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Terminal Value Problem For Implicit Katugampola Fractional Differential Equations In b-Metric Spaces*

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Abstract

This paper mainly concerns a class of Caputo-Hadamard implicit fractional differential equations in b-metric spaces. The existence results are derived using the $\alpha - \phi$-Geraghty type contraction and the fixed point theory. An application is also considered to illustrate the novelty of the main result in the last section. The results obtained in this paper extend several contributions in this field.

1 Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, and other applied sciences. Considerable attention has been given to the existence of solutions to initial and boundary value problems for fractional differential equations (see [9, 11, 12]).

The problem of implicit fractional differential equations has received specific attention recently. For instance, see the papers [3, 13] and references therein.

In the process of rapid development of fixed point theory, many new spaces have emerged with their applications and the study of the newly emerged spaces has been an interesting topic among the mathematical research community. A very interesting notion of b-metric space was introduced by Czerwik [6, 7]. The existence of fixed points for the various classes of operators in the setting of b-metric spaces has been investigated extensively. For more details, we refer the reader to [4, 5, 8, 10] and the references therein. Our article is motivated by the paper [3] which deals with the nonlinear implicit fractional differential equations with the Katugampola derivative. More precisely, they proved the existence and uniqueness of solutions for a class of nonlinear implicit fractional differential equations via the Katugampola fractional derivatives with an initial condition. The arguments for their study are based upon the Banach contraction principle, Schauders fixed point theorem and the nonlinear alternative of Leray-Schauder type. For this reason, motivated by the above contributions, in particular by [4, 5], our goal in this paper is to investigate the existence of solutions for the following class of boundary value problems of Caputo-Hadamard implicit fractional differential equations, namely

$$\begin{cases} ({}^{Hc}D_1^r u)(t) = f\big(t, u(t), ({}^{Hc}D_1^r u)(t)\big);\ t \in I := [1, T], \\ u(1) = u_1,\ u'(T) = u_T, \end{cases} \tag{1}$$

where $T > 1$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{Hc}D_1^r$ is the Hadamard-Caputo fractional derivative of order $r \in (1, 2]$ and $u_1, u_T \in \mathbb{R}$.

For the sake of completeness, we recall some basic notions, notations and fundamental results.

2 Preliminaries

By $C(I)$, we denote the Banach space of all continuous functions froms I into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup_{t\in I} |u(t)|.$$

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Let $L^1(I)$ be the Banach space of measurable functions $u : I \to \mathbb{R}$ endowed with norm

$$\|u\|_{L^1} = \int_0^T |u(t)|dt.$$

Definition 1 *The Hadamard fractional integral of order r, of a function $h : [1, \infty) \to \mathbb{R}$ is defined as*

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} h(s)\frac{ds}{s}, \quad r > 0,$$

provided that the integral exists.

Definition 2 *For at least n-times differentiable function $h : [1, \infty) \to \mathbb{R}$, the Caputo-type Hadamard derivative of fractional order r is defined as follows:*

$$D^r h(t) = \frac{1}{\Gamma(n-r)} \int_1^t \left(\log \frac{t}{s}\right)^{n-r-1} \delta^n h(s)\frac{ds}{s},$$

where $\delta = t\left(\frac{d}{dt}\right)$, $\log(.) = \log_e(.)$, and $[r]$ denotes the integer part of the positive number r.

Lemma 1 ([9]) *Let $u \in AC^n_\delta[a, b]$ or $C^r_\delta[a, b]$ and $r > 0$, where*

$$X^n_\delta[a, b] = \Big\{h : [a, b] \to C : \delta^{n-1}h(t) \in X[a, b]\Big\}.$$

Then, one has

$$I^r(D^r)u(t) = u(t) - \sum_{k=0}^{n-1} C_k (\log t)^k,$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n-1$, $(n = [r] + 1)$.

Lemma 2 *Let $h \in C(I)$, and $\alpha \in (1, 2]$. Then the unique solution of the following problem*

$$\begin{cases} ({}^{Hc}D^r_1 u)(t) = h(t);\ t \in I, \\ u(1) = u_1,\ u'(T) = u_T, \end{cases}$$

is given by:

$$u(t) = u_1 + Tu_T \log t + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s}ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{h(s)}{s}ds. \tag{2}$$

Proof. Solving the equation

$$({}^{Hc}D^r_1 u)(t) = h(t),$$

we get

$$u(t) = {}^H I^r_1 h(t) + c_0 + c_1 \log t.$$

Thus

$$u'(t) = {}^H I^{r-1}_1 h(t) + \frac{c_1}{t}.$$

In view of the boundary conditions, it is obvious that

$$c_0 = u_1, \quad \text{and} \quad c_1 = Tu_T - {}^H I^{r-1}_1 h(T).$$

Hence, we obtain (2).

Conversely, if u satisfies the integral equation (2), then

$$\begin{cases} ({}^{Hc}D^r_1 u)(t) = h(t);\ t \in I, \\ u(1) = u_1, \quad u'(T) = u_T. \end{cases}$$

Thus the proof of Lemma 2 is complete. ■

Definition 3 **([1, 2])** *Let M be a nonempty set and $c \geq 1$ be a given real number. A mapping $d : M \times M \to \mathbb{R}_+^*$ is said to be a b-metric if for all $\mu, \nu, \xi \in M$, the following conditions are fulfilled:*

(bM1) $d(\mu, \nu) = 0$ *if and only if* $\mu = \nu$;

(bM2) $d(\mu, \nu) = d(\nu, \mu)$;

(bM3) $d(\mu, \xi) \leq c[d(\mu, \nu) + d(\nu, \xi)]$.

In this case, the pair (M, d) is called a b-metric space (with constant c).

Example 1 **([1, 2])** *Let $d : C(I) \times C(I) \to \mathbb{R}_+^*$ be defined by*

$$d(u, v) = \|(u - v)^2\|_\infty := \sup_{t \in I} \|u(t) - v(t)\|^2 \quad \text{for all } u, v \in C(I).$$

It is clear that d is a b-metric with $c = 2$.

Example 2 **([1, 2])** *Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}_+^*$ be defined by*

$$d(x, y) = |x^2 - y^2| \quad \text{for all } x, y \in X.$$

It is obvious that d is not a metric. However, it is easy to see that d is a b-metric space with $c \geq 2$.

Let Φ be the set of all increasing and continuous function $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ satisfying the property:

$$\phi(c\mu) \leq c\phi(\mu) \leq c\mu, \quad \text{for } c > 1 \text{ and } \phi(0) = 0.$$

We denote by $\mathcal{F}$ the family of all nondecreasing functions $\lambda : \mathbb{R}_+^* \to [0, \frac{1}{c^2})$ for some $c \geq 1$.

Definition 4 **([1, 2])** *Let (M, d) be a b-metric space and let $T : M \to M$ be a self-map. We say that T is a generalized α-ϕ-Geraghty mapping whenever there exist $\alpha : M \times M \to \mathbb{R}_+^*$, and some $L \geq 0$ such that for*

$$D(x, y) = \max\left\{d(x, y), d(x, T(x)), d(y, T(y)), \frac{d(x, T(y)) + d(y, T(x))}{2s}\right\},$$

$$N(x, y) = \min\{d(x, y), d(x, T(x)), d(y, T(y))\},$$

we have

$$\alpha(\mu, \nu)\phi(c^3 d(T(\mu), T(\nu)) \leq \lambda(\phi(D(\mu, \nu))\phi(D(\mu, \nu)) + L\psi(N(\mu, \nu); \tag{3}$$

for all $\mu, \nu \in M$, where $\lambda \in \mathcal{F}$, $\phi, \psi \in \Phi$.

Remark 1 *In the case when $L = 0$ in Definition 4, and the fact that*

$$d(x, y) \leq D(x, y) \quad \text{for all } x, y \in M,$$

the inequality (3) implies

$$\alpha(\mu, \nu)\phi(c^3 d(T(\mu), T(\nu)) \leq \lambda(\phi(d(\mu, \nu))\phi(d(\mu, \nu)). \tag{4}$$

Definition 5 **([1, 2])** *Let M be a nonempty set and $\alpha : M \times M \to \mathbb{R}_+^*$. A mapping T is said to be $\alpha-$admissible if it satisfies, for all $\mu, \nu \in M$,*

$$\alpha(\mu, \nu) \geq 1 \Rightarrow \alpha(T(\mu), T(\nu)) \geq 1.$$

Definition 6 **([1, 2])** *Let (M, d) be a b-metric space and let $\alpha : M \times M \to \mathbb{R}_+^*$ be a function. M is said to be α-regular if for every sequence $\{x_n\}_{n \in \mathbb{N}}$ in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}_{k \in \mathbb{N}}$ of $\{x_n\}_n$ with $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

The following fixed point theorem plays a key role in the proof of our main results.

Theorem 1 ([1, 2]) *Let (M, d) be a complete b-metric space and $T : M \to M$ be a generalized α-ϕ-Geraghty contraction type mapping such that*

(i) T is α-admissible;

(ii) there exists $\mu_0 \in M$ such that $\alpha(\mu_0, T(\mu_0)) \geq 1$;

(iii) either T is continuous or M is $\alpha-$regular.

Then T has a fixed point. Moreover, if

(iv) for all fixed points μ, ν of T, either $\alpha(\mu, \nu) \geq 1$ or $\alpha(\nu, \mu) \geq 1$,

then T has a unique fixed point.

Now, we are ready to state and prove our main results.

3 Main Results

Define $d : C(I) \times C(I) \to \mathbb{R}_+^*$ by

$$d(u, v) = \left\|(u-v)^2\right\|_C = \sup_{t\in I}\left|u(t) - v(t)\right|^2.$$

Then $(C(I), d)$ is a b-metric space with $c = 2$.

Definition 7 *We say that u is a solution of the problem (1) if $u \in C(I)$ and satisfies the equation*

$$u(t) = u_1 + Tu_T \log t + \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{g(s)}{s}ds - \frac{T\log t}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\frac{g(s)}{s}ds,$$

where $g \in C(I)$, and $g(t) = f(t, u(t), g(t))$.

The following hypotheses will be used in the sequel.

(H_1) There exist $\phi \in \Phi$, $p : C(I) \times C(I) \to (0, \infty)$ and $q : I \to (0, 1)$ such that for each $u, v, u_1, v_1 \in C(I)$, and $t \in I$,the following:

$$\left|f(t, u, v) - f(t, u_1, v_1)\right| \leq p(u, u_1)\left|u - u_1\right| + q(t)\left|v - v_1\right|,$$

is fulfilled with

$$\left\|\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{p(u,v)}{1-q*}\frac{ds}{s}\right\|_C^2 + \left\|\frac{T\log t}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\frac{p(u,v)}{1-q*}\frac{ds}{s}\right\|_C^2 \leq \phi(\|(u-v)^2\|_C).$$

(H_2) There exist $\mu_0 \in C(I)$ and a function $\theta : C(I) \times C(I) \to \mathbb{R}$, such that

$$\theta\left(\mu_0(t), u_1 + Tu_T\log t + \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{g(s)}{s}ds - \frac{T\log t}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\frac{g(s)}{s}ds\right) \geq 0,$$

where $g \in C(I)$, with $g(t) = f(t, \mu_0(t), g(t))$.

(H_3) For each $t \in I$, and $u, v \in C(I)$, we have $\theta(u(t), v(t)) \geq 0$, implies that

$$\theta \left(u_1 + Tu_T \log t + \tfrac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds, \right.$$
$$\left. v_1 + Tv_T \log t + \tfrac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{h}{s} ds \right) \geq 0,$$

where $g, h \in C(I)$, with $g(t) = f(t, u(t), g(t))$ and $h(t) = f(t, v(t), h(t))$.

(H_4) If $(u_n)_{n\in N} \subset C(I)$ with $u_n \to u$ and $\theta(u_n, u_{n+1}) \geq$, then $\theta(u_n, u) \geq 1$.

The main result of the paper is the following:

Theorem 2 *Assume that hypotheses (H_1)–(H_4) hold. Then the problem (1) has at least one solution defined on I.*

Proof. We consider the operator $N : C(I) \to C(I)$ defined by

$$(Nu)(t) = u_1 + Tu_T \log t + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds,$$

where $g \in C(I)$, with $g(t) = f(t, u(t), g(t))$.

Using Lemma 2, it is clear that the fixed points of the operator N are solutions of (1). Let $\alpha : C(I) \times C_(I) \to (0, \infty)$ be defined by

$$\begin{cases} \alpha(u, v) = 1 & \text{if } \theta(u(t), v(t)) \geq 0, \ t \in I, \\ \alpha(u, v) = 0 & \text{else.} \end{cases}$$

First, we prove that N is a generalized α-ϕ-Geraghty operator. For any $u, v \in C(I)$ and each $t \in I$, we have

$$\big|(Nu)(t) - (Nv)(t)\big| \leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{|g(s) - h(s)|}{s} ds + \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{|g(s) - h(s)|}{s} ds,$$

where $g, h \in C(I)$, with $g(t) = f(t, u(t), g(t))$, and $h(t) = f(t, v(t), h(t))$.

Taking into account (H_1), one has

$$\begin{aligned} \big|g(t) - h(t)\big| &= \big|f(t, u(t), g(t)) - f(t, v(t), h(t))\big| \leq p(u, v)\big|u(t) - v(t)\big| + q(t)\big|g(t) - h(t)\big| \\ &\leq p(u, v)\big|(u(t) - v(t))^2\big|^{\frac{1}{2}} + q(t)\big|g(t) - h(t)\big|. \end{aligned}$$

Thus,

$$\big|g(t) - h(t)\big| \leq \frac{p(u, v)}{1 - q*} \big\|(u - v)^2\big\|_C^{\frac{1}{2}},$$

where $q* = \sup_{t\in I} |q(t)|$. Next, we have

$$\begin{aligned} \big|(Nu)(t) - (Nv)(t)\big| \ \leq \ & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{p(u, v)}{1 - q*} \|(u - v)^2\|_C^{\frac{1}{2}} \frac{ds}{s} \\ & + \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{p(u, v)}{1 - q*} \|(u - v)^2\|_C^{\frac{1}{2}} \frac{ds}{s}, \end{aligned}$$

we deduce that

$$\begin{aligned} \alpha(u, v)\big|(Nu)(t) - (Nv)(t)\big|^2 \ \leq \ & \big\|(u - v)^2\big\|_C \, \alpha(u, v) \left\| \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{p(u, v)}{1 - q*} \frac{ds}{s} \right\|_C^2 \\ & + \big\|(u - v)^2\big\|_C \alpha(u, v) \left\| \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{p(u, v)}{1 - q*} \frac{ds}{s} \right\|_C^2 \\ \leq \ & \big\|(u - v)^2\big\|_C \phi(\|(u - v)^2\|_C). \end{aligned}$$

This yields

$$\alpha(u,v)\phi(2^3 d(N(u),N(v)) \leq \lambda(\phi(d(u,v))\phi(d(u,v)),$$

where $\lambda \in F$, $\phi \in \Phi$, with $\lambda(t) = \frac{1}{8}t$, and $\phi(t) = t$. Then it is clear that, N is a generalized α-ϕ-Geraghty operator. Let $u, v \in C_{r,\rho}(I)$ such that $\alpha(u,v) \geq 1$, thus, for each $t \in I$, we have $\theta(u(t), v(t)) \geq 0$. This implies from (H_3) that $\theta(Nu(t), Nv(t)) \geq 0$, which gives $\alpha(N(u), N(v)) \geq 1$. Hence, N is a α-admissible.

Now, from (H_2), there exists $\mu_0 \in C_{r,\rho}(I)$ such that

$$\alpha(\mu_0, N(\mu_0)) \geq 1.$$

Finally, from (H_4), If $(\mu_n)_{n\in N} \subset M$ with $\mu_n \to \mu$ and $\alpha(\mu_n, \mu_{n+1}) \geq 1$, then

$$\alpha(\mu_n, \mu) \geq 1.$$

It follows from Theorem 1, that N has a fixed point u which is a solution of problem (1). The proof of Theorem 2 is completed. ■

4 An Example

Let $(C([0,1]), d, 2)$ be the complete b-metric space, with $d : C([0,1]) \times C([0,1]) \to \mathbb{R}_+^*$ is given by

$$d(u,v) = \left\|(u-v)^2\right\|_C.$$

Consider the following fractional differential problem

$$\begin{cases} ({}^{Hc}D_1^r u)(t) = f\big(t, u(t), ({}^{Hc}D_1^r u)(t)\big);\ t \in [0,1], \\ u(1) = 2, \end{cases} \tag{5}$$

where

$$f(t,u(t),v(t)) = \frac{\big(1+\sin(|u(t)|)\big)}{4\big(1+|u(t)|\big)} + \frac{e^{-t}}{2\big(1+|v(t)|\big)};\ t \in [0,1];\ u, v \in C([0,1]).$$

For this reason, we will split our consideration into two cases.

Case 1: If $|u(t)| \leq |v(t)|$, we can see that

$$\begin{aligned}
& \big|f(t,u(t),u_1(t)) - f(t,v(t),v_1(t))\big| \\
= \;& t\left|\frac{1+\sin(|u(t)|)}{4(1+|u(t)|)} - \frac{1+\sin(|v(t)|)}{4(1+|v(t)|)}\right| + \left|\frac{e^{-t}}{2(1+|u_1(t)|)} - \frac{e^{-t}}{2(1+|v_1(t)|)}\right| \\
\leq \;& \frac{1}{4}\big||u(t)|-|v(t)|\big| + \frac{1}{4}\big|\sin(|u(t)|)-\sin(|v(t)|)\big| \\
& + \frac{1}{4}\big||u(t)|\sin(|v(t)|) - |v(t)|\sin(|u(t)|)\big| + \frac{e^{-t}}{2}\big|u_1(t)-v_1(t))\big| \\
\leq \;& \frac{1}{4}\big|u(t)-v(t)\big| + \frac{1}{4}\big|\sin(|u(t)|)-\sin(|v(t)|)\big| \\
& + \frac{1}{4}\big||v(t)|\sin(|v(t)|) - |v(t)|\sin(|u(t)|)\big| + \frac{e^{-t}}{2}\big|u_1(t)-v_1(t))\big| \\
= \;& \frac{1}{4}\big|u(t)-v(t)\big| + \frac{1}{4}\big(1+|v(t)|\big)\big|\sin(|u(t)|)-\sin(|v(t)|)\big| + \frac{e^{-t}}{2}\big|u_1(t)-v_1(t))\big| \\
\leq \;& \frac{1}{4}\big|u(t)-v(t)\big| + \frac{1}{2}(1+|v(t)|) \\
& \times \left|\sin\left(\frac{\big||u(t)|-|v(t)|\big|}{2}\right)\right|\left|\cos\left(\frac{|u(t)|+|v(t)|}{2}\right)\right| + \frac{e^{-t}}{2}\big|u_1(t)-v_1(t)\big| \\
\leq \;& \frac{1}{4}\Big(2+|v(t)|\Big)\big|u(t)-v(t)\big| + \frac{e^{-t}}{2}\big|u_1(t)-v_1(t)\big|.
\end{aligned}$$

Case 2: If $|v(t)| \leq |u(t)|$, we obtain

$$\big|f(t,u(t)) - f(t,v(t))\big| \leq \frac{1}{4}\Big(2 + |u(t)|\Big)\big|u(t) - v(t)\big| + \frac{e^{-t}}{2}\big|u_1(t) - v_1(t)\big|.$$

Hence

$$\big|f(t,u(t)) - f(t,v(t))\big| \leq \frac{1}{4}\min_{t\in I}\Big\{2 + |u(t)|, 2 + |v(t)|\Big\}\big|u(t) - v(t)\big| + \frac{e^{-t}}{2}\big|u_1(t) - v_1(t)\big|.$$

Clearly, the hypothesis (H_1) is satisfied with

$$p(u,v) = \frac{1}{4}\min_{t\in I}\Big\{2 + |u(t)|, 2 + |v(t)|\Big\},$$

and

$$q(t) = \frac{1}{2}e^{-t}.$$

Now, we introduce the functions $\lambda(t) = \frac{1}{8}t$, $\phi(t) = t$, $\alpha : C([0,1]) \times C([0,1]) \to \mathbb{R}_+^*$ such that

$$\begin{cases} \alpha(u,v) = 1; \ if \ \delta(u(t),v(t)) \geq 0, \ t \in I, \\ \alpha(u,v) = 0; \ else, \end{cases}$$

and

$$\delta : C([0,1]) \times C([0,1]) \to \mathbb{R} \quad \text{with } \delta(u,v) = \|u - v\|_C.$$

It is clear to observe that Hypothesis (H_2) is satisfied with $\mu_0(t) = u_0$. Moreover, (H_3) holds from the definition of the function δ. We easily deduce by Theorem 2 that the problem (5) has at least one solution defined on $[0,1]$.

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