) \text{ and } \mathcal{W}_2(T(w), \epsilon) \leq \mathcal{W}_1\left(w, \frac{\epsilon}{M}\right).$$

Take $\delta = \frac{\epsilon}{M}$. Then

$$\begin{aligned} \mathcal{G}_2(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w) - T(\theta), \epsilon) &\geq \mathcal{G}_1(u_1, u_2, ..., u_{n-1}, w - \theta, \delta), \\ \mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w) - T(\theta), \epsilon\right) &\leq \mathcal{B}_2\left(u_1, u_2, ..., u_{n-1}, w - \theta, \delta\right), \\ \mathcal{Y}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T(w) - T(\theta), \epsilon\right) &\leq \mathcal{Y}_2\left(u_1, u_2, ..., u_{n-1}, w - \theta, \delta\right). \end{aligned}$$

Therefore, T is strongly nf-continuous on U. ■

Theorem 6 *If a linear operator $T : U \to V$ is sequentially nf-continuous at u_0 in U then it is sequentially nf-continuous on U.*

Proof. Suppose that $T : U \to V$ is sequentially nf-continuous at w_0 in U. We shall show that T is sequentially nf-continuous on U. Let $w \in U$ be any arbitrary and (w_n) be any sequence converging to w w.r.t $N_1\left(\mathcal{G}_1, \mathcal{B}_1, \mathcal{W}_1\right)$ then, we have for all $\eta > 0$, $u_1, u_2, ..., u_{n-1} \in U$ and $T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right) \in V$,

$$\lim_{n\to\infty} \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w, \eta\right) = 1,$$

$$\lim_{n\to\infty} \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w, \eta\right) = 0,$$

and

$$\lim_{n\to\infty} \mathcal{W}_1\left(u_1, u_2, ..., u_{n-1}, w_n - w, \eta\right) = 0.$$

This implies that

$$\lim_{n\to\infty} \mathcal{G}_1\left(u_1, u_2, ..., u_{n-1}, (w_n - w + w_0) - w_0, \eta\right) = 1,$$

$$\lim_{n\to\infty} \mathcal{B}_1\left(u_1, u_2, ..., u_{n-1}, (w_n - w + w_0) - w_0, \eta\right) = 0,$$

and

$$\lim_{n\to\infty} \mathcal{W}_1\left(u_1, u_2, ..., u_{n-1}, (w_n - w + w_0) - w_0, \eta\right) = 0.$$

Since T is sequentially nf-continuous at w_0, we get

$$\lim_{n\to\infty} \mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n - w + w_0\right) - T\left(w_0\right), \eta\right) = 1,$$

$$\lim_{n\to\infty} \mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n - w + w_0\right) - T\left(w_0\right), \eta\right) = 0,$$

and

$$\lim_{n\to\infty} \mathcal{W}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n - w + w_0\right) - T\left(w_0\right), \eta\right) = 0.$$

This gives for each $\eta > 0$

$$\lim_{n\to\infty} \mathcal{G}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w\right), \eta\right) = 1,$$

$$\lim_{n\to\infty} \mathcal{B}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w\right), \eta\right) = 0,$$

and

$$\lim_{n\to\infty} \mathcal{W}_2\left(T\left(u_1\right), T\left(u_2\right), ..., T\left(u_{n-1}\right), T\left(w_n\right) - T\left(w\right), \eta\right) = 0.$$

This denotes that $(T(w_n)) \to T(w)$ w.r.t $N_2(\mathcal{G}_2, \mathcal{B}_2, \mathcal{W}_2)$ and so T is sequentially nf-continuous on U. ∎

The proof of the following two Theorems is omitted as it can be obtained analogously to the proofs of Theorem 4 and Theorem 5.

Theorem 7 *A linear operator $T : U \to V$ is weakly nf-continuous on U if T is weakly nf-continuous at a point u_0 in U.*

Proof. The proof is easy so it is omitted. ∎

Theorem 8 *A linear operator $T : U \to V$ is weakly nf-continuous if and only if T is weakly neutrosophic bounded.*

Proof. Omitted as it follows from the proof of Theorem 5. ∎

4 Conclusions

In this study, we have explored the concepts of continuity and boundedness in the context of NFnNS, focusing on linear operators. Our research reveals a distinct disparity between sequential continuity and strong continuity within these spaces, challenging conventional expectations and emphasizing the complexity of this mathematical framework.

Furthermore, we have introduced the notions of weak and strong boundedness for operators, extending our understanding of bounded operators to this intricate setting. The study uncovers valuable insights into the interplay between continuity and boundedness within NFnNS.

In conclusion, our findings advance the understanding of mathematical structures in operator theory, providing a foundation for future research and applications in various scientific and engineering domains. The relationships between continuity and boundedness in these spaces offer rich opportunities for further exploration and the potential to enhance our grasp of complex mathematical concepts.

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