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Approximations By The Fractional Function Of The Sum Of Two Functions Converging To e*

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Abstract

In this paper, we establish sharp inequality related to the sum of the functions $(1+\frac{1}{x})^x$ and $(1-\frac{1}{x})^{-x}$: for $x > 1$, we have

$$\frac{e(2x^\alpha - 1)}{x^\alpha - 1} < \left(1 + \frac{1}{x}\right)^x + \left(1 - \frac{1}{x}\right)^{-x} < \frac{e(2x^\beta - 1)}{x^\beta - 1},$$

where the constants $\alpha = e$ and $\beta = 2$ are the best possible. Moreover, we present two conjectures related to the inequality.

1 Introduction

In this paper, we present sharp inequalities related to the function $(1 + \frac{1}{x})^x + (1 - \frac{1}{x})^{-x}$. The function $(1 + \frac{1}{x})^x$ is strictly increasing for $x > 1$ and converges to e, also many results are known about the speed of convergence to e, the fractional function approximation [1–4, 8, 9]. On the other hand, the function $(1-\frac{1}{x})^{-x}$ is strictly decreasing for $x > 1$ and converges to e, and the sum of these functions is strictly decreasing for $x > 1$ and converges to $2e$. The study of inequalities involving sums of functions with different monotonicity is interesting and Wilker's inequality [7] is known as an example of such an inequality. Our main theorem is a new result not known until now and we present two conjectures in the end of this paper.

Theorem 1 *For $x > 1$, we have*

$$\frac{e(2x^\alpha - 1)}{x^\alpha - 1} < \left(1 + \frac{1}{x}\right)^x + \left(1 - \frac{1}{x}\right)^{-x} < \frac{e(2x^\beta - 1)}{x^\beta - 1},$$

where the constants $\alpha = e$ and $\beta = 2$ are the best possible.

2 Preliminaries

In this section, we will show some lemmas to prove Theorem 1.

Lemma 1 *For $x \geq 1$, we have*

$$e\left(1 - \frac{7}{14x + 12}\right) < \left(1 + \frac{1}{x}\right)^x < e\left(1 - \frac{6}{12x + 11}\right).$$

That is,

$$\frac{e(5 + 14x)}{2(6 + 7x)} < \left(1 + \frac{1}{x}\right)^x < \frac{e(12x + 5)}{12x + 11}.$$

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The Lemma 1 is proved by Xie and Zhong in [8].

Lemma 2 *For $x > 1$, we have*

$$\left(1-\frac{1}{x}\right)^{-x} < \frac{e(2x-1)}{2(x-1)}.$$

Proof. We set

$$\begin{aligned} f(x) &= \ln\left(1-\frac{1}{x}\right)^{-x} - \ln\frac{e(2x-1)}{2(x-1)} \\ &= -x\ln(x-1) + x\ln x - 1 - \ln(2x-1) + \ln(x-1) + \ln 2\,. \end{aligned}$$

The derivatives of $f(x)$ are

$$f'(x) = \frac{2}{1-2x} - \ln(x-1) + \ln x$$

and

$$f''(x) = -\frac{1}{(x-1)x(2x-1)^2} < 0.$$

Hence, $f'(x)$ is strictly decreasing for $x > 1$. By $\lim_{x\to\infty} f'(x) = 0$, we have $f'(x) > 0$ for $x > 1$ and $f(x)$ is strictly increasing for $x > 1$. From $\lim_{x\to\infty} f(x) = 0$, we obtain $f(x) < 0$ for $x > 1$. ■

Lemma 3 *For $1 < x < \frac{26}{25}$, we have*

$$\left(1-\frac{1}{x}\right)^{-x} < \frac{e(x-1)+1}{\left(1-\sqrt{x-1}\right)(x-1)}.$$

Proof. If $t = \sqrt{x-1}$, then the inequality to prove is

$$\left(\frac{t^2}{t^2+1}\right)^{-t^2-1} < \frac{et^2+1}{t^2(1-t)} \quad \text{for } 0 < t < \frac{1}{5}.$$

We set

$$\begin{aligned} f(t) &= \ln\left(\frac{t^2}{t^2+1}\right)^{-t^2-1} - \ln\frac{et^2+1}{t^2(1-t)} \\ &= -2t^2\ln t + t^2\ln\left(t^2+1\right) + \ln\left(t^2+1\right) - \ln\left(et^2+1\right) + \ln(1-t) \end{aligned}$$

and the derivative of $f(t)$ is

$$\begin{aligned} f'(t) &= t\left(\frac{et(t-2)-1}{(1-t)t\left(et^2+1\right)} + 2\ln\left(t^2+1\right) - 4\ln t\right) \\ &< t\left(\frac{2t(t-2)-1}{(1-t)t\left(3t^2+1\right)} + 2\ln\left(1^2+1\right) - 4\ln t\right) = tg(t). \end{aligned}$$

The derivative of $g(t)$ is

$$g'(t) = \frac{h(t)}{(t-1)^2t^2\left(3t^2+1\right)^2},$$

where

$$\begin{aligned} h(t) &= -36t^7 + 72t^6 - 48t^5 + 6t^4 - 16t^3 + 15t^2 - 6t + 1 \\ &> -36\left(\frac{1}{5}\right)t^6 + 72t^6 - 48\left(\frac{1}{5}\right)t^4 + 6t^4 - 16\left(\frac{1}{5}\right)t^2 + 15t^2 - 6t + 1 \\ &= \frac{324}{5}t^6 - \frac{18}{5}t^4 + \frac{59}{5}t^2 - 6t + 1 > \frac{324}{5}t^6 - \frac{18}{5}\left(\frac{1}{5}\right)^2 t^2 + \frac{59}{5}t^2 - 6t + 1 \\ &> \frac{1457}{125}t^2 - 6t + 1 = \frac{1457}{125}\left(t - \frac{375}{1457}\right)^2 + \frac{332}{1457} > 0. \end{aligned}$$

Since $g(t)$ is strictly increasing for $0 < t < \frac{1}{5}$ and $g(\frac{1}{5}) = -\frac{1075}{112} + 2\ln 2 + 4\ln 5 \cong -1.77417 < 0$, we have $f'(t) < 0$ for $0 < t < \frac{1}{5}$ and $f(t)$ is strictly decreasing for $0 < t < \frac{1}{5}$. Since $\lim_{t\to 0+0} t\ln t = 0$ and $\lim_{t\to 0+0} f(t) = 0$, we have $f(t) < 0$ for $0 < t < \frac{1}{5}$. ■

Lemma 4 *For $x > 1$, we have*

$$\frac{ex - e + 1}{x - 1} < \left(1 - \frac{1}{x}\right)^{-x}.$$

Proof. We set

$$\begin{aligned} f(x) &= \ln \frac{ex - e + 1}{x - 1} - \ln \left(1 - \frac{1}{x}\right)^{-x} \\ &= \ln(ex - e + 1) - \ln(x - 1) + x\ln(x - 1) - x\ln x. \end{aligned}$$

The derivatives of $f(x)$ are

$$f'(x) = \frac{e}{ex - e + 1} + \ln(x - 1) - \ln x$$

and

$$f''(x) = \frac{e(2 - e)x + e^2 - 2e + 1}{x(x - 1)(ex - e + 1)^2}.$$

From $f''(x) > 0$ for $1 < x < \frac{1-2e+e^2}{e(e-2)} \cong 1.51217$ and $f''(x) < 0$ for $x > \frac{1-2e+e^2}{e(e-2)}$, $f'(x)$ is strictly increasing for $1 < x < \frac{1-2e+e^2}{e(e-2)}$ and strictly decreasing for $x > \frac{1-2e+e^2}{e(e-2)}$. By $\lim_{x\to 1+0} f'(x) = -\infty$ and $\lim_{x\to\infty} f'(x) = 0$, there exists a unique real number x_0 such that $f'(x) < 0$ for $1 < x < x_0$ and $f'(x) > 0$ for $x > x_0$. Hence, $f(x)$ is strictly decreasing for $1 < x < x_0$ and strictly increasing for $x > x_0$. By $\lim_{x\to 1+0} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$, we obtain $f(x) < 0$ for $x > 1$. ■

Lemma 5 *For $x > 1$, we have*

$$\frac{e(12x - 5)}{12x - 11} < \left(1 - \frac{1}{x}\right)^{-x}.$$

Proof. We set

$$\begin{aligned} f(x) &= \ln \frac{e(12x - 5)}{12x - 11} - \ln \left(1 - \frac{1}{x}\right)^{-x} \\ &= 1 + \ln(12x - 5) - \ln(12x - 11) + x\ln(x - 1) - x\ln x. \end{aligned}$$

The derivatives of $f(x)$ are

$$f'(x) = \frac{144x^2 - 264x + 127}{(x - 1)(12x - 11)(12x - 5)} + \ln(x - 1) - \ln x$$

and

$$f''(x) = \frac{-4320x^2 + 7296x - 3025}{(x - 1)^2 x(12x - 11)^2(12x - 5)^2}.$$

Since $g(x) = -4320x^2 + 7296x - 3025$ is convex upwards and takes the maximum value at $x = \frac{38}{45}$, so we have $g(x) < -4320 \cdot 1^2 + 7296 \cdot 1 - 3025 = -49 < 0$. Hence, we have $f''(x) < 0$ and $f'(x)$ is strictly decreasing for $x > 1$. By $\lim_{x\to\infty} f'(x) = 0$, we obtain $f'(x) > 0$ for $x > 1$ and $f(x)$ is strictly increasing for $x > 1$. From $\lim_{x\to\infty} f(x) = 0$, we obtain $f(x) < 0$ for $x > 1$. ■

Lemma 6 *For $1 < x < 2$, we have*

$$\frac{14ex^2 + (14 - 9e)x - 5e + 12}{(14 - 7e)x + 7e + 12} < x^e.$$

Proof. We note that

$$14ex^2 + (14-9e)x - 5e + 12 > 14e \cdot 1^2 + (14-9e)2 - 5e + 12 = 40 - 9e > 0$$

and

$$(14-7e)x + 7e + 12 > (14-7e)2 + 7e + 12 = 40 - 7e > 0.$$

Here, we set

$$\begin{aligned} f(x) &= \ln \frac{14ex^2 + (14-9e)x - 5e + 12}{(14-7e)x + 7e + 12} - \ln x^e \\ &= \ln(14ex^2 + (14-9e)x - 5e + 12) - \ln((14-7e)x + 7e + 12) - e \ln x \end{aligned}$$

and the derivative of $f(x)$ is

$$f'(x) = \frac{e(x-1)g(x)}{x\left((14-7e)x + 7e + 12\right)\left(14ex^2 + (14-9e)x - 5e + 12\right)},$$

where

$$g(x) = 98(e-2)(e-1)x^2 - 21(e-2)(8+3e)x - 35e^2 + 24e + 144.$$

From we have

$$g(1) = 676 - 312e \cong -172.104 < 0,$$

$$g(2) = 1600 - 1236e + 231e^2 \cong -52.9244 < 0$$

and $g(x)$ is convex downward, hence we have $g(x) < 0$ for $1 < x < 2$. $f(x)$ is strictly decreasing for $1 < x < 2$. From $\lim_{x \to 1} f(x) = 0$, we can get $f(x) < 0$ for $1 < x < 2$. ■

Lemma 7 *For $x \geq 2$, we have*

$$\frac{168x^2 - 10x + 17}{149} < x^{\frac{5}{2}}.$$

Proof. We set

$$f(x) = 168x^2 - 10x + 17 - 149x^{\frac{5}{2}}$$

and the derivative of $f(x)$ is

$$f'(x) = \frac{1}{2}\left(-745x^{\frac{3}{2}} + 672x - 20\right) < 0$$

for $x \geq 2$. Since $f(x)$ is strictly decreasing for $x > 2$ and $f(2) = 669 - 596\sqrt{2} \cong -173.871 < 0$, we have $f(x) < 0$ for $x \geq 2$. ■

3 Proof of Theorem 1

Proof of Theorem 1. We consider the equation

$$\frac{e(2x^c - 1)}{x^c - 1} = \left(1 + \frac{1}{x}\right)^x + \left(1 - \frac{1}{x}\right)^{-x},$$

then

$$c = \frac{\ln\left(\left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x} - e\right) - \ln\left(\left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x} - 2e\right)}{\ln x} = F(x).$$

Here, we will show that $2 < F(x) < e$ for $x > 1$, $\lim_{x\to 1+0} F(x) = e$ and $\lim_{x\to\infty} F(x) = 2$. We set

$$G(x,y) = \frac{\ln(y-e) - \ln(y-2e)}{\ln x} \quad \text{for } x > 1 \text{ and } y > 2e.$$

Then the derivative of $G(x,y)$ for y is

$$\frac{\partial G(x,y)}{\partial y} = -\frac{e}{(y-2e)(y-e)\ln x} < 0.$$

Therefore, $G(x,y)$ is strictly decreasing for $y > 2e$. First we will prove $F(x) > 2$ for $x > 1$. From Lemmas 1 and 2, we have

$$\left(1+\frac{1}{x}\right)^{x} + \left(1-\frac{1}{x}\right)^{-x} < \frac{e(12x+5)}{12x+11} + \frac{e(2x-1)}{2(x-1)}$$

and

$$\begin{aligned}
F(x) &= G\left(x, \left(1+\frac{1}{x}\right)^{x} + \left(1-\frac{1}{x}\right)^{-x}\right) > G\left(x, \frac{e(12x+5)}{12x+11} + \frac{e(2x-1)}{2(x-1)}\right) \\
&= \frac{\ln\left(\frac{e(12x+5)}{12x+11} + \frac{e(2x-1)}{2(x-1)} - e\right) - \ln\left(\frac{e(12x+5)}{12x+11} + \frac{e(2x-1)}{2(x-1)} - 2e\right)}{\ln x} \\
&= \frac{\ln\frac{e\left(24x^2-2x+1\right)}{2(x-1)(12x+11)} - \ln\frac{23e}{2(x-1)(12x+11)}}{\ln x} = \frac{\ln\left(24x^2-2x+1\right) - \ln 23}{\ln x} \\
&= \frac{\ln\left(x^2 + \frac{(x-1)^2}{23}\right)}{\ln x} > 2
\end{aligned}$$

for $x > 1$. Hence, we obtain $F(x) > 2$ for $x > 1$. Next we will prove $F(x) < e$ for $x > 1$. By Lemmas 1, 4 and 6, we have

$$\frac{e(5+14x)}{2(6+7x)} + \frac{ex-e+1}{x-1} < \left(1+\frac{1}{x}\right)^{x} + \left(1-\frac{1}{x}\right)^{-x}$$

for $x > 1$ and

$$\begin{aligned}
F(x) &= G\left(x, \left(1+\frac{1}{x}\right)^{x} + \left(1-\frac{1}{x}\right)^{-x}\right) < G\left(x, \frac{e(5+14x)}{2(6+7x)} + \frac{ex-e+1}{x-1}\right) \\
&= \frac{\ln\left(\frac{e(5+14x)}{2(6+7x)} + \frac{ex-e+1}{x-1} - e\right) - \ln\left(\frac{e(5+14x)}{2(6+7x)} + \frac{ex-e+1}{x-1} - 2e\right)}{\ln x} \\
&= \frac{\ln\frac{14ex^2-9ex+14x-5e+12}{2(x-1)(7x+6)} - \ln\frac{-7ex+14x+7e+12}{2(x-1)(7x+6)}}{\ln x} \\
&= \frac{\ln\frac{14ex^2+(14-9e)x-5e+12}{(14-7e)x+7e+12}}{\ln x} < \frac{\ln x^e}{\ln x} = e
\end{aligned}$$

for $1 < x < 2$. Moreover, by Lemmas 1, 5 and 7, we have

$$\frac{e(5+14x)}{2(6+7x)} + \frac{e(12x-5)}{12x-11} < \left(1+\frac{1}{x}\right)^{x} + \left(1-\frac{1}{x}\right)^{-x}$$

for $x > 1$ and

$$\begin{aligned} F(x) =&G\left(x, \left(1+\frac{1}{x}\right)^{x}+\left(1-\frac{1}{x}\right)^{-x}\right) < G\left(x, \frac{e(5+14x)}{2(6+7x)}+\frac{e(12x-5)}{12x-11}\right)\\ =&\frac{\ln\left(\frac{e(5+14x)}{2(6+7x)}+\frac{e(12x-5)}{12x-11}-e\right)-\ln\left(\frac{e(5+14x)}{2(6+7x)}+\frac{e(12x-5)}{12x-11}-2e\right)}{\ln x}\\ =&\frac{\ln\frac{e\left(168x^2-10x+17\right)}{2(7x+6)(12x-11)}-\ln\frac{149e}{2(7x+6)(12x-11)}}{\ln x}\\ =&\frac{\ln\frac{168x^2-10x+17}{149}}{\ln x} < \frac{\ln x^{\frac{5}{2}}}{\ln x}=\frac{5}{2} \end{aligned}$$

for $x \geq 2$. Hence, we obtain $F(x) < e$ for $x > 1$. Next we will prove $\lim_{x\to\infty} F(x) = 2$ and $\lim_{x\to 1+0} F(x) = e$. From L'Hopital's theorem [6], we have

$$\begin{aligned} \lim_{x\to\infty} F(x) \leq \lim_{x\to\infty}\frac{\ln\frac{168x^2-10x+17}{149}}{\ln x} &= \lim_{x\to\infty}\frac{\ln\left(168x^2-10x+17\right)-\ln 149}{\ln x}\\ &= \lim_{x\to\infty}\frac{336x^2-10x}{168x^2-10x+17}=2\,. \end{aligned}$$

Thus, we obtain $\lim_{x\to\infty} F(x) = 2$. By Lemmas 1 and 3, we have

$$\left(1+\frac{1}{x}\right)^{x}+\left(1-\frac{1}{x}\right)^{-x} < \frac{e(12x+5)}{12x+11}+\frac{e(x-1)+1}{\left(1-\sqrt{x-1}\right)(x-1)}$$

and

$$\begin{aligned} F(x) &= G\left(x, \left(1+\frac{1}{x}\right)^{x}+\left(1-\frac{1}{x}\right)^{-x}\right) > G\left(x, \frac{e(12x+5)}{12x+11}+\frac{e(x-1)+1}{\left(1-\sqrt{x-1}\right)(x-1)}\right)\\ &= \frac{\ln\left(\frac{e(12x+5)}{12x+11}+\frac{e(x-1)+1}{\left(1-\sqrt{x-1}\right)(x-1)}-e\right)-\ln\left(\frac{e(12x+5)}{12x+11}+\frac{e(x-1)+1}{\left(1-\sqrt{x-1}\right)(x-1)}-2e\right)}{\ln x} \end{aligned}$$

for $1 < x < \frac{26}{25}$. If $t = \sqrt{x-1}$, then we can get

$$\begin{aligned} F(x) &> \frac{\ln\left(\frac{e\left(12t^2+17\right)}{12t^2+23}+\frac{et^2+1}{t^2-t^3}-e\right)-\ln\left(\frac{e\left(12t^2+17\right)}{12t^2+23}+\frac{et^2+1}{t^2-t^3}-2e\right)}{\ln(t^2+1)}\\ &= \frac{\ln\left(\frac{12et^4+6et^3+17et^2+12t^2+23}{(1-t)t^2(12t^2+23)}\right)-\ln\left(\frac{12et^5+29et^3-6et^2+12t^2+23}{(1-t)t^2(12t^2+23)}\right)}{\ln\left(t^2+1\right)}\\ &= \frac{\ln\left(12et^4+6et^3+17et^2+12t^2+23\right)-\ln\left(12et^5+29et^3-6et^2+12t^2+23\right)}{\ln\left(t^2+1\right)} \end{aligned}$$

for $0 < t < \frac{1}{5}$. From L'Hopital's theorem [6], we obtain

$$\begin{aligned}
&\lim_{x\to 1+0} F(x)\\
&\geq \lim_{t\to 0+0} \frac{\ln\left(12et^4+6et^3+17et^2+12t^2+23\right)-\ln\left(12et^5+29et^3-6et^2+12t^2+23\right)}{\ln\left(t^2+1\right)}\\
&= \lim_{t\to 0+0} \frac{\frac{48et^3+18et^2+34et+24t}{12et^4+6et^3+17et^2+12t^2+23}-\frac{60et^4+87et^2-12et+24t}{12et^5+29et^3-6et^2+12t^2+23}}{\frac{2t}{t^2+1}}\\
&= \lim_{t\to 0+0} \frac{e\left(t^2+1\right)H(t)}{2\left(12et^4+6et^3+17et^2+12t^2+23\right)\left(12et^5+29et^3-6et^2+12t^2+23\right)}\\
&= \frac{e\cdot 1\cdot H(0)}{2\cdot 23\cdot 23} = e,
\end{aligned}$$

where

$$\begin{aligned}
H(t) &= -144et^7-144et^6-264et^5-432t^5-144et^4+288t^4-529et^3\\
&\quad -1656t^3+1104t^2-1587t+1058.
\end{aligned}$$

Therefore, we obtain $\lim_{x\to 1+0} F(x) = e$ and the proof of Theorem 1 is complete. ■

4 Conjectures

We present the following some conjectures related to the function $(1+\frac{1}{x})^x + (1-\frac{1}{x})^{-x}$.

Conjecture 1 *For $x > 1$, we have*

$$\frac{\alpha}{x^2-1} < \left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x} - \frac{e(2x^2-1)}{x^2-1} < \frac{\beta}{x^2-1},$$

where the constants $\alpha = 2-e \cong -0.718282$ and $\beta = -\frac{e}{12} \cong -0.226523$ are the best possible.

Conjecture 2 *For $x > 1$, we have*

$$\frac{\alpha}{x^2-1} < \left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x} - \frac{e(2x^e-1)}{x^e-1} < \frac{\beta}{x^2-1},$$

where the constants $\alpha = 0$ and $\beta = \frac{11e}{12} \cong 2.49176$ are the best possible.

Conjecture 3 *For $2 < p < e$, there exists a unique number x_p with $x_p > 1$ such that*

$$\left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x} < \frac{e(2x^p-1)}{x^p-1} \quad \textit{for } 1 < x < x_p,$$

and

$$\left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x} > \frac{e(2x^p-1)}{x^p-1} \quad \textit{for } x > x_p.$$

The conjecture 3 can be proved to be true if $F(x)$ in the proof shows strictly decreasing for $x > 1$. Although not applicable to Conjecture 3, Malešević and Mihailović [5] is known work on the monotonicity of such functions.

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