A Short Note On Lyapunov-Type Inequalities For Hilfer Fractional Boundary Value Problems*

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Abstract

This paper deals with fractional boundary value problems involving the Hilfer fractional differential operator of order $1 < \alpha \leq 2$ and type $0 \leq \beta \leq 1$. We derive the corresponding Lyapunov-type inequalities for two prominent classes of Hilfer fractional boundary value problems (HFBVPs) involving separated and anti-periodic boundary conditions. For this purpose, we construct the associated Green's functions and deduce their important properties.

1 Introduction

Fractional differential equations (FDEs) have proved to be valuable tools in modelling many phenomena in various fields of science and engineering. There has been a significant development in the theory and applications of FDEs in the last few decades; see the monographs of Podlubny [11], Kilbas et al. [7], Hilfer [5] and the references therein.

Lyapunov [8] proved an inequality for Hill's equation associated with conjugate boundary conditions, known as Lyapunov's inequality. Cheng [2] developed the discrete analogue of the inequality of Lyapunov for the first time. Many improvements in Lyapunov's inequality have been carried out due to its wide applications in oscillation theory, disconjugacy, eigenvalue problems etc. The study of Lyapunov-type inequalities for fractional boundary value problems has recently begun. In [3], Ferreira replaced the second-order derivative with the αth-order, $\alpha \in (1, 2]$, Riemann–Liouville fractional derivative and obtained a Lyapunov-type inequality for the corresponding conjugate boundary value problem. Following this work, many authors gave valuable contributions to this topic. For an excellent introduction to the evolution of Lyapunov-type inequalities for ordinary differential equations, ordinary difference equations and fractional differential equations, we refer to [1, 9, 10, 12, 13] and the references therein.

Recently, Ntouyas et al. [9, 10] surveyed several generalizations of Lyapunov's inequality for fractional boundary value problems involving a variety of fractional derivative operators and boundary conditions. Motivated by these works, in this paper, we establish Lyapunov-type inequalities for the following fractional boundary value problems involving the Hilfer fractional differential operator of order $1 < \alpha \leq 2$ and type $0 \leq \beta \leq 1$, of the form

$$
\begin{cases}
\left(D_a^{\alpha,\beta} y\right)(t) + q(t)y(t) = 0, \quad a < t < b, \\
l\left(I_a^{(2-\alpha)(1-\beta)} y\right)(a) - m\left(DI_a^{(2-\alpha)(1-\beta)} y\right)(a) = 0, \\
n\left(I_a^{(2-\alpha)(1-\beta)} y\right)(b) + p\left(DI_a^{(2-\alpha)(1-\beta)} y\right)(b) = 0,
\end{cases}
\tag{1}
$$

and

$$
\begin{cases}
\left(D_a^{\alpha,\beta} y\right)(t) + q(t)y(t) = 0, \quad a < t < b, \\
\left(I_a^{(2-\alpha)(1-\beta)} y\right)(a) + \left(I_a^{(2-\alpha)(1-\beta)} y\right)(b) = 0, \\
\left(DI_a^{(2-\alpha)(1-\beta)} y\right)(a) + \left(DI_a^{(2-\alpha)(1-\beta)} y\right)(b) = 0,
\end{cases}
\tag{2}
$$

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where l, m, n, p are constants such that $l^2 + m^2 > 0$ and $n^2 + p^2 > 0$, $q : [a, b] \to \mathbb{R}$ is a continuous function, $D_a^{\alpha,\beta}$ denotes the Hilfer fractional differential operator of order $1 < \alpha \leq 2$ and type $0 \leq \beta \leq 1$ and D denotes the first-order differential operator.

2   Preliminaries

In this section, we present some important definitions and results of fractional calculus which will be useful in the next section.

Definition 1 ([7]) *The Euler gamma function is given by*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

Using the reduction formula

$$\Gamma(z + 1) = z\Gamma(z), \quad \Re(z) > 0,$$

the Euler gamma function can be extended to the half-plane $\Re(z) \leq 0$ except for $z = 0, -1, -2, \ldots$

Definition 2 ([7]) *Let a, $b \in \mathbb{R}$ and $\alpha > 0$. The αth-order Riemann–Liouville fractional integral of a function $y : [a, b] \to \mathbb{R}$ is defined by*

$$(I_a^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} y(s) ds, \quad t > a,$$

provided the right-hand side exists. For $\alpha = 0$, we define I_a^α to be the identity map.

Definition 3 ([6]) *Let a, $b \in \mathbb{R}$, $\alpha > 0$, $0 \leq \beta \leq 1$ and choose $n \in \mathbb{N}$ such that $n - 1 < \alpha \leq n$. The αth-order and β^{th}-type Hilfer fractional derivative of a function $y : [a, b] \to \mathbb{R}$ is defined by*

$$(D_a^{\alpha,\beta} y)(t) = \left(I_a^{\beta(n-\alpha)} D^n I_a^{(n-\alpha)(1-\beta)} y \right)(t), \quad t > a,$$

if the right-hand side exists. Here $D^n = \frac{d^n}{dt^n}$ denotes the classical n^{th}-order differential operator.

Definition 4 ([7]) *We denote by $L(a, b)$ the space of Lebesgue measurable functions $y : [a, b] \to \mathbb{R}$ for which*

$$\|y\|_L = \int_a^b |y(t)| dt < \infty.$$

Definition 5 ([7]) *We denote by $C[a, b]$ the space of continuous functions $y : [a, b] \to \mathbb{R}$ with the norm*

$$\|y\|_C = \max_{t \in [a,b]} |y(t)|.$$

Definition 6 ([7]) *Let $AC[a, b]$ be the space of real-valued functions y which are absolutely continuous on $[a, b]$. We denote by $AC^n[a, b]$ the space of real-valued functions y which have continuous derivatives up to order $n - 1$ on $[a, b]$ such that $y^{(n-1)} \in AC[a, b]$. In particular, $AC^1[a, b] = AC[a, b]$.*

Definition 7 ([4]) *Let a, $b \in \mathbb{R}$, $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $\gamma = \alpha + 2\beta - \alpha\beta$ and $\delta = 2 - \gamma = (2 - \alpha)(1 - \beta) \in [0, 1)$. We denote by $C_\delta[a, b]$ the weighted space of continuous functions*

$$C_\delta[a, b] = \left\{ y : (a, b] \to \mathbb{R} : (t - a)^\delta y(t) \in C[a, b] \right\},$$

with the norm

$$\|y\|_{C_\delta} = \max_{t \in [a,b]} |(t - a)^\delta y(t)|.$$

Lemma 1 ([6]) *For $a \in \mathbb{R}$, $\alpha > 0$, $0 \le \beta \le 1$ and $\mu > -1$, we have*

$$D_a^{\alpha,\beta}(t-a)^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}(t-a)^{\mu-\alpha}, \quad t > a.$$

Lemma 2 *For $a \in \mathbb{R}$, $1 < \alpha \le 2$ and $0 \le \beta \le 1$, we have*

$$I_a^{(2-\alpha)(1-\beta)}(t-a)^{-(2-\alpha)(1-\beta)} = \Gamma(1-(2-\alpha)(1-\beta)), \quad t > a,$$

$$I_a^{(2-\alpha)(1-\beta)}(t-a)^{1-(2-\alpha)(1-\beta)} = (t-a)\Gamma(2-(2-\alpha)(1-\beta)), \quad t > a.$$

Lemma 3 ([14]) *Let $y \in L(a,b)$, $n-1 < \alpha \le n$, $n \in \mathbb{N}$, $\beta \in [0,1]$ and $I_a^{(n-\alpha)(1-\beta)}y \in AC^k[a,b]$. Then, we have*

$$\left(I_a^\alpha D_a^{\alpha,\beta}y\right)(t) = y(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \lim_{t \to a^+} \frac{d^k}{dt^k}\left(I_a^{(n-\alpha)(1-\beta)}y\right)(t),$$

for $t > a$.

Proposition 1 *Let f and g be two nonnegative real-valued functions defined on a set S. Assume that f and g attain their maximum values in S. Then, for each fixed $t \in S$,*

$$|f(t) - g(t)| \le \max\{f(t), g(t)\} \le \max\left\{\max_{t \in S} f(t), \max_{t \in S} g(t)\right\}.$$

3   Main Results

In this section, we establish Lyapunov-type inequalities for the HFBVPs (1) and (2). For this purpose, we consider the corresponding linear HFBVPs:

$$\begin{cases} \left(D_a^{\alpha,\beta}y\right)(t) + h(t) = 0, & a < t < b, \\ l\left(I_a^{(2-\alpha)(1-\beta)}y\right)(a) - m\left(DI_a^{(2-\alpha)(1-\beta)}y\right)(a) = 0, \\ n\left(I_a^{(2-\alpha)(1-\beta)}y\right)(b) + p\left(DI_a^{(2-\alpha)(1-\beta)}y\right)(b) = 0, \end{cases} \tag{3}$$

and

$$\begin{cases} \left(D_a^{\alpha,\beta}y\right)(t) + h(t) = 0, & a < t < b, \\ \left(I_a^{(2-\alpha)(1-\beta)}y\right)(a) + \left(I_a^{(2-\alpha)(1-\beta)}y\right)(b) = 0, \\ \left(DI_a^{(2-\alpha)(1-\beta)}y\right)(a) + \left(DI_a^{(2-\alpha)(1-\beta)}y\right)(b) = 0, \end{cases} \tag{4}$$

where $h : [a,b] \to \mathbb{R}$ is a continuous function.

Theorem 1 *Assume that $mn + lp + ln(b-a) \ne 0$. The unique solution of the HFBVP (3) is given by*

$$y(t) = \int_a^b G(t,s)h(s)ds, \quad a < t < b, \tag{5}$$

where

$$G(t,s) = \begin{cases} G_1(t,s), & a < s \le t < b, \\ G_2(t,s), & a < t \le s < b. \end{cases} \tag{6}$$

Here

$$G_1(t,s) = G_2(t,s) - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)},$$

and

$$G_2(t,s) = (t-a)^{-(2-\alpha)(1-\beta)}(b-s)^{-\beta(2-\alpha)} \frac{[l(t-a) + m(\alpha-1+\beta(2-\alpha))][n(b-s) + p(1-2\beta+\alpha\beta)]}{[mn+lp+ln(b-a)]\Gamma(2-2\beta+\alpha\beta)\Gamma(2-(2-\alpha)(1-\beta))}.$$

Proof. Applying the αth-order Riemann–Liouville fractional integration operator on both sides of (3) and applying the Lemma 3, we get

$$y(t) = C_1 \frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(1-(2-\alpha)(1-\beta))} + C_2 \frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds. \tag{7}$$

Now, using Lemma 2 to (7), we have

$$\left(I_a^{(2-\alpha)(1-\beta)} y \right)(t) = C_1 + C_2(t-a) - \int_a^t \frac{(t-s)^{1-2\beta+\alpha\beta}}{\Gamma(2-2\beta+\alpha\beta)} h(s)ds \tag{8}$$

and

$$\left(DI_a^{(2-\alpha)(1-\beta)} y \right)(t) = C_2 - \int_a^t \frac{(t-s)^{-2\beta+\alpha\beta}}{\Gamma(1-2\beta+\alpha\beta)} h(s)ds. \tag{9}$$

Using the boundary conditions

$$l \left(I_a^{(2-\alpha)(1-\beta)} y \right)(a) - m \left(DI_a^{(2-\alpha)(1-\beta)} y \right)(a) = 0$$

and

$$n \left(I_a^{(2-\alpha)(1-\beta)} y \right)(b) + p \left(DI_a^{(2-\alpha)(1-\beta)} y \right)(b) = 0$$

to (8) and (9), we get

$$C_1 = \frac{m}{[mn+lp+ln(b-a)]\,\Gamma(2-2\beta+\alpha\beta)} \int_a^b (b-s)^{-\beta(2-\alpha)} \left[n(b-s) + p(1-2\beta+\alpha\beta) \right] h(s)ds$$

and

$$C_2 = \frac{l}{[mn+lp+ln(b-a)]\,\Gamma(2-2\beta+\alpha\beta)} \int_a^b (b-s)^{-\beta(2-\alpha)} \left[n(b-s) + p(1-2\beta+\alpha\beta) \right] h(s)ds.$$

Now, substituting the values of C_1 and C_2 in (7) and rearranging the terms, we obtain (5). The proof is complete. ∎

Theorem 2 *The unique solution of the HFBVP* (4) *is given by*

$$y(t) = \int_a^b \bar{G}(t,s)h(s)ds, \quad a < t < b, \tag{10}$$

where

$$\bar{G}(t,s) = \begin{cases} \bar{G}_1(t,s), & a < s \le t < b, \\ \bar{G}_2(t,s), & a < t \le s < b. \end{cases} \tag{11}$$

Here

$$\bar{G}_1(t,s) = \bar{G}_2(t,s) - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)},$$

and

$$\bar{G}_2(t,s) = \frac{(t-a)^{-(2-\alpha)(1-\beta)}(b-s)^{-\beta(2-\alpha)} \left[\frac{(t-a)}{(1-(2-\alpha)(1-\beta))} + \frac{(b-s)}{(1-\beta(2-\alpha))} - \frac{(b-a)}{2} \right]}{2\Gamma(1-2\beta+\alpha\beta)\Gamma(1-(2-\alpha)(1-\beta))}.$$

Proof. Using the boundary conditions

$$\left(I_a^{(2-\alpha)(1-\beta)} y \right)(a) + \left(I_a^{(2-\alpha)(1-\beta)} y \right)(b) = 0$$

and

$$\left(DI_a^{(2-\alpha)(1-\beta)}y\right)(a) + \left(DI_a^{(2-\alpha)(1-\beta)}y\right)(b) = 0$$

to (8) and (9), we get

$$C_1 = -\frac{(b-a)}{4}\int_a^b \frac{(b-s)^{-2\beta+\alpha\beta}}{\Gamma(1-2\beta+\alpha\beta)}h(s)ds + \frac{1}{2}\int_a^b \frac{(b-s)^{1-2\beta+\alpha\beta}}{\Gamma(2-2\beta+\alpha\beta)}h(s)ds$$

and

$$C_2 = \frac{1}{2}\int_a^b \frac{(b-s)^{-2\beta+\alpha\beta}}{\Gamma(1-2\beta+\alpha\beta)}h(s)ds.$$

Now, substituting the values of C_1 and C_2 in (7) and rearranging the terms, we obtain (10). The proof is complete. ■

Theorem 3 *Assume that l, m, n, $p \geq 0$ and $mn + lp + ln(b-a) > 0$. Denote by*

$$H(t,s) = (t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}G(t,s)$$

$$= \begin{cases} (t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}G_1(t,s), & a \leq s \leq t \leq b, \\ (t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}G_2(t,s), & a \leq t \leq s \leq b. \end{cases}$$

$$= \begin{cases} H_1(t,s), & a \leq s \leq t \leq b, \\ H_2(t,s), & a \leq t \leq s \leq b. \end{cases}$$

Then,

$$|H(t,s)| \leq \max\left\{\Omega, \frac{(b-a)}{\Gamma(\alpha)}\right\}, \quad (t,s) \in [a,b] \times [a,b],$$

where

$$\Omega = \frac{[l(b-a)+m(\alpha-1+\beta(2-\alpha))]\,[n(b-a)+p(1-2\beta+\alpha\beta)]}{[mn+lp+ln(b-a)]\,\Gamma(2-2\beta+\alpha\beta)\Gamma(2-(2-\alpha)(1-\beta))}.$$

Proof. We have

$$H_1(t,s) = H_2(t,s) - \frac{(t-s)^{\alpha-1}(t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}}{\Gamma(\alpha)}, \quad a \leq s \leq t \leq b,$$

and

$$H_2(t,s) = \frac{[l(t-a)+m(\alpha-1+\beta(2-\alpha))]\,[n(b-s)+p(1-2\beta+\alpha\beta)]}{[mn+lp+ln(b-a)]\,\Gamma(2-2\beta+\alpha\beta)\Gamma(2-(2-\alpha)(1-\beta))},$$

for $a \leq t \leq s \leq b$. Clearly,

$$H_2(t,s) \geq 0, \quad (t,s) \in [a,b] \times [a,b],$$

and

$$\frac{(t-s)^{\alpha-1}(t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}}{\Gamma(\alpha)} \geq 0, \quad a \leq s \leq t \leq b.$$

Now, we apply Proposition 1. For $a \leq s \leq t \leq b$, we obtain

$$|H_1(t,s)| = \left|H_2(t,s) - \frac{(t-s)^{\alpha-1}(t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}}{\Gamma(\alpha)}\right|$$

$$\leq \max\left\{\max_{a\leq s\leq t\leq b} H_2(t,s), \max_{a\leq s\leq t\leq b}\left[\frac{(t-s)^{\alpha-1}(t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}}{\Gamma(\alpha)}\right]\right\}.$$

Denote by

$$K(t,s) = \frac{(t-s)^{\alpha-1}(t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}}{\Gamma(\alpha)}, \quad a \leq s \leq t \leq b.$$

For a fixed $t \in [a, b]$, consider

$$\frac{\partial}{\partial s} K(t, s) = -\frac{(t-a)^{(2-\alpha)(1-\beta)}}{\Gamma(\alpha)} \left[\beta(2-\alpha)(t-s)^{\alpha-1}(b-s)^{\beta(2-\alpha)-1}(\alpha-1)(t-s)^{\alpha-2}(b-s)^{\beta(2-\alpha)} \right].$$

Since $1 < \alpha \le 2$, $0 \le \beta \le 1$, $a \le s < t \le b$, we have

$$\frac{\partial}{\partial s} K(t, s) < 0, \qquad a \le s \le t \le b,$$

implying that $K(t, s)$ is a decreasing function of s for a fixed $t \in [a, b]$. Then,

$$K(t, s) \le K(t, a), \quad a \le s \le t \le b.$$

Denote by

$$f(t) = K(t, a) = \frac{(t-a)^{\alpha-1}(t-a)^{(2-\alpha)(1-\beta)}(b-a)^{\beta(2-\alpha)}}{\Gamma(\alpha)}, \quad a \le t \le b.$$

Consider

$$f'(t) = \frac{(1 - \beta(2-\alpha))(t-a)^{-\beta(2-\alpha)}(b-a)^{\beta(2-\alpha)}}{\Gamma(\alpha)}, \quad a < t \le b.$$

Since $1 < \alpha \le 2$, $0 \le \beta \le 1$, $a \le t \le b$, we have

$$f'(t) > 0, \quad a < t \le b,$$

implying that f is an increasing function of t for $t \in [a, b]$. Thus,

$$f(t) \le f(b), \quad a \le t \le b.$$

Therefore,

$$K(t, s) \le K(b, a), \quad a \le s \le t \le b.$$

That is,

$$\max_{a \le s \le t \le b} K(t, s) = \frac{(b-a)}{\Gamma(\alpha)}.$$

We also know that

$$\max_{(t,s) \in [a,b] \times [a,b]} H_2(t, s) = \Omega.$$

Therefore,

$$|H_1(t, s)| \le \max \left\{ \Omega, \frac{(b-a)}{\Gamma(\alpha)} \right\}, \quad a \le s \le t \le b.$$

Hence, for all $(t, s) \in [a, b] \times [a, b]$,

$$|H(t, s)| = \max \left\{ \max_{a \le t \le s \le b} |H_2(t, s)|, \max_{a \le s \le t \le b} |H_1(t, s)| \right\} = \max \left\{ \Omega, \frac{(b-a)}{\Gamma(\alpha)} \right\}.$$

The proof is complete. ∎

Theorem 4 *Denote by*

$$\bar{H}(t, s) = (t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}\bar{G}(t, s)$$

$$= \begin{cases} (t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}\bar{G}_1(t, s), & a \le s \le t \le b, \\ (t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}\bar{G}_2(t, s), & a \le t \le s \le b. \end{cases}$$

$$= \begin{cases} \bar{H}_1(t, s), & a \le s \le t \le b, \\ \bar{H}_2(t, s), & a \le t \le s \le b. \end{cases}$$

Then,

$$\left|\bar{H}(t,s)\right| \leq \frac{(b-a)}{A}\left[\frac{1}{(1-(2-\alpha)(1-\beta))} + \frac{1}{(1-\beta(2-\alpha))} - \frac{1}{2}\right] + \frac{(b-a)}{\Gamma(\alpha)},$$

for all $(t,s) \in [a,b] \times [a,b]$, *where*

$$A = 2\Gamma(1-2\beta+\alpha\beta)\Gamma(1-(2-\alpha)(1-\beta)).$$

Proof. We have

$$\bar{H}_1(t,s) = \bar{H}_2(t,s) - \frac{(t-s)^{\alpha-1}(t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}}{\Gamma(\alpha)}, \quad a \leq s \leq t \leq b,$$

and

$$\bar{H}_2(t,s) = \frac{1}{A}\left[\frac{(t-a)}{(1-(2-\alpha)(1-\beta))} + \frac{(b-s)}{(1-\beta(2-\alpha))} - \frac{(b-a)}{2}\right],$$

for $a \leq t \leq s \leq b$. Clearly, $A > 0$ and

$$\frac{(t-s)^{\alpha-1}(t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}}{\Gamma(\alpha)} \geq 0, \quad a \leq s \leq t \leq b.$$

For a fixed $s \in [a,b]$, we have

$$\frac{\partial}{\partial t}\bar{H}_2(t,s) = \frac{1}{A(1-(2-\alpha)(1-\beta))} > 0, \tag{12}$$

implying that $\bar{H}_2(t,s)$ is an increasing function of t. Thus, we have

$$\max_{a \leq t \leq s}\left|\bar{H}_2(t,s)\right| = \max\left\{\left|\bar{H}_2(a,s)\right|, \left|\bar{H}_2(s,s)\right|\right\}.$$

We observe that $\bar{H}_2(s,s)$ is an increasing function of s for $0 \leq \beta \leq \frac{1}{2}$ and it is a decreasing function of s for $\frac{1}{2} \leq \beta \leq 1$, since

$$\frac{d}{ds}\bar{H}_2(s,s) = \frac{1}{A}\left[\frac{1}{(1-(2-\alpha)(1-\beta))} - \frac{1}{(1-\beta(2-\alpha))}\right]$$

is positive for $0 \leq \beta \leq \frac{1}{2}$ and is negative for $\frac{1}{2} \leq \beta \leq 1$. Therefore, we have

$$\max_{a \leq s \leq b}\bar{H}_2(s,s) = \max\left\{\left|\bar{H}_2(a,a)\right|, \left|\bar{H}_2(b,b)\right|\right\}$$

$$= \frac{1}{A}\max\left\{\left|\frac{(b-a)}{(1-\beta(2-\alpha))} - \frac{(b-a)}{2}\right|, \left|\frac{(b-a)}{(1-(2-\alpha)(1-\beta))} - \frac{(b-a)}{2}\right|\right\}$$

$$= \frac{(b-a)}{A}\max\left\{\left|\frac{1}{(1-\beta(2-\alpha))} - \frac{1}{2}\right|, \left|\frac{1}{(1-(2-\alpha)(1-\beta))} - \frac{1}{2}\right|\right\}.$$

Since $1 < \alpha \leq 2$ and $0 \leq \beta \leq 1$, we have

$$1 - \beta(2-\alpha) < 1, \quad 1 - (2-\alpha)(1-\beta) < 1,$$

implying that

$$\frac{1}{(1-\beta(2-\alpha))} > 1, \quad \frac{1}{(1-(2-\alpha)(1-\beta))} > 1. \tag{13}$$

So,

$$\max_{a \leq s \leq b}\bar{H}_2(s,s) = \frac{(b-a)}{A}\max\left\{\frac{1}{(1-\beta(2-\alpha))} - \frac{1}{2}, \frac{1}{(1-(2-\alpha)(1-\beta))} - \frac{1}{2}\right\} = \frac{(b-a)}{A}\left[B - \frac{1}{2}\right], \tag{14}$$

where

$$B = \max \left\{ \frac{1}{(1 - \beta(2 - \alpha))}, \frac{1}{(1 - (2 - \alpha)(1 - \beta))} \right\}.$$

Now, we consider

$$\frac{d}{ds} \bar{H}_2(a, s) = -\frac{1}{A} \left[\frac{1}{(1 - \beta(2 - \alpha))} \right] < 0.$$

So, $\bar{H}_2(a, s)$ is a decreasing function of s. Thus, from (13), we obtain

$$\max_{a \leq s \leq b} \bar{H}_2(a, s) = \max \left\{ \left| \bar{H}_2(a, a) \right|, \left| \bar{H}_2(a, b) \right| \right\}$$

$$= \frac{1}{A} \max \left\{ \left| \frac{(b - a)}{(1 - \beta(2 - \alpha))} - \frac{(b - a)}{2} \right|, \left| -\frac{(b - a)}{2} \right| \right\}$$

$$= \frac{(b - a)}{A} \max \left\{ \frac{1}{(1 - \beta(2 - \alpha))} - \frac{1}{2}, \frac{1}{2} \right\}$$

$$= \frac{(b - a)}{A} \left[\frac{1}{(1 - \beta(2 - \alpha))} - \frac{1}{2} \right].$$

Hence, we have

$$\max_{a \leq t \leq s \leq b} \left| \bar{H}_2(t, s) \right| = \frac{(b - a)}{A} \left[B - \frac{1}{2} \right].$$

Now, for $a \leq s \leq t \leq b$, consider $\bar{H}_2(t, s)$. For a fixed $s \in [a, b]$, it follows from (12) that $\bar{H}_2(t, s)$ is an increasing function of t. Thus, we have

$$\max_{s \leq t \leq b} \left| \bar{H}_2(t, s) \right| = \max \left\{ \left| \bar{H}_2(b, s) \right|, \left| \bar{H}_2(s, s) \right| \right\}.$$

Consider

$$\frac{d}{ds} \bar{H}_2(b, s) = -\frac{1}{A} \left[\frac{1}{(1 - \beta(2 - \alpha))} \right] < 0.$$

So, $\bar{H}_2(b, s)$ is a decreasing function of s. Thus, we have

$$\max_{a \leq s \leq b} \bar{H}_2(b, s) = \max \left\{ \left| \bar{H}_2(b, a) \right|, \left| \bar{H}_2(b, b) \right| \right\}$$

$$= \frac{1}{A} \max \left\{ \left| \frac{(b - a)}{(1 - (2 - \alpha)(1 - \beta))} + \frac{(b - a)}{(1 - \beta(2 - \alpha))} - \frac{(b - a)}{2} \right|, \right.$$

$$\left. \left| \frac{(b - a)}{(1 - (2 - \alpha)(1 - \beta))} - \frac{(b - a)}{2} \right| \right\}$$

$$= \frac{(b - a)}{A} \max \left\{ \left| \frac{1}{(1 - (2 - \alpha)(1 - \beta))} + \frac{1}{(1 - \beta(2 - \alpha))} - \frac{1}{2} \right|, \right.$$

$$\left. \left| \frac{1}{(1 - (2 - \alpha)(1 - \beta))} - \frac{1}{2} \right| \right\}$$

$$= \frac{(b - a)}{A} \max \left\{ \frac{1}{(1 - (2 - \alpha)(1 - \beta))} + \frac{1}{(1 - \beta(2 - \alpha))} - \frac{1}{2}, \right.$$

$$\left. \frac{1}{(1 - (2 - \alpha)(1 - \beta))} - \frac{1}{2} \right\} \quad \text{(By (13))}$$

$$= \frac{(b - a)}{A} \left[\frac{1}{(1 - (2 - \alpha)(1 - \beta))} + \frac{1}{(1 - \beta(2 - \alpha))} - \frac{1}{2} \right].$$

Then, it follows from (14) that

$$\max_{a \leq s \leq t \leq b} |\bar{H}_2(t, s)| = \frac{(b-a)}{A} \left[\frac{1}{(1 - (2-\alpha)(1-\beta))} + \frac{1}{(1 - \beta(2-\alpha))} - \frac{1}{2} \right].$$

Denote by

$$K(t, s) = \frac{(t-s)^{\alpha-1}(t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}}{\Gamma(\alpha)}, \quad a \leq s \leq t \leq b.$$

It follows from the proof of the above theorem, we obtain

$$\max_{a \leq s \leq t \leq b} K(t, s) = \frac{(b-a)}{\Gamma(\alpha)}.$$

Now, for $a \leq s \leq t \leq b$, consider

$$\begin{aligned}
|\bar{H}_1(t, s)| &= |\bar{H}_2(t, s) - K(t, s)| \\
&\leq |\bar{H}_2(t, s)| + |K(t, s)| \\
&\leq \frac{(b-a)}{A} \left[\frac{1}{(1 - (2-\alpha)(1-\beta))} + \frac{1}{(1 - \beta(2-\alpha))} - \frac{1}{2} \right] + \frac{(b-a)}{\Gamma(\alpha)}.
\end{aligned}$$

Therefore, for all $(t, s) \in [a, b] \times [a, b]$,

$$\begin{aligned}
|\bar{H}(t, s)| &= \max \left\{ \max_{a \leq t \leq s \leq b} |\bar{H}_2(t, s)|, \max_{a \leq s \leq t \leq b} |\bar{H}_1(t, s)| \right\} \\
&= \frac{(b-a)}{A} \left[\frac{1}{(1 - (2-\alpha)(1-\beta))} + \frac{1}{(1 - \beta(2-\alpha))} - \frac{1}{2} \right] + \frac{(b-a)}{\Gamma(\alpha)}.
\end{aligned}$$

The proof is complete. ∎

Now, we are able to develop Lyapunov-type inequalities for (1) and (2).

Theorem 5 *Assume that l, m, n, $p \geq 0$ and $mn + lp + ln(b-a) > 0$. If the HFBVP (1) has a nontrivial solution, then*

$$\int_a^b (s-a)^{(\alpha-2)(1-\beta)}(b-s)^{\beta(\alpha-2)}|q(s)|ds \geq \frac{1}{\Lambda}, \tag{15}$$

where

$$\Lambda = \max \left\{ \Omega, \frac{(b-a)}{\Gamma(\alpha)} \right\}.$$

Proof. We know that $\mathfrak{B} = C_{(2-\alpha)(1-\beta)}[a, b]$ is a Banach space with the norm

$$\|y\|_{C_{(2-\alpha)(1-\beta)}} = \max_{t \in [a,b]} \left| (t-a)^{(2-\alpha)(1-\beta)}y(t) \right|.$$

It follows from Theorem 1 that y is a solution of (1) if and only if y is a solution of the Fredholm integral equation

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds.$$

Consider

$$
\begin{aligned}
\|y\|_{C_{(2-\alpha)(1-\beta)}} &= \max_{t\in[a,b]} \left| (t-a)^{(2-\alpha)(1-\beta)} y(t) \right| \\
&= \max_{t\in[a,b]} \left| (t-a)^{(2-\alpha)(1-\beta)} \int_a^b G(t,s)q(s)y(s)ds \right| \\
&\leq \max_{t\in[a,b]} \left[(t-a)^{(2-\alpha)(1-\beta)} \int_a^b |G(t,s)||q(s)||y(s)|ds \right] \\
&= \max_{t\in[a,b]} \left[(t-a)^{(2-\alpha)(1-\beta)} \int_a^b (s-a)^{-(2-\alpha)(1-\beta)}|G(t,s)||q(s)| \right. \\
&\qquad\qquad \left. \left[(s-a)^{(2-\alpha)(1-\beta)}|y(s)| \right] ds \right] \\
&\leq \|y\|_{C_{(2-\alpha)(1-\beta)}} \max_{t\in[a,b]} \left[(t-a)^{(2-\alpha)(1-\beta)} \int_a^b (s-a)^{-(2-\alpha)(1-\beta)}|G(t,s)||q(s)|ds \right] \\
&\leq \|y\|_{C_{(2-\alpha)(1-\beta)}} \int_a^b \max_{t\in[a,b]} \left| (t-a)^{(2-\alpha)(1-\beta)}(b-s)^{\beta(2-\alpha)}G(t,s) \right| \\
&\qquad\qquad (s-a)^{-(2-\alpha)(1-\beta)}(b-s)^{-\beta(2-\alpha)}|q(s)|ds \\
&= \|y\|_{C_{(2-\alpha)(1-\beta)}} \int_a^b \left[\max_{t\in[a,b]} |H(t,s)| \right] (s-a)^{-(2-\alpha)(1-\beta)}(b-s)^{-\beta(2-\alpha)}|q(s)|ds \\
&\leq \Lambda \|y\|_{C_{(2-\alpha)(1-\beta)}} \int_a^b (s-a)^{-(2-\alpha)(1-\beta)}(b-s)^{-\beta(2-\alpha)}|q(s)|ds
\end{aligned}
$$

implying that

$$
1 \leq \Lambda \int_a^b (s-a)^{-(2-\alpha)(1-\beta)}(b-s)^{-\beta(2-\alpha)}|q(s)|ds.
$$

The proof is complete. ∎

Theorem 6 *If the HFBVP* (2) *has a nontrivial solution, then*

$$
\int_a^b (s-a)^{(\alpha-2)(1-\beta)}(b-s)^{\beta(\alpha-2)}|q(s)|ds \geq \frac{1}{\Theta}, \tag{16}
$$

where

$$
\Theta = \frac{(b-a)}{A}\left[\frac{1}{(1-(2-\alpha)(1-\beta))} + \frac{1}{(1-\beta(2-\alpha))} - \frac{1}{2} \right] + \frac{(b-a)}{\Gamma(\alpha)}.
$$

Proof. The proof is similar to the proof of Theorem 5. So, we omit it. ∎

Conclusion

In this article, we derived the corresponding Lyapunov-type inequalities for two prominent classes of HFBVPs (1) and (2) involving separated and anti-periodic boundary conditions, respectively. For this purpose, we constructed the associated Green's functions and deduced their important properties.

References

[1] R. P. Agarwal, M. Bohner and A. Özbekler, Lyapunov Inequalities and Applications, Springer, Cham, 2021.

[2] S. S. Cheng, A discrete analogue of the inequality of Lyapunov, Hokkaido Math. J., 12(1983), 105–112.

[3] R. A. C. Ferreira, A Lyapunov-type inequality for a fractional boundary value problem, Fract. Calc. Appl. Anal., 16(2013), 978–984.

[4] K. M. Furati, M. D. Kassim and N. E. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, Comput. Math. Appl., 64(2012), 1616–1626.

[5] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[6] R. Hilfer, Y. Luchko and Ž. Tomovski, Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives, Fract. Calc. Appl. Anal., 12(2009), 299–318.

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[8] A. Liapounoff, Problème général de la stabilité du mouvement, Ann. Fac. Sci. Toulouse: Math., 9(1907), 203–474.

[9] S. K. Ntouyas, B. Ahmad and T. P. Horikis, Recent developments of Lyapunov-type inequalities for fractional differential equations, In: Differential and Integral Inequalities, Springer, Cham, 2019, 619–686.

[10] S. K. Ntouyas, B. Ahmad and J. Tariboon, A survey on recent results on Lyapunov-type inequalities for fractional differential equations, Fractal Fract., 6(2022), 1–35.

[11] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[12] J. P. Pinasco, Lyapunov-type Inequalities: With Applications to Eigenvalue Problems, SpringerBriefs in Mathematics, Springer, New York, 2013.

[13] A. Tiryaki, Recent developments of Lyapunov-type inequalities, Adv. Dyn. Syst. Appl., 5(2010), 231–248.

[14] Ž. Tomovski, Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator, Nonlinear Anal., 75(2012), 3364–3384.