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On The Zero Bounds Of Quaternionic Polynomials With Restricted Coefficients*

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Abstract

The main aim of this paper is to study the extensions of the classical Eneström-Kakeya theorem and its various generalizations about the distribution of zeros of polynomials from complex to the quaternionic setting.

1 Introduction

In mathematics, polynomial zeros have a long and illustrious history. The study of the distribution of zeros of polynomials in the geometric function theory is a problem of interest both in mathematics and in the application areas such as physical systems. In addition to having numerous applications, this study has been the inspiration for much more research, both from the theoretical point of view, as well as from the practical point of view. The zeros of a polynomial are continuous functions of its coefficients, in general, it is quite complicated to derive bounds on the norm of zeros of a general algebraic polynomial. Therefore, for attaining better and sharp bounds it is desirable to put restrictions on the coefficients of the polynomial. The subject dates back to around the time when the geometric representation of complex numbers was introduced into mathematics, and the first contributors to the subject were Gauss and Cauchy [3]. A classical result due to Cauchy [3] on the distribution of zeros of a polynomial may be stated as follows:

Theorem 1 *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then all the zeros of p lie in*

$$|z| < 1 + \max_{1\le v\le n-1} \left|\frac{a_v}{a_n}\right|.$$

Although there are other results in the literature about the bounds for polynomial zeros (see, e.g. [19], [20]), the striking property of the bound in Theorem 1 that distinguishes it from other such bounds is its ease of computation. This simplicity, however, comes at the expense of precision. The following elegant result concerning the distribution of zeros of a polynomial when its coefficients are restricted is known in the literature as Eneström-Kakeya theorem (for reference, see [5], [19], [20]).

Theorem 2 *If $p(z) = \sum\limits_{v=0}^{n} a_v z^v$, is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_n \ge a_{n-1} \ge ... \ge a_1 \ge a_0 \ge 0,$$

then all the zeros of $p(z)$ lie in

$$|z| \le 1.$$

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It seems that G. Eneström was the first to get a result of this nature when he was studying a problem in the theory of pension funds. In essence, the above result appeared for the first time in a little circulated paper of Eneström [4]. Later, Eneström made the significant parts of his earlier paper accessible to the international mathematical community and mentioned it in his publications of 1893-95. Independently, in 1912, the result was obtained by S. Kakeya [17] with a purely geometrical approach and in a more general form. The Eneström-Kakeya theorem is particularly important in the study of stability of numerical methods for differential equations and, subsequently it has been extended in various ways, even to complex coefficients with restricted arguments. In the literature, for example see ([1], [9], [13], [14], [15], [16]), there exist various extensions and generalizations of the Eneström-Kakeya theorem. We refer the reader to the comprehensive books of Marden [19] and Milovanović et al. [20] for an exhaustive survey of extensions and refinements of this well-known result. In 1967, Joyal, Labelle and Rahman [16] published a result which might be considered the foundation of the studies which we are currently studying. The Eneström-Kakeya theorem, stated as Theorem 2 above deals with polynomials with non-negative coefficients which form a monotone sequence. In fact, Joyal, Labelle and Rahman generalized Theorem 2 by dropping the condition of non-negativity and maintaining the condition of monotonicity. Namely, they proved the following result.

Theorem 3 *If $p(z) = \sum_{v=0}^{n} a_v z^v$, is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0,$$

then all the zeros of $p(z)$ lie in

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Of course, when $a_0 \geq 0$, Theorem 3 reduces to Theorem 2. Various estimates for the location of zeros in terms of coefficients, with emphasis on the distribution of zeros of the algebraic polynomials with restricted coefficients has been intensively studied since the second half of the nineteenth century, and substantial breakthroughs have been achieved. The Eneström-Kakeya Theorem 1 and its various generalizations as mentioned above are the classic and significant examples of this kind. Provided such a richness of the complex setting, a natural question is to ask what kind of results in the quaternionic setting can be obtained. In this paper we consider this problem and present extensions to the quaternionic setting of some classical results of Eneström-Kakeya type as discussed above.

2 Background

In order to introduce the framework in which we will work, let us introduce some preliminaries on quaternions which will be useful in the sequel. Quaternions are essentially a generalization of complex numbers to four dimensions (one real and three imaginary parts) and were first studied and developed by Sir Rowan William Hamilton in 1843. This number system of quaternions is denoted by $\mathbb{H}$ in honor of Hamilton. This theory of quaternions is by now very well developed in many different directions, and we refer the reader to [10], [11], [18], [12] and [21] for the basic features of quaternions and quaternionic functions. Before we proceed further, we need to introduce some preliminaries on quaternions. The set of quaternions denoted by $\mathbb{H}$ is a noncommutative division ring. It consists of elements of the form $q = \alpha + \beta i + \gamma j + \delta k,\ \alpha, \beta, \gamma, \delta \in \mathbb{R}$, where the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = ijk = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$. Every element $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$ is composed by the real part $\text{Re}(q) = \alpha$ and the imaginary part $\text{Im}(q) = \beta i + \gamma j + \delta k$. The conjugate of q is denoted by $\overline{q}$ and is defined as $\overline{q} = \alpha - \beta i - \gamma j - \delta k$ and the norm of q is $|q| = \sqrt{q\overline{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The inverse of each non zero element q of $\mathbb{H}$ is defined as $q^{-1} = |q|^{-2}\overline{q}$. We define the ball $B(0, r) = \{q \in \mathbb{H}; |q| < r\}$, for $r > 0$.

Very recently, Carney et al. [2] proved the following extension of Theorem 2 for the quaternionic polynomial $p(q)$. More prescisely, they proved the following result.

Theorem 4 *If* $p(q) = \sum_{v=0}^{n} q^v a_v$, *is a polynomial of degree* n *(where* q *is a quaternionic variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $p(q)$ *lie in*

$$|q| \leq 1.$$

They also proved the following result similar to **Theorem D** but instead of polynomials with monotone increasing real coefficients, it considers quaternionic polynomials with monotone increasing real and imaginary parts and thus giving the quaternionic analogue of **Theorem D**.

Theorem 5 *If* $p(q) = \sum_{v=0}^{n} q^v a_v$, *is a polynomial of degree* n *(where* q *is a quaternionic variable) with quaternionic coefficients, where* $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ *for* $v = 0, 1, 2, ..., n$, *satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq ... \geq \beta_1 \geq \beta_0,$$

$$\gamma_n \geq \gamma_{n-1} \geq ... \geq \gamma_1 \geq \gamma_0,$$

$$\delta_n \geq \delta_{n-1} \geq ... \geq \delta_1 \geq \delta_0,$$

then all the zeros of $p(q)$ *lie in*

$$|q| \leq \frac{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)}{|a_n|}.$$

In the meantime, Tripathi [22] besides proving some other results also established the following generalization of Theorem 5.

Theorem 6 *Let* $p(q) = \sum_{v=0}^{n} q^v a_v$ *be a polynomial of degree* n *(where* q *is a quaternionic variable) with quaternionic coefficients, where* $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ *for* $v = 0, 1, 2, ..., n$, *satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_l,$$

$$\beta_n \geq \beta_{n-1} \geq ... \geq \beta_l,$$

$$\gamma_n \geq \gamma_{n-1} \geq ... \geq \gamma_l,$$

$$\delta_n \geq \delta_{n-1} \geq ... \geq \delta_l,$$

for $0 \leq l \leq n$. *Then all the zeros of* $p(q)$ *lie in*

$$|q| \leq \frac{1}{|a_n|}\Big[|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (\alpha_n - \alpha_l) + (\beta_n - \beta_l) + (\gamma_n - \gamma_l) + (\delta_n - \delta_l) + M_l\Big],$$

where

$$M_l = \sum_{v=1}^{l}\Big[|\alpha_v - \alpha_{v-1}| + |\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|\Big].$$

Remark 1 *For* $l = 0$, *Theorem 6 reduces to Theorem 5.*

3 Main Results

In this section, we state our main results and their proofs are given in the next section. We begin with the following result which as a special case gives generalization of **Theorem D**.

Theorem 7 *If* $p(q)=\sum_{v=0}^{n} q^v a_v$, *is a quaternionic polynomial of degree* n *with real coefficients* a_v, $v=0,1,2,...,n$ *and for some* $k_v\geq 1$, $v=0,1,2,...,r$, $0\leq r\leq n-1$, *and we have*

$$k_0a_n\geq k_1a_{n-1}\geq k_2a_{n-2}\geq ...\geq k_{r-1}a_{n-r+1}\geq k_ra_{n-r}\geq a_{n-r-1}\geq ...\geq a_1\geq a_0,$$

then all the zeros of $p(q)$ *lie in*

$$|q|\leq \frac{1}{|a_n|}\left\{k_0(|a_n|+a_n)+2\sum_{v=1}^{r}(k_v-1)|a_{n-v}|-a_0+|a_0|-|a_n|\right\}.$$

If we take $k_v=1$, $v=0,1,2,...,r$ in Theorem 7, we obtain the following result which is an extension of **Theorem D** from the complex to quaternionic setting.

It is important to mention that this corollary is a special case of a result due to Tripathi ([22], Theorem 3.9).

Corollary 8 *If* $p(q)=\sum_{v=0}^{n} q^v a_v$, *is a quaternionic polynomial of degree* n *with real coefficients* a_v, $v=0,1,2,...,n$, *and satisfying*

$$a_n\geq a_{n-1}\geq a_{n-2}\geq ...\geq a_1\geq a_0,$$

then all the zeros of $p(q)$ *lie in*

$$|q|\leq \frac{1}{|a_n|}(a_n-a_0+|a_0|).$$

Setting $a_0>0$ in Corollary 8, we get Theorem 4.

Theorem 9 *If* $p(q)=\sum_{v=0}^{n} q^v a_v$, *is a quaternionic polynomial of degree* n *with quaternionic coefficients* $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k$ *for* $v=0,1,2,...,n$, *and for some* $k_v\geq 1$, $v=0,1,2,...,r$, $0\leq r\leq n-1$, *and we have*

$$k_0\alpha_n\geq k_1\alpha_{n-1}\geq k_2\alpha_{n-2}\geq ...\geq k_{r-1}\alpha_{n-r+1}\geq k_r\alpha_{n-r}\geq \alpha_{n-r-1}\geq ...\geq \alpha_1\geq \alpha_0,$$

then all the zeros of $p(q)$ *lie in*

$$|q|\leq \frac{1}{|a_n|}\left\{k_0(|\alpha_n|+\alpha_n)+2\sum_{v=1}^{r}(k_v-1)|\alpha_{n-v}|-\alpha_0+|\alpha_0|-|\alpha_n|+L\right\},$$

where

$$L=2\sum_{v=0}^{n}(|\beta_v|+|\gamma_v|+|\delta_v|).$$

On setting $\beta_v=\gamma_v=\delta_v=0$ for $v=0,1,2,...,n$ in Theorem 9, we recover Theorem 7. Similarly by taking $k_v=1$, $v=0,1,2,...,r$ in Theorem 9, we obtain the following result.

Corollary 10 *If* $p(q)=\sum_{v=0}^{n} q^v a_v$, *is a quaternionic polynomial of degree* n *where* $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k$ *for* $v=0,1,2,...,n$, *satisfying*

$$\alpha_n\geq \alpha_{n-1}\geq \alpha_{n-2}\geq ...\geq \alpha_1\geq \alpha_0,$$

then all the zeros of $p(q)$ *lie in*

$$|q|\leq \frac{1}{|a_n|}\left\{\alpha_n-\alpha_0+|\alpha_0|+L\right\},$$

where L *is defined in Theorem 9.*

If in Corollary 10, we assume $\alpha_0 > 0$ and use the fact that $\alpha_n \leq |a_n|$, we get the following generalization of Theorem 4 (see also Carney et al. [2], Theorem 7).

Corollary 11 *If* $p(q) = \sum_{v=0}^{n} q^v a_v$, *is a quaternionic polynomial of degree* n *where* $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ *for* $v = 0, 1, 2, ..., n$, *satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0 > 0,$$

then all the zeros of $p(q)$ *lie in*

$$|q| \leq 1 + \frac{2}{\alpha_n} \sum_{v=0}^{n} (|\beta_v| + |\gamma_v| + |\delta_v|).$$

4 Lemma

We need the following lemma due to Gentili and Stoppato [6] for the proofs of the main results.

Lemma 12 *If* $f(q) = \sum_{v=0}^{\infty} q^v a_v$ *and* $g(q) = \sum_{v=0}^{\infty} q^v b_v$ *be two given quaternionic power series with radii of convergence greater than* R. *The regular product of* $f(q)$ *and* $g(q)$ *is defined as*

$$(f \star g)(q) = \sum_{v=0}^{\infty} q^v c_v, \quad \textit{where } c_v = \sum_{l=0}^{v} a_l b_{v-l}.$$

Let $|q_0| < R$. *Then* $(f \star g)(q_0) = 0$ *if and only if either* $f(q_0) = 0$ *or* $f(q_0) \neq 0$ *implies* $g(f(q_0)^{-1} q_0 f(q_0)) = 0$.

5 Proofs of the Main Results

Proof of Theorem 7. Consider the polynomial

$$f(q) = \sum_{v=1}^{n} q^v (a_v - a_{v-1}) + a_0.$$

We have $p(q) \star (1 - q) = f(q) - q^{n+1} a_n$. Therefore by Lemma 12, $p(q) \star (1 - q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ implies $p(q)^{-1} q p(q) - 1 = 0$, that is, $p(q)^{-1} q p(q) = 1$. If $p(q) \neq 0$, then $q = 1$. Therefore, the only zeros of $p(q) \star (1 - q)$ are $q = 1$ and the zeros of $p(q)$.

For $|q| = 1$, we have

$$\begin{aligned}
|f(q)| &= |q^n(a_n - a_{n-1}) + ... + q^{n-r}(a_{n-r} - a_{n-r-1}) + ... + q(a_1 - a_2) + a_0| \\
&= \Big| q^n \big(k_0 a_n - k_1 a_{n-1} - (k_0 - 1) a_n + (k_1 - 1) a_{n-1}\big) \\
&\quad + q^{n-1} \big(k_1 a_{n-1} - k_2 a_{n-2} - (k_1 - 1) a_{n-1} + (k_2 - 1) a_{n-2}\big) \\
&\quad + ... + q^{n-r-1} \big(k_{r-1} a_{n-r+1} - k_r a_{n-r} - (k_{r-1} - 1) a_{n-r+1} + (k_r - 1) a_{n-r}\big) \\
&\quad + q^{n-r} \big(k_r a_{n-r} - a_{n-r-1} - (k_r - 1) a_{n-r}\big) + q^{n-r-1} (a_{n-r-1} - a_{n-r-2}) \\
&\quad + ... + q^2 (a_2 - a_1) + q(a_1 - a_0) + a_0 \Big| \\
&= \Big| - (k_0 - 1) q^n a_n + q^n (k_0 a_n - k_1 a_{n-1}) + (k_1 - 1) q^n a_{n-1} + q^{n-1} (k_1 a_{n-1} - k_2 a_{n-2}) \\
&\quad - q^{n-1} (k_1 - 1) a_{n-1} + (k_2 - 1) q^{n-1} a_{n-2} + ... + q^{n-r+1} (k_{r-1} a_{n-r+1} - k_r a_{n-r}) \\
&\quad - (k_{r-1} - 1) q^{n-r+1} a_{n-r+1} + (k_r - 1) q^{n-r-1} a_{n-r} + q^{n-r} (k_r a_{n-r} - a_{n-r-1})
\end{aligned}$$

$$
\begin{aligned}
& -(k_r-1)q^{n-r}a_{n-r}+q^{n-r-1}(a_{n-r-1}-a_{n-r-2})+\ldots+q^2(a_2-a_1)+q(a_1-a_0)+a_0\Big| \\
\leq\ & (k_0-1)|a_n|+k_0a_n-k_1a_{n-1}+(k_1-1)|a_{n-1}|+k_1a_{n-1}-k_2a_{n-2}+(k_1-1)|a_{n-1}| \\
& +(k_2-1)|a_{n-2}|+\ldots+k_{r-1}a_{n-r+1}-k_ra_{n-r}+(k_{r-1}-1)|a_{n-r+1}|+(k_r-1)|a_{n-r}| \\
& +k_ra_{n-r}-a_{n-r+1}+(k_r-1)|a_{n-r}|+a_{n-r-1}-a_{n-r-2}+\ldots+a_2-a_1+a_1-a_0+|a_0| \\
=\ & k_0(|a_n|+a_n)+2\sum_{v=1}^{r}(k_v-1)|a_{n-v}|-a_0+|a_0|-|a_n|.
\end{aligned}
$$

Since

$$\max_{|q|=1}\left|q^n\star f\left(\frac{1}{q}\right)\right|=\max_{|q|=1}\left|f\left(\frac{1}{q}\right)\right|=\max_{|q|=1}|f(q)|,$$

we see that $q^n\star f\left(\frac{1}{q}\right)$ has the same bound on $|q|=1$ as $f(q)$, that is

$$\left|q^n\star f\left(\frac{1}{q}\right)\right|\leq k_0(|a_n|+a_n)+2\sum_{v=1}^{r}(k_v-1)|a_{n-v}|-a_0+|a_0|-|a_n| \quad \text{for } |q|=1.$$

Applying maximum modulus theorem ([7], Theorem 3.4), it follows that

$$\left|q^n\star f\left(\frac{1}{q}\right)\right|\leq k_0(|a_n|+a_n)+2\sum_{v=1}^{r}(k_v-1)|a_{n-v}|-a_0+|a_0|-|a_n| \quad \text{for } |q|\leq 1.$$

Replacing q by $\frac{1}{q}$, we get for $|q|\geq 1$

$$|f(q)|\leq\left\{k_0(|a_n|+a_n)+2\sum_{v=1}^{r}(k_v-1)|a_{n-v}|-a_0+|a_0|-|a_n|\right\}|q|^n. \tag{1}$$

But

$$|p(q)\star(1-q)|=|f(q)-q^{n+1}a_n|\geq|a_n||q|^{n+1}-|f(q)|.$$

Using (1), we have for $|q|\geq 1$

$$|p(q)\star(1-q)|\geq|a_n||q|^{n+1}-\left\{k_0(|a_n|+a_n)+2\sum_{v=1}^{r}(k_v-1)|a_{n-v}|-a_0+|a_0|-|a_n|\right\}|q|^n.$$

This implies that $|p(q)\star(1-q)|>0$, i.e., $p(q)\star(1-q)\neq 0$ if

$$|q|>\frac{1}{|a_n|}\left\{k_0(|a_n|+a_n)+2\sum_{v=1}^{r}(k_v-1)|a_{n-v}|-a_0+|a_0|-|a_n|\right\}.$$

Since the only zeros of $p(q)\star(1-q)$ are $q=1$ and the zeros of $p(q)$, we see that $p(q)\neq 0$ for

$$|q|>\frac{1}{|a_n|}\left\{k_0(|a_n|+a_n)+2\sum_{v=1}^{r}(k_v-1)|a_{n-v}|-a_0+|a_0|-|a_n|\right\}.$$

Hence all the zeros of $p(q)$ lie in

$$|q|\leq\frac{1}{|a_n|}\left\{k_0(|a_n|+a_n)+2\sum_{v=1}^{r}(k_v-1)|a_{n-v}|-a_0+|a_0|-|a_n|\right\}.$$

This completes the proof of Theorem 7. ■

Proof of Theorem 9. Consider the polynomial

$$f(q)=\sum_{v=1}^{n}q^{v}(a_v-a_{v-1})+a_0.$$

We have $p(q)\star(1-q)=f(q)-q^{n+1}a_n$. So by Lemma 12, $p(q)\star(1-q)=0$ if and only if either $p(q)=0$ or $p(q)\neq 0$ implies $p(q)^{-1}qp(q)-1=0$, that is, $p(q)^{-1}qp(q)=1$. If $p(q)\neq 0$, then $q=1$. Therefore the only zeros of $p(q)\star(1-q)$ are $q=1$ and the zeros of $p(q)$.

For $|q|=1$, we have

$$\begin{aligned}
|f(q)| &\le |a_0|+\sum_{v=1}^{n}|a_v-a_{v-1}|\\
&\le |\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+\sum_{v=1}^{n}|\alpha_v-\alpha_{v-1}|\\
&\quad+\sum_{v=1}^{n}\Big\{|\beta_v-\beta_{v-1}|+|\gamma_v-\gamma_{v-1}|+|\delta_v-\delta_{v-1}|\Big\}\\
&\le |\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+\sum_{v=0}^{n-1}|\alpha_{n-v}-\alpha_{n-v-1}|\\
&\quad+\sum_{v=1}^{n}\Big\{|\beta_v|+|\beta_{v-1}|+|\gamma_v|+|\gamma_{v-1}|+|\delta_v|+|\delta_{v-1}|\Big\}\\
&= |\alpha_0|+\sum_{v=r+1}^{n-1}|\alpha_{n-v}-\alpha_{n-v-1}|+2\sum_{v=0}^{n}\Big\{|\beta_v|+|\gamma_v|+|\delta_v|\Big\}\\
&\quad+\sum_{v=0}^{r}|k_v\alpha_{n-v}-k_{v+1}\alpha_{n-v-1}-(k_v-1)\alpha_{n-v}+(k_{v+1}-1)\alpha_{n-v-1}|,\quad k_{r+1}=1\\
&\le |\alpha_0|+\sum_{v=r+1}^{n-1}|\alpha_{n-v}-\alpha_{n-v-1}|+2\sum_{v=0}^{n}\Big\{|\beta_v|+|\gamma_v|+|\delta_v|\Big\}\\
&\quad+\sum_{v=0}^{r}|k_v\alpha_{n-v}-k_{v+1}\alpha_{n-v-1}|+\sum_{v=0}^{r}|(k_v-1)\alpha_{n-v}|+\sum_{v=0}^{r}|(k_{v+1}-1)\alpha_{n-v-1}|,\quad k_{r+1}=1\\
&= |\alpha_0|+\sum_{v=r+1}^{n-1}(\alpha_{n-v}-\alpha_{n-v-1})+2\sum_{v=0}^{n}\Big\{|\beta_v|+|\gamma_v|+|\delta_v|\Big\}\\
&\quad+\sum_{v=0}^{r}(k_v\alpha_{n-v}-k_{v+1}\alpha_{n-v-1})+(k_0-1)|\alpha_n|+2\sum_{v=1}^{r}(k_v-1)|\alpha_{n-v}|,\quad k_{r+1}=1\\
&= k_0(|\alpha_n|+\alpha_n)+2\sum_{v=1}^{r}(k_v-1)|\alpha_{n-v}|-\alpha_0+|\alpha_0|-|\alpha_n|+L,
\end{aligned}$$

where $L=2\sum_{v=0}^{n}(|\beta_v|+|\gamma_v|+|\delta_v|)$.

Since

$$\max_{|q|=1}\left|q^n\star f\Big(\frac{1}{q}\Big)\right|=\max_{|q|=1}\left|f\Big(\frac{1}{q}\Big)\right|=\max_{|q|=1}|f(q)|,$$

we see that $q^n\star f\Big(\frac{1}{q}\Big)$ has the same bound on $|q|=1$ as $f(q)$, that is

$$\left|q^n\star f\Big(\frac{1}{q}\Big)\right|\le k_0(|\alpha_n|+\alpha_n)+2\sum_{v=1}^{r}(k_v-1)|\alpha_{n-v}|-\alpha_0+|\alpha_0|-|\alpha_n|+L\quad\text{for}\quad |q|=1.$$

After few steps as in the proof of Theorem 7, we conclude that all the zeros of $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\left\{k_0(|\alpha_n| + \alpha_n) + 2\sum_{v=1}^{r}(k_v - 1)|\alpha_{n-v}| - \alpha_0 + |\alpha_0| - |\alpha_n| + L\right\}.$$

This completes the proof of Theorem 9. ■

6 Conclusions

Some new generalizations of the Eneström-Kakeya Theorem 5 for quaternionic polynomials has been established that are benefical in determining the regions containing all the zeros of a polynomial.

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