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A General Approach To A Familiar Class Of Slowly Diverging Series*

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Abstract

In this note, we prove a general theorem related to the divergence of series. A direct consequence of this theorem is the divergence of a familiar class of slowly diverging series.

1 Introduction

A familiar class of slowly diverging series includes $\sum \frac{1}{k}, \sum \frac{1}{k\ln k}, \sum \frac{1}{k \ln k \ln\ln k}$, etc. In general, denoting by

$$
\begin{aligned}
\ln^{(0)} x &= x, \\
\ln^{(1)} x &= \ln x, \\
\ln^{(2)} x &= \ln\ln x, \\
&\vdots \\
\ln^{(m)} x &= \underbrace{\ln\ln\cdots\ln}_{m\ times} x,
\end{aligned}
$$

it is well-known that the following series diverges:

$$\sum_{k\geq l} \frac{1}{\ln^{(0)} k \ln^{(1)} k \ln^{(2)} k \cdots \ln^{(m)} k} = \sum_{k\geq l} \frac{1}{k \ln k \ln^{(2)} k \cdots \ln^{(m)} k}, \tag{1}$$

where l is the smallest integer such that $\ln^{(m)} l > 0$ for a given finite number m.

The standard method for proving that the series in Eq. (1) diverges involves the integral test, although there are other methods in the literature that show the divergence of this series, for example, the Cauchy's condensation test as well as the Abel-Dini theorem (see, e.g., [1] and [2]). However, in this note, we will establish a general theorem that the divergence of the series in Eq. (1) is a direct consequence thereof. We prove the main theorem in the next section.

2 Main results

In this section, we aim to present our main results. We prove the following theorem:

Theorem 1 *Let A_k be a strictly increasing sequence of positive numbers tending to infinity. Then, for each fixed nonnegative integer m, the series $\sum_{k\geq l} \frac{A_{k+1}-A_k}{A_k \ln A_k \ln^{(2)} A_k \cdots \ln^{(m)} A_k}$ diverges, where l is the smallest integer such that $\ln^{(m)} A_l$ is positive.*

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Proof. It can be shown that if k is large enough such that $\ln^{(m+1)} A_k$ is positive, then

$$\begin{aligned}\ln^{(m+1)} A_{k+1} &\leq \ln^{(m+1)} A_k + \frac{A_{k+1} - A_k}{\ln^{(0)} A_k \ln^{(1)} A_k \ln^{(2)} A_k \cdots \ln^{(m)} A_k} \\ &= \ln^{(m+1)} A_k + \frac{A_{k+1} - A_k}{A_k \ln A_k \ln^{(2)} A_k \cdots \ln^{(m)} A_k}. \end{aligned} \tag{2}$$

Indeed, to prove inequality (2), we must first show that if inequality (2) holds for $m-1$, then it holds for m. This can be done by taking the natural logarithm of both sides of inequality (2) for $m-1$, and using the well-known inequality $\ln(1+x) \leq x$ for $x \geq 0$. Therefore,

$$\begin{aligned} &\overbrace{\ln(\ln^{(m)} A_{k+1})}^{\ln^{(m+1)} A_{k+1}} \leq \ln\left(\ln^{(m)} A_k + \frac{A_{k+1} - A_k}{A_k \ln A_k \ln^{(2)} A_k \cdots \ln^{(m-1)} A_k}\right) \\ = \quad & \ln^{(m+1)} A_k + \ln\left(1 + \frac{A_{k+1} - A_k}{A_k \ln A_k \ln^{(2)} A_k \cdots \ln^{(m-1)} A_k \ln^{(m)} A_k}\right) \\ \leq \quad & \ln^{(m+1)} A_k + \frac{A_{k+1} - A_k}{A_k \ln A_k \ln^{(2)} A_k \cdots \ln^{(m-1)} A_k \ln^{(m)} A_k}. \end{aligned}$$

Now, it is enough to show that inequality (2) holds for $m = 0$ (then, as shown above, it holds for $m = 1$, then for $m = 2$, and so on). Since $\ln(1+x) \leq x$ for $x \geq 0$, if we set $x = \frac{A_{k+1}-A_k}{A_k}$, then we get $\ln A_{k+1} \leq \ln A_k + \frac{A_{k+1}-A_k}{A_n}$, that is, inequality (2) for $m = 0$. Thus, inequality (2) is proved.

From inequality (2), we have

$$\frac{A_{k+1} - A_k}{A_k \ln A_k \ln^{(2)} A_k \cdots \ln^{(m)} A_k} \geq \ln \frac{\ln^{(m)} A_{k+1}}{\ln^{(m)} A_k}. \tag{3}$$

Putting $k = l, l+1, \ldots, n$ in inequality (3) and adding, gives

$$\sum_{k=l}^{n} \frac{A_{k+1} - A_k}{A_k \ln A_k \ln^{(2)} A_k \cdots \ln^{(m)} A_k} \geq \sum_{k=l}^{n} \ln \frac{\ln^{(m)} A_{k+1}}{\ln^{(m)} A_k} = \ln \prod_{k=l}^{n} \frac{\ln^{(m)} A_{k+1}}{\ln^{(m)} A_k} = \ln \frac{\ln^{(m)} A_{n+1}}{\ln^{(m)} A_l}.$$

Since $\ln^{(m)} A_{n+1}$ tends to infinity as $n \to \infty$, therefore the series $\sum_{k=l}^{\infty} \frac{A_{k+1}-A_k}{A_k \ln A_k \ln^{(2)} A_k \cdots \ln^{(m)} A_k}$ is divergent. The proof is complete. ∎

Corollary 1 *For $A_k = k$, Theorem 1 implies the divergence of the series in Eq. (1).*

It can be shown that Theorem 1 can produce some divergent series that diverge more slowly than the series in Eq. (1). In this regard, let us consider the following example.

Example 1 *For $A_k = \sqrt{k}$ and $m = 1$, Theorem 1 implies that the series $\sum_{k=2}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\ln\sqrt{k}} = 2\sum_{k=2}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\ln k}$ is divergent, that is, the series $\sum_{k=2}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\ln k}$ diverges. On the other hand, it can be seen that*

$$\frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k}\ln k} = \frac{\sqrt{1 + \frac{1}{k}} - 1}{\ln k} < \frac{\frac{1}{k}}{\ln k} = \frac{1}{k \ln k} \quad \text{for } k > 1.$$

Thus, this gives another proof (by comparison test) of the divergence of the familiar series $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$.

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References

[1] D. D. Bonar and M. J. Khoury Jr., Real infinite series, Mathematical Association of America, Washington, DC, 2006.

[2] K. Knopp, Theory and application of infinite series, Dover, New York, 1990.