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Iterative Schemes For Solving Higher Order Hemivariational Inequalities*

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Abstract

In this paper, we introduce and study a new class of variational inequalities, known as higher order hemivariational inequality. We use the auxiliary principle technique to suggest and analyze some iterative schemes for higher order hemivariational inequalities. We prove the convergence of these iterative methods under some weak conditions. Our method of proof of the convergence criteria is simple as compared with other techniques. Several important special cases are discussed as applications. Results obtained in this paper continue to hold for new and known classes of variational inequalities and related optimization problems. The ideas and techniques of this paper may inspire further research in various branches of pure and applied sciences.

1 Introduction

Variational inequalities theory, which was introduced by Stampachia [27] in 1964, provides us with a simple, general and unified framework to study a wide class of problems arising in pure and applied sciences. Variational inequalities have been extended and generalized in several directions using novel and innovative techniques. Hemivariational inequalities [11, 23, 24] involving the nonlinear Lipschitz continuous functions can be viewed as novel and important generalization of variational inequalities. It have been shown [23, 24] that, if a nonsmooth and nonconvex superpotential of a structure is quasidifferentiable, then these problems can be studied via the hemivariational inequalities. In passing we remark that if the nonlinear Lipschitz continuous function is a differentiable convex function, then hemivariational inequalities coincide with the mildly nonlinear variational inequalities introduced and studied by Noor [8] in 1975.

It is well known that the minimum of the differentiable convex function can be characterized by the variational inequalities. To be more precise, let K be a convex set in the Hilbert space H. Then the minimum $u \in K$ of the differentiable convex function F is equivalent to finding $u \in K$ such that

$$\langle F'(u), v-u\rangle \geq 0, \quad \forall v \in K, \tag{1}$$

which is called the variational inequality (1). Here $F'(u)$ is the Frechet derivative of the convex function F at $u \in K$ in the direction $v-u$. Motivated by this result, Noor and Noor [12] proved that the optimality conditions of the differentiable strongly convex function with modulus $\eta > 0$ is equivalent to finding $u \in K$such that

$$\langle F'(u), v-u\rangle + \eta\|v-u\|^p \geq 0, \quad \forall v \in K, \quad p \geq 1,$$

which is called the higher order variational inequality. For numerical methods and applications of the higher order variational inequalities, see [7, 12, 13, 14, 15, 16, 17, 20] and the references therein
We would like to emphasize that the hemivariational inequalities and higher order variational inequalities are quite different generalizations of the variational inequalities and related optimizations problems. It is natural to study these different problems in a unified framework. This motivated us to introduce and consider some

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classes of higher order hemivariational inequality. These higher order hemivariational inequalities are more general than and include variational inequalities and related optimization problems as special case. Due to structure of these inequalities, it is not possible to extend the usual projection and resolvent techniques for solving heimivariational inequalities. However, these difficulties can be overcome by using the auxiliary principle, which is mainly due to Lions et al. [6] and Glowinski et al. [4]. Noor [8, 10, 11] and Noor et al. [12, 13, 14, 15, 16, 17, 20, 21] have used this technique to develop some iterative schemes for solving various classes of variational inequalities and equilibrium problems. We point out that this technique does not involve any projection and resolvent of the operator and is flexible. In this paper, we show that the auxiliary principle technique can be used to suggest and analyze a class of inertial iterative methods for solving regularized hemivariational inequalities. It iw worth mentioning that the inertial type methods was suggested by Polyak [26] to speed up the convergence of iterative methods. We also prove that the convergence of these new methods requires pseudomonotonicity, which is weaker conation than monotonicity. As special cases, one obtain several known and new results for hemivariational inequalities, variational inequalities and related optimization problems. Results obtained in this paper, represent an improvement and refinement of the known results for nonconvex variational inequalities.

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $j : H \longrightarrow R$ be a locally Lipschitz continuous function.

First of all, we recall the following concepts and results from nonsmooth analysis, see [1].

Definition 1 ([1]) *Let j be locally Lipschitz continuous at a given point $x \in H$ and v be any other vector in H. The Clarke's generalized directional derivative of j at x in the direction v, denoted by $j^0(x; v)$, is defined as*

$$f^0(x; v) = \lim_{t \to 0^+} \sup_{h \to 0} \frac{f(x + h + tv) - f(x + h)}{t}.$$

The generalized gradient of j at x, denoted $\partial j(x)$, is defined to be subdifferential of the function $j^0(x; v)$ at 0. That is

$$\partial j(x) = \{w \in H : \langle w, v \rangle \leq j^0(x; v), \quad \forall v \in H\}.$$

Lemma 1 *Let j be a locally Lipschitz continuous at a given point $x \in H$ with a constant L. Then*

(i) $\partial j(x)$ *is a none-empty compact subset of H and $\|\xi\| \leq L$ for each $\xi \in \partial j(x)$.*

(ii) For every $v \in H$, $\quad j^0(x; v) = \max\{\langle \xi, c \rangle : \xi \in \partial j(x)\}$.

(iii) The function $v \longrightarrow j^0(x; v)$ is finite, positively homogeneous, subadditive, convex and continuous.

(iv) $j^0(x; -v) = (-j)^0(x; v)$.

(v) $j^0(x; v)$ *is upper semicontinuous as a function of $(x; v)$.*

(vi) $\forall x \in H$, *there exists a constant $\alpha > 0$ such that*

$$|j^0(x; v)| \leq \alpha \|v\|, \quad \forall v \in H.$$

If j is convex on K and locally Lipschitz continuous at $x \in K$, then $\partial j(x)$ coincides with the subdifferential $j'(x)$ of j at x in the sense of convex analysis , and $j^0(x; v)$ coincides with the directional derivative $j'(x; v)$ for each $v \in H$, that is, $j^0(x; v) = \langle j'(x), v \rangle$.

In this section, we introduce and consider the higher order general variational inequalities (HOGVI). First of all, we recall the following concepts and basic results, which are mainly due to Noor et al. [12, 13, 14, 15] and Mohsen et al. [7].

Definition 2 *A function F on the convex set K is said to be higher order strongly convex, if there exists a constant η, such that*

$$\begin{aligned} F(u+t(v-u)) &\leq (1-t)F(u)+tF(v) \\ &\quad -\eta\{t^p(1-t)+t(1-t)^p\}\|v-u\|^p, \quad \forall u,v\in K,\ t\in[0,1]. \end{aligned}$$

A function F is said to higher order concave function, if and only if, $-F$ is higher order convex function. If $t=\frac{1}{2}$, then

$$F\left(\frac{u+v}{2}\right)\leq \frac{F(u)+F(v)}{2}-\eta\frac{1}{2^p}\|v-u\|^p, \quad \forall u,v\in K,\ t\in[0,1].$$

The function F is said to be higher order J-convex function.

We nw consider the energy functional $I[v]$ defined as:

$$I[v]=F(v)+\phi(v), \tag{2}$$

where $F(v)$ and $\phi(v)$ are convex functions.

Theorem 1 *Let F be a differentiable higher order convex function with modulus $\zeta>0$ and let $\phi(v)$ be a directionally differentiable convex functions on the convex set K. If $u\in K$ is the minimum of the functional $I[v]$ defined by (2), then*

$$F(v)+\phi(v)-(F(u)+\phi(u))\geq\langle F'(u),v-u\rangle+\phi'(u;v-u)+\eta\|v-u\|^p, \quad \forall v,u\in K. \tag{3}$$

Proof. Let $u\in K$ be a minimum of the functional $I[v]$. Then

$$I[u]\leq I[v], \quad \forall v\in K,$$

which implies that

$$F(u)+\phi(u)\leq F(v)+\phi(v), \quad \forall v\in K. \tag{4}$$

Since K is a convex set, we see that

$$v_t=(1-\lambda)u+\lambda v\in K, \quad \forall u,v\in K,\ \lambda\in[0,1].$$

Taking $v=v_t$ in (4), we have

$$F(u)+\phi(u)\leq F(v_t)+\phi(v_t), \quad \forall v\in K. \tag{5}$$

Since F is differentiable strongly convex function, we see that

$$\begin{aligned} F(u+\lambda(v-u)) &\leq F(u)+\lambda(v-u) \\ &\quad -\eta\{\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|v-u\|^p, \quad \forall u,v\in K,\ p\geq 1, \end{aligned}$$

from which, using (5), we have

$$\begin{aligned} F(v)-F(u) &\geq \lim_{\lambda\to 0}\{\frac{F(u+\lambda(v-u))-F(u)}{\lambda}\}+\eta\|v-u\|^p \\ &= \langle F'(u),v-u\rangle+\eta\|v-u\|^p, \quad p\geq 1. \end{aligned} \tag{6}$$

In a similar way,

$$\phi(v)-\phi(u)\geq\lim_{\lambda\to 0}\{\frac{\phi(u+\lambda(v-u))-\phi(u)}{\lambda}\}=\phi'(u;v-u). \tag{7}$$

From (7) and (6), we have

$$F(v)+\phi(v)-(F(u)+\phi(u))\geq=\langle F'(u),v-u\rangle+\phi'(u;v-u)+\eta\|v-u\|^p,$$

the required result (3). ■

Remark 1 *We would like to point out that, if $u \in K$ satisfies the inequality*

$$\langle F'(u), v-u\rangle + \phi'(u; v-u) + \eta\|\nu - h(\mu)\|^p \geq 0, \quad \forall u, v \in K, \tag{8}$$

then $u \in K$ is the minimum of the functional $I[v] = F(v) + \phi(v)$. The inequality of the type (8) is called the higher order hemivariational inequality.

In many applications, the inequality of the type (8) may not arise as the minimum of the sum of the two differentiable convex functions. These facts motivated us to consider more general hemivariaonal inequality, which contains the inequality (8) as a special case.

For given nonlinear continuous operators $T, A : H \longrightarrow H$, we consider the problem of finding $u \in K$ such that

$$\langle Tu, v-u\rangle + A(u; v-u) + \zeta\|v-u\|^p \geq 0, \quad p \geq 1, \quad \forall v \in K, \tag{9}$$

which is called the *higher order hemivariational inequality.*

We now discuss some new and known classes of variational inequalities and related optimization problems.

(i) If $A(u; v-u) = \phi'(u; v-u)$ denotes directional derivative of the convex function ϕ. u in the direction $v-u$, then problem (9) reduces to finding $u \in K$, such that

$$\langle Tu, v-u\rangle + \phi'(u; v-u) + \zeta\|v-u\|^p \geq 0, \quad p \geq 1, \quad \forall v \in K, \tag{10}$$

which is also called the higher order hemivariational inequality.

(ii) For $A(u; v-u) = J^0(u; v-u)$ and $\zeta = 0$, the problem (9) reduces to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + J^0(u; v-u) \geq 0, \quad \forall v \in K, \tag{11}$$

which is known as hemivariational inequality introduced and studied by Panagiotopoulos [23, 24]. Hemivariational inequalities have important applications in superpotential analysis of elasticity and structural analysis.

(iii) If the operator T is linear, symmetric, positive and $j(.)$ is locally Lipschitz continuous function, then one can easily show that the problem (11) is equivalent to finding the minimum of the functional on the convex set K, where

$$I[v] = \langle Tv, v\rangle + 2j(v),$$

is known as the nonlinear energy functional associated with hemivariational inequalities.

(iv) If $\phi(.)$ is a smooth and convex function, then $\phi'(u; v-u) = \langle \phi'(u), v-u\rangle$, and consequently problem (10) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \langle \phi'(u), v-u\rangle + \zeta\|v-u\|^p, \quad p \geq 1, \quad \forall v \in K, \quad \geq 0$$

which is called the higher order mildly nonlinear variational inequality.

(v) If $\zeta = 0$, and $A(u; v-u) = \langle Au, v-u\rangle$, then the problem (9) reduces to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \langle A(u), v-u\rangle \geq 0, \quad \forall v \in K, \tag{12}$$

which is called the mildly nonlinear variational inequality, introduced and studied by Noor [8] in 1975. It is worth mentioning the problem (12) can be viewed as difference of two monotone operators, see [19].

(vi) If $K^\star = \{u \in H : \langle u, v\rangle \geq 0, \quad \forall v \in K\}$ is a polar convex of the convex K, then problem (12) is equivalent to fining $u \in H$, such that

$$u \in K, \quad Tu + A(u) \in K^\star, \quad \langle Tu + A(u), u\rangle = 0,$$

is called the generalized complementarity problem, see Noor [9]. For the applications, numerical methods and other aspects of complementarity problems, see [2, 5, 9, 10, 20, 22] and the references therein.

(vii) If $K = H$, then problem (12) is equivalent to fining $u \in H$, such that

$$\langle Tu + A(u), v\rangle = 0, \quad \forall v \in H,$$

which is called the weak formulation of the mildly nonlinear boundary value problem. One can easily show that the system of absolute value equations [18, 19] is a special case of the problem (12) and complementarity problems, see [2, 9].

(viii) If $A(.;.) = 0$, then problem (9) reduces to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \zeta\|v-u\|^p \geq 0, \quad \forall v \in K, \quad p \geq 1, \tag{13}$$

is called the higher order variational inequality introduced and studied in [14].

(ix) If $\zeta = 0$, then problem (13) is exactly the variational inequality problem of finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K,$$

which is known as the classical variational inequality. For the recently applications, numerical methods, sensitivity analysis and local uniqueness of solutions of variational inequalities, see [2–27] and the references therein.

This show that the problem (9) is quite and unified one. Due to the structure and nonlinearity involved, one has to consider its own. It is an open problem to develop unified numerical implementation numerical methods for solving the higher order hemivariational inequalities.

3 Main Results

In this section, we use the auxiliary principle technique, which is mainly due to Glowinski, Lions and Tremolieres [4] as developed in [10–21], to suggest and analyze some inertial iterative methods for solving higher order hemivariational inequalities (9).

For a given $u \in K$ satisfying (9), consider the problem of finding $w \in K$ such that

$$\langle \rho T(w + \eta(w-u)) + w - u, v - w\rangle + A(w; v-w) + \zeta\|v-w\|^p \geq 0, \quad \forall v \in K,\ p \geq 1,$$

where $\rho > 0$, $\eta, \zeta \in [0,1]$ are constants.

This inequality is called the auxiliary higher order hemivariational inequality. If $w = u$, then w is a solution of (9). This simple observation enables us to suggest the following iterative method for solving (9).

Algorithm 1 *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} & \langle \rho T(u_{n+1} + \eta(u_n - u_{n+1})) + u_{n+1} - u_n, v - u_{n+1}\rangle \\ \geq \quad & -A(u_{n+1}; v - u_{n+1}) - \zeta\|u_{n+1} - u_n\|^p, \quad \forall v \in K,\ p \geq 1. \end{aligned}$$

Algorithm 1 is called the hybrid proximal point algorithm for solving higher order hemivariational inequalities(9).

Special Cases

We now consider some cases of Algorithm 1.

(I). For $\eta = 0$, Algorithm 1 reduces to:

Algorithm 2 *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T u_{n+1} + u_{n+1} - u_n, v - u_{n+1}\rangle \geq -A(u_{n+1}; v - u_{n+1}) - \zeta\|u_{n+1} - u_n\|^p, \quad \forall v \in K,\ p \geq 1. \tag{14}$$

(II). If $\eta = 1$, then Algorithm 1 reduces to:

Algorithm 3 *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T u_n + u_{n+1} - u_n, v - u_{n+1}\rangle \geq -A(u_{n+1}; v - u_{n+1}) - \zeta\|u_{n+1} - u_n\|^p, \quad \forall v \in K,\ p \geq 1.$$

(III). If $\eta = \frac{1}{2}$, then Algorithm 1 collapses to:

Algorithm 4 *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} & \langle \rho T(\frac{u_{n+1} + u_n}{2}) + \eta(u_n - u_{n+1})) + u_{n+1} - u_n, v - u_{n+1}\rangle \\ \geq \quad & -A(u_{n+1}; v - u_{n+1}) - \zeta\|u_{n+1} - u_n\|^p, \quad \forall v \in K,\ p \geq 1, \end{aligned}$$

which is called the mid-point proximal method for solving the problem (9).

If $A(.;.) = 0$, then Algorithm 1 reduces to:

Algorithm 5 *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_{n+1} + \eta(u_n - u_{n+1})) + u_{n+1} - u_n, v - u_{n+1}\rangle + \zeta\|v - u_{n+1}\|^p \geq 0, \quad \forall v \in K,\ p \geq 1$$

for solving order order variational inequality.

For the convergence analysis of Algorithm 2, we recall the following concepts and results.

Definition 3 *$\forall u, v, z \in H$, an operator $T : H \to H$ is said to be:*

(i) monotone, iff,

$$\langle Tu - Tv, u - v\rangle \geq 0.$$

(ii) pseudomonotone with respect to $A(.;.) + \eta\|v - u\|^p, \}$, if and only if,

$$\begin{aligned} & \langle Tu, v - u\rangle + A(u; v - u) + \zeta\|v - u\|^p \geq 0 \\ \Longrightarrow \quad & \langle Tv, v - u\rangle - A(v; v - u) - \zeta\|v - u\|^p \geq 0. \end{aligned}$$

(iii) partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, z - v\rangle \geq -\alpha\|z - u\|^2.$$

Note that for $z = u$, partially relaxed strongly monotonicity reduces to monotonicity. It is known [17] that cocoercivity implies partially relaxed strongly monotonicity, but the converse is not true. It is known that monotonicity implies pseudomonotonicity; but the converse is not true. Consequently, the class of pseudomonotone operators is bigger than the one of monotone operators.

Definition 4 *The operator $A(.;.)$ is called monotone, iff*

$$A(u; v - u) + A(v; u - v) \leq 0, \quad \forall u, v \in H.$$

Lemma 2 *$\forall u, v \in H$,*

$$2\langle u, v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2. \tag{15}$$

We now consider the convergence criteria of Algorithm 2. The analysis is in the spirit of Noor [10]. We include the proof for the sake of completeness and to convey an idea of the technique involved.

Theorem 2 *Let $u \in K$ be a solution of (9) and let u_{n+1} be the approximate solution obtained from Algorithm 2. If the operator T is pseudomonotone with respect to $A(.;.) + \zeta\|v-u\|^p$, then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{16}$$

Proof. Let $u \in K$ be a solution of (9). Then

$$\langle Tv, v-u\rangle - A(v; v-u) - \zeta\|v-u\|^p \geq 0, \quad \forall v \in K. \tag{17}$$

since T is a pseudomonotone operator with respect to $A(.;.) + \eta\|v-u\|^p$. Now taking $v = u_{n+1}$ in (17), we have

$$\langle Tu, u_{n+1} - u\rangle - A(u_{n+1}; u_{n+1} - u) - \zeta\|u_{n+1} - u\|^p \geq 0. \tag{18}$$

Taking $v = u$ in (14), we get

$$\langle \rho Tu_{n+1} + u_{n+1} - u_n, u - u_{n+1}\rangle + \rho A(u_{n+1}; u - u_{n+1}) \geq -\zeta\|u - u_{n+1}\|^P,$$

which can be written as

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq \langle \rho Tu_{n+1}, u_{n+1} - u\rangle - \rho A(u_{n+1}; u - u_{n+1}) - \rho\zeta\|u - u_{n+1}\|^p \geq 0, \tag{19}$$

where we have used (18).

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (15), we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u - u_{n+1}\|^2 - \|u_{n+1} - u_n\|^2. \tag{20}$$

Combining (19) and (20), we have

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^p.$$

■

Theorem 3 *Let H be a finite dimensional space and all the assumptions of Theorem 2 hold. Then the sequence $\{u_n\}_1^\infty$ given by Algorithm 2 converges to a solution u of (9).*

Proof. Let $u \in K$ be a solution of (9). From (16), it follows that the sequence $\{||u - u_n||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{21}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_1^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_1^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (21), taking the limit $n_j \longrightarrow \infty$ and using (21), we have

$$\langle T\hat{u}, v - \hat{u}\rangle + A(\hat{u}; v - \hat{u}) + \zeta\|v - \hat{u}\|^p \geq 0, \quad \forall v \in K,$$

which implies that $\hat{u}$ solves the higher order hemivariational inequality (9) and

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

■

We again consider the auxiliary principle technique to suggest some hybrid inertial proximal point methods for solving the problem (9). For a given $u \in K$ satisfying (9), consider the problem of finding $w \in K$ such that

$$\begin{aligned}&\langle \rho T(w+\eta(w-u)) + w - u + \alpha(u-u), v-w\rangle\\&+A((w+\xi(w-u)); v-w) + \zeta\|v-w\|^p \geq 0, \quad \forall v \in K,\ p \geq 1,\end{aligned} \tag{22}$$

where $\rho > 0$, $\alpha, \xi, \eta, \zeta \in [0,1]$ are constants.

Clearly, for $w = u$, w is a solution of (9). This fact motivated us to to suggest the following iterative method for solving (9).

Algorithm 6 *For given $u_0, u_1 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho T(u_{n+1} + \eta(u_n - u_{n+1})) + u_{n+1} - u_n + \alpha(u_n - u_{n-1}, v - u_{n+1}\rangle\\&\geq -A((u_{n+1+\xi(u_n-u_{n+1})}; v - u_{n+1}) - \zeta\|u_{n+1} - u_n\|^p, \quad \forall v \in K,\ p \geq 1.\end{aligned}$$

which is known as the inertial iterative method.

Note that for $\alpha = 0$ and $\xi = 0$, Algorithm 6 is exactly the Algorithm 1. Using essentially the technique of Theorem 2 and Noor [10], one can study the convergence analysis of Algorithm 6.

For different and appropriate values of the parameters, ξ, η, ζ, α, the operators T, A and spaces, one can obtain a wide class of inertial type iterative methods for solving the higher order hemivaritaional inequalities and related optimization problems.

Conclusion: In this paper, we have shown that the auxiliary principle technique can be extended for solving higher order hemivariational inequalities with suitable modifications. We note that this technique is independent of the projection and the resolvent of the operator. Moreover, we have studied the convergence analysis of these new methods under weaker conditions. We have only considered the theoretical aspects of the hybrid inertial iterative methods. It is an interesting problem to implement these methods numerically and compare with other iterative schemes.

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