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Optimal Control Of Pollution Rate In A Spatiotemporal Bioeconomic Model*

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Abstract

The management of plankton production constitutes a major challenge for the development of aquaculture. To attend this objective, *chlorophyll-a*, a pigment present in all photosynthetic organisms, is generally and historically used as an estimator of the biomass of planktonic organisms. In this work, we use the data of *chlorophyll-a* and we choose two controls strategies to minimize the pollution mortality rate. By using **Sea-Das** software we obtain raster maps. These maps show the distribution of *Chlorophyll-a* in Moroccan maritime areas throughout the month of May 2019 and 2020. We notice that we choose these two maps precisely because we noticed that, during the international lockdown period (caused by Covid-19 pandemic), a significant number of marine resources have come to light. The aim purpose of this article is to propose and analyze mathematically a bioeconomic model of plankton organism taking into account the negative effect of pollution. We seek to control the mortality pollution rate and to clarify the impact of the pollution in the reproduction of marine populations.

1 Introduction

Marine research has indicated that the products released into the seas and oceans results marine pollution [1, 2, 3, 4, 5, 6]. This pollution is mainly related to human activity and arrives in the marine environment through the vector of river routes, winds, low latitude air where it is directly discharged into the sea. Waste injure and hinder the mobility of different marine populations, transport invasion species or even cause asphyxiation of the seabed. In general, it threatens aquatic ecosystems. The pollution of the seas and oceans has a profound impact on all aquatic life. So, once in the ocean, the waste has multiple impacts on aquatic life and also on humans.

According to a recent study conducted by experts, in 2050 there will be more plastic than fish in the seas and oceans. Actually, more than 817 marine and coastal species have been identified as victims of marine litter pollution due to ingestion, strangulation or entanglement. This number of species threatened by marine litter continues to increase.

For this reason, in this work we search to show mathematically the negative influence of pollution on the evolution biomass of different marine species and to control the mortality pollution rates. We apply our study for planktonic organisms. These microscopic plants constitute the basis of the marine food chain. According

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to a study by an international team of scientists, different types of pollution affect plankton. For example, pollution by petroleum products, by creating a film on the surface of the water, prevents some of the light from entering the marine environment, which can decrease the photosynthetic activities of phytoplankton, and by preventing exchanges gas (oxygen, carbon dioxide) between the atmosphere and seawater, will disrupt the metabolism of organisms, plants or animals.

This article is situated in this general context. In this work we present and study a bioeconomic model for phytoplankton and zooplankton organisms in continues time and to make the study close to reality we consider the spatiotemporal discrete bioeconomic model associated. The two planktons are in predation and they are supposed to be exploited by fishing fleet.

In the literature, authors have already study biological and bioeconomic models of marine species in the continues time but they have not included the negative effect of pollution in there studies.

We can cite for examples [7, 8, 9]. In these works, authors have studied bioeconomic models of harvesting concerning marine populations fishery for the optimal management of renewable resources.

Other interesting examples are [10, 11]. In paper [10], author has discussed bioeconomic analysis and different management strategies in fisheries and in paper [11] the authors have studied a combined harvesting model associated to one predator and two prey marine populations. Further important examples in this context are [12, 13].

In this context, we can also refer to [14, 15, 16, 17]. In paper [14], authors have proposed a bioeconomic equilibrium system for many fishermen exploiting three competing species. In paper [15], authors have tacked in consideration the fact that the price of fish populations depends on the quantity harvested and they have modeled a bioeconomic model concerning competition-predation species.

In the same research field, it is worth mentioning the works [18, 19, 20, 21, 22, 23, 24, 25, 27, 28]. In these works authors have proposed and analysed a different bioeconomic models. Also, they have discussed the influence of biological and economic parameters on the fishing efforts, catches and profits. Concerning optimal control harvesting bioeconomic models we can cite for example [29, 30, 31, 32, 33, 34, 35]. Let us note that most of these models deals with single species or two species communities.

In the discrete-time prey-predator model proposed, the growth's evolution of planktonic creatures according to a spatial diffusion is described in a uniformly sized area of global interest. Based on the works [36, 37], we define the area of subdomains of global interest by cells containing the two planktonic organisms. The aim of this part is to consider an optimal control problem concerning the pollution rate of a prey-predator bioeconomic model in Morocco's fishing areas.

In this work, we focus on the management of plankton production, which constitutes a major challenge for the development of aquaculture. To attend this objective, *chlorophyll-a*, a pigment present in all photosynthetic organisms, is generally and historically used as an estimator of the biomass of planktonic organisms. In oceanography, *chlorophyll-a* gives as an idea about the quantity of phytoplankton present in the ocean. In other terms, a higher concentration of *chlorophyll-a* means a higher biomass of phytoplankton. Thanks to satellites the concentration of *chlorophyll-a* in the surface layers of the ocean can be evaluated from space.

To achieve our objective, we use the data of *chlorophyll-a* and we choose two controls strategies. The main purpose of these strategies is to minimize the level of pollution rates and maximize the biomasses of phytoplankton and zooplankton creatures under their exploitation.

The paper is structured as follows. In Section 2, we propose and analyse the bioeconomic model of the two planktons under the exploitation in the fishing zone. In section 3, we describe the discrete bioeconomic model without and with control. In Section 4, we give the objective functional and we analyse the optimal control. In section 5, we present numerical simulations and we discuss the results in both cases without and with control. Finally, we close this paper with a conclusion.

2 Bioeconomic Model Construction and Analysis

In this section, we assume the existence of two marine planktonic species, namely phytoplankton and zooplankton, denoted by x and y, respectively. We denoted that the abundance of plankton has a very strong relationship with the concentration of *chlorophyll-a*. We suppose that these two planktonic populations follow logistic growth with an intrinsic growth rate. It is noted by r_1 for phytoplanktonic organisms and r_2 for zooplankton organisms. The carrying capacity of each population is noted by K_1 and K_2, respectively. Phytoplankton populations are preys for zooplankton. Therefore, the parameter α_{12} represents the predation rate coefficient of phytoplankton and the parameter α_{21} represents the conversion rates of phytoplankton into zooplankton. The natural mortality of phytoplankton is designated by d_1 and that of zooplankton by d_2. It should be mentioned that pollution negatively influences the reproduction of plankton, which leads to their degradation and mortality. Let δ_1 and δ_2 the mortality rate from pollution associated to phytoplankton and zooplankton, respectively. In order to make our model closer to reality, we add as a constraint the exploitation of these two plankton by fishing fleets. Therefore, we introduce this fishing activity by considering the harvesting functions $H_1 = q_1E_1x$ linked to phytoplankton and $H_2 = q_2E_2y$ linked to zooplankton. Here q_1 and q_2 are the catchability coefficients of phytoplankton and zooplankton, respectively. E_1 and E_2 are fishing efforts linked to the exploitation of phytoplankton and zooplankton.

Mathematically, the bioeconomic model that represents the evolution of phytoplankton with these considerations and hypothesis is given by the following system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = r_1x\left(1 - x/K_1\right) - \left(d_1 + \delta_1\right)x - \alpha_{12}xy - q_1E_1x, \\ \dfrac{dy}{dt} = r_2y\left(1 - y/K_2\right) - \left(d_2 + \delta_2\right)y + \alpha_{21}xy - q_2E_2y, \end{cases} \tag{1}$$

with positive initial conditions

The explanations of parameters are presented on this table

Parameters	Explanations
x	Biomass density of phytoplankton
y	Biomass density of zooplankton
r_1	Intrinsic growth rate of phytoplankton
r_2	Intrinsic growth rate of zooplankton
K_1	Carrying capacity of phytoplankton
K_2	Carrying capacity of zooplankton
α_{12}	Predation rate coefficients of phytoplankton
α_{21}	Conversion rates of phytoplankton into zooplankton
d_1	Natural mortality by degradation rate of phytoplankton
d_2	Natural mortality by degradation rate of zooplankton
δ_1	Mortality rate from pollution of phytoplankton
δ_2	Mortality rate from pollution of zooplankton
q_1	Catchability coefficients of phytoplankton
q_2	Catchability coefficients of zooplankton
E_1	Fishing efforts linked to the exploitation of phytoplankton
E_2	Fishing efforts linked to the exploitation of zooplankton

2.1 Positivity and Boundedness of the Solutions

To prove the positivity and the uniform boundedness of the system we use the following theorem.

Theorem 1 *All solutions $(x(t), y(t))$ of the system (1) with positive initial conditions are positive for all $t \geq 0$, and the region of attraction for all solutions initiating in the interior of the positive octant is the ensemble*

$$E = \left\{(x, y) \in \mathbb{R}_+^2 : \alpha_{21}x + \alpha_{12}y \leqslant \frac{Y}{\beta}\right\}$$

where β is a positive constant and $Y = \left[\alpha_{21}K_1(r_1+\beta)^2/4r_1\right] + \left[\alpha_{12}K_2(r_2+\beta)^2/4r_2\right]$.

Proof. The differential equation system (1) can be given by the form

$$\begin{cases} \dfrac{dx}{dt} = x\left[r_1\left(1 - x/K_1\right) - \left(d_1 + \delta_1\right) - \alpha_{12}y - q_1E_1\right] = xf_1\left(x, y\right), \\ \dfrac{dy}{dt} = y\left[r_2\left(1 - y/K_2\right) - \left(d_2 + \delta_2\right) + \alpha_{21}x - q_2E_2\right] = yf_2\left(x, y\right), \end{cases} \quad (2)$$

with the positive initial conditions.

The system (1) is defined on the set $\left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\right\}$. The equations of the system (2) together with the positive initial conditions gives

$$\begin{cases} x(t) = x(0)\exp\left(\int_0^t f_1\left(x(s), y(s)\right)\right) ds > 0, \\ y(t) = y(0)\exp\left(\int_0^t f_2(x(s), y(s))\right) ds > 0, \end{cases}$$

proving the positivity of the solutions.

To prove the boundedness of the solutions we pose $X(t) = \alpha_{21}x\left(t\right) + \alpha_{12}y\left(t\right)$. Then, by introducing a positive constant β we obtain

$$\begin{aligned} \frac{dX}{dt} + \beta X &= \alpha_{21}r_1x\left(1 - x/K_1\right) + \alpha_{12}r_2y\left(1 - y/K_2\right) - \alpha_{21}q_1E_1x - \alpha_{12}q_2E_2y \\ &\quad -\alpha_{21}\left(d_1 + \delta_1\right) - \alpha_{12}\left(d_2 + \delta_2\right) + \beta\alpha_{21}x\left(t\right) + \beta\alpha_{12}y\left(t\right) \\ &\leq \alpha_{21}K_1(r_1 + \beta)^2/4r_1 + \alpha_{12}K_2(r_2 + \beta)^2/4r_2 = Y. \end{aligned}$$

Using the theory of inequality [20], we obtain

$$X < \frac{Y}{\beta} - \left(\frac{Y}{\beta} - Y\left(0\right)\right)\exp\left(-\beta t\right).$$

Therefore, when t tends to infinity we have $0 < X \leq \dfrac{Y}{\beta}$, proving the theorem. ■

2.2 Mathematical Analysis

The bioeconomic model (1) has four solutions. The trivial equilibrium point $P_0 = (0, 0)$. The axial equilibrium points

$$P_1 = \left(\frac{K_1\left(r_1 - d_1 - \delta_1 - q_1E_1\right)}{r_1}, 0\right) \quad \text{and} \quad P_2 = \left(0, \frac{K_2\left(r_2 - d_2 - \delta_2 - q_2E_2\right)}{r_2}\right).$$

The interior equilibrium point $P^* = (x^*, y^*)$, where

$$x^* = \frac{\alpha_{12}K_1K_2\left(r_2 - d_2 - \delta_2 - E_2q_2\right) + K_1r_2\left(r_1 - d_1 - \delta_1 - E_1q_1\right)}{r_1r_2 + \alpha_{12}\alpha_{21}K_1K_2}$$

and

$$y^* = \frac{\alpha_{21}K_1K_2\left(r_1 - d_1 - \delta_1 - E_1q_1\right) + K_2r_1\left(r_2 - \delta_2 - d_2 - E_2q_2\right)}{r_1r_2 + \alpha_{12}\alpha_{21}K_1K_2}.$$

The local stability of equilibrium points is presented in the following theorem.

Theorem 2 *The local stability of the equilibrium points P_0, P_1, P_2 and P^* of the bioeconomic model (1) is given by*

i) P_0 is locally asymptotically stable if $r_1 < d_1 + \delta_1 + q_1E_1$ and $r_2 < d_2 + \delta_2 + q_2E_2$;

ii) P_1 is locally asymptotically stable if $r_1 > d_1 + \delta_1 + q_1E_1$;

iii) P_2 *is locally asymptotically stable if* $r_2 > d_2 + \delta_2 + q_2E_2$;

iv) P^* *is locally asymptotically stable if* $r_1 > d_1 + \delta_1 + q_1E_1$ *and* $r_2 > d_2 + \delta_2 + q_2E_2$.

Proof. The Jacobian matrix of system (1) is

$$\begin{bmatrix} r_1(1-2x/K_1) - d_1 - \delta_1 - \alpha_{12}y - q_1E_1 & -\alpha_{12}x \\ \alpha_{21}y & r_2(1-2y/K_2) - d_2 - \delta_2 + \alpha_{21}x - q_2E_2 \end{bmatrix}.$$

i) Evaluating the Jacobian matrix at P_0 gives

$$\begin{bmatrix} r_1 - d_1 - \delta_1 - q_1E_1 & 0 \\ 0 & r_2 - d_2 - \delta_2 - q_2E_2 \end{bmatrix}.$$

The eigenvalues are $r_1 - d_1 - \delta_1 - q_1E_1$ and $r_2 - d_2 - \delta_2 - q_2E_2$. Clearly for this point to be locally asymptotically stable we should have $r_1 < d_1 + \delta_1 + q_1E_1$ and $r_2 < d_2 + \delta_2 + q_2E_2$.

ii) Evaluating the Jacobian matrix at P_1 and replacing

$$r_1(1-2x/K_1) - d_1 - \delta_1 - \alpha_{12}y - q_1E_1 \text{ and } r_2(1-2y/K_2) - d_2 - \delta_2 + \alpha_{21}x - q_2E_2$$

by $-r_1x/K_1$ and $-r_2y/K_2$, respectively, gives

$$\begin{bmatrix} d_1 + \delta_1 + q_1E_1 - r_1 & \alpha_{12}K_1(d_1 + \delta_1 + q_1E_1 - r_1)/r_1 \\ 0 & 0 \end{bmatrix}.$$

this point is locally asymptotically stable if $r_1 > d_1 + \delta_1 + q_1E_1$.

iii) Evaluating the Jacobian matrix at P_2 gives

$$\begin{bmatrix} 0 & 0 \\ \dfrac{\alpha_{21}K_2(r_2 - d_2 - \delta_2 - q_2E_2)}{r_2} & d_2 + \delta_2 + q_2E_2 - r_2 \end{bmatrix}.$$

This point is locally asymptotically stable if $r_2 > d_2 + \delta_2 + q_2E_2$.

iv) Evaluating the Jacobian matrix at P^* gives

$$\begin{bmatrix} -r_1x^*/K_1 & -\alpha_{12}x^* \\ \alpha_{21}y^* & -r_2y^*/K_2 \end{bmatrix}.$$

The trace of the Jacobian matrix is $trace(J) = -r_1x^*/K_1 - r_2y^*/K_2$ and its determinant is

$$\det(J) = (r_1r_2 + \alpha_{12}\alpha_{21}K_1K_2)\,x^*y^*/K_1K_2.$$

Now, if $r_1 > d_1 + \delta_1 + q_1E_1$ and $r_2 > d_2 + \delta_2 + q_2E_2$ then P^* is locally asymptotically stable. Figure 1 illustrates the local stability of equilibrium point P^*.

■

3 Discrete-Time Model Description

In this section we search to make our study closer to reality. We suggest the discrete-time (XY) model (3) associated to the bioeconomic one (1) that show the spatial-temporal evolution of planktonic organisms in

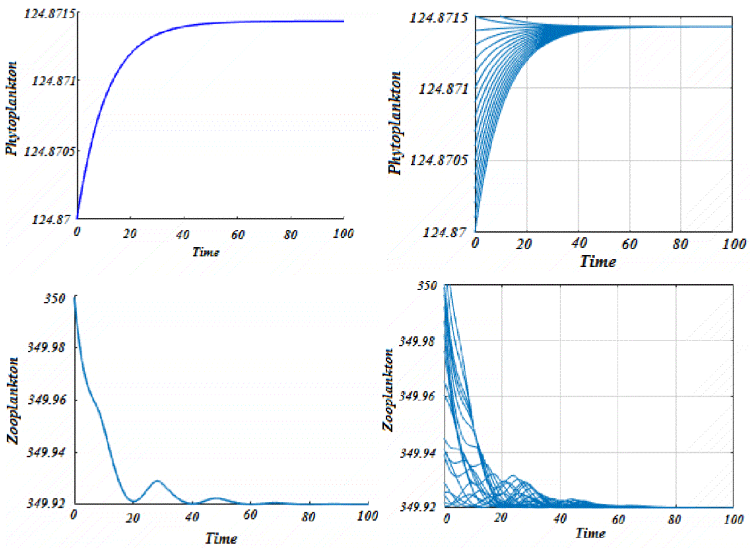


Figure 1: Behaviors and phase portrait of system (1) at P^*. We take as initial value the point $(124.87, 350)$ and the following values: $r_1 = 0.37$, $r_2 = 0.5$, $K_1 = 225$, $K_2 = 700$, $\alpha_{12} = 0.21$, $\alpha_{21} = 0.19$, $d_1 = 0.001$, $d_2 = 0.0015$, $\delta_1 = 0.2$, $\delta_2 = 0.3$, $q_1 = 0.13$, $q_2 = 0.18$, $E_1 = 30$, $E_2 = 50$.

a global area of interest $\mathfrak{G}$ with a uniform size. The global area of interest $\mathfrak{G}$ is divided to R^2 region and it is expressed as the union of compartment $\mathfrak{G} = \bigcup_{i,j=1}^{M} S_{ij}$, with S_{ij} a subdomain or region.

$$\begin{cases} x_{k+1}^{S_{ij}} = r_1^{S_{ij}} x_k^{S_{ij}} \left(1 - \dfrac{x_k^{S_{ij}}}{K_1^{S_{ij}}}\right) - \left(d_1^{S_{ij}} + \delta_1^{S_{ij}}\right) x_k^{S_{ij}} - \alpha_{12}^{S_{ij}} x_k^{S_{ij}} y_k^{S_{ij}} - q_1^{S_{ij}} x_k^{S_{ij}} E_1^{S_{ij}}, \\ y_{k+1}^{S_{ij}} = r_2^{S_{ij}} y_k^{S_{ij}} \left(1 - \dfrac{y_k^{S_{ij}}}{K_2^{S_{ij}}}\right) - \left(d_2^{S_{ij}} + \delta_2^{S_{ij}}\right) y_k^{S_{ij}} + \alpha_{21}^{S_{ij}} x_k^{S_{ij}} y_k^{S_{ij}} - q_2^{S_{ij}} y_k^{S_{ij}} E_2^{S_{ij}}, \end{cases} \tag{3}$$

with the initial conditions $x_0^{S_{ij}}$ and $y_0^{S_{ij}}$ the region S_{ij}. All coefficients of system (3) are non-negative.

We keep the same descriptions of variables and parameters of system (1) to describe system (3), in the last one we classify the planktonic organisms into two ranges in the region S_{ij}. We suppose that there exist interactions from time unit k to time $k+1$. We add that the unit of time k can correspond to days, months or years, it depends on the frequency of data collection and statistics. In this paper, the collected data concerning the concentration of *chlorophyll-a* and the biomasses of plankton organism are given in monthly frequency.

3.1 Description of the Bioeconomic Model with Controls

In this section, we search to describe the spatial-temporal evolution of the proposed bioeconomic model (3) with controls. The two controls $u_k^{S_{ij}}$ and $v_k^{S_{ij}}$ are introduced in the predefined model

$$\begin{cases} x_{k+1}^{S_{ij}} = r_1^{S_{ij}} x_k^{S_{ij}} \left(1 - \dfrac{x_k^{S_{ij}}}{K_1^{S_{ij}}}\right) - \left(d_1^{S_{ij}} + u_k^{S_{ij}} \delta_1^{S_{ij}}\right) x_k^{S_{ij}} - \alpha_{12}^{S_{ij}} x_k^{S_{ij}} y_k^{S_{ij}} - E_1^{S_{ij}} q_1^{S_{ij}} x_k^{S_{ij}}, \\ y_{k+1}^{S_{ij}} = r_2^{S_{ij}} y_k^{S_{ij}} \left(1 - \dfrac{y_k^{S_{ij}}}{K_2^{S_{ij}}}\right) - \left(d_2^{S_{ij}} + v_k^{S_{ij}} \delta_2^{S_{ij}}\right) y_k^{S_{ij}} + \alpha_{21}^{S_{ij}} x_k^{S_{ij}} y_k^{S_{ij}} - E_2^{S_{ij}} q_2^{S_{ij}} y_k^{S_{ij}}. \end{cases} \tag{4}$$

4 Optimal-Control Problem

In this section, we choose the following objective functional

$$J(u^{S_{ij}}, v^{S_{ij}}) = \Gamma_1 x_N^{S_{ij}} + \Gamma_2 y_N^{S_{ij}} + \sum_{k=1}^{N-1} \left(\Gamma_1 x_k^{S_{ij}} + \Gamma_2 y_k^{S_{ij}} - \frac{W_1}{2} \left(u_k^{S_{ij}}\right)^2 - \frac{W_2}{2} \left(v_k^{S_{ij}}\right)^2 \right) \tag{5}$$

subject to system (4). We consider Γ_1 and Γ_2 positive constants to maintain a well-balanced level in the size of $x_k^{S_{ij}}$ and $y_k^{S_{ij}}$, respectively. The positive weight parameters associated to the controls $u_k^{S_{ij}}$ and $v_k^{S_{ij}}$ are given, in the objective functional, by W_1 and W_2.

The main objective of this section is to maximize the biomass of planktons $x_k^{S_{ij}}$ and fish populations $y_k^{S_{ij}}$, or, to maximize the objective functional (5). To achieve this objective, we must determine the optimal controls $\left(u_k^{S_{ij}*}\right)$ and $\left(v_k^{S_{ij}*}\right)$ such that:

$$J\left(u^{S_{ij}*}, v^{S_{ij}*}\right) = \max\left\{ J_{ij}\left(u^{S_{ij}}, v^{S_{ij}}\right), u^{S_{ij}} \in \mathcal{U}, \quad v^{S_{ij}} \in \mathcal{V} \right\} \tag{6}$$

where $\mathcal{U}$ and $\mathcal{V}$ are the ensembles of admissible controls described by

$$\mathcal{U} = \{(u) | u^{min} \leq u_k \leq u^{max}, k \in \{0, \ldots, N-1\}\}$$

and

$$\mathcal{V} = \{(v) | v^{min} \leq v_k \leq v^{max}, k \in \{0. \ldots, N-1\}\},$$

where (u^{min}, u^{max}) and (v^{min}, v^{max}) are confined in $(]0,1[)^2$.

The existence of the predefined optimal control is given in the following theorem.

Theorem 3 (Sufficient conditions) *The optimal control problem (6) with the state equations of system (4) admits the controls $\left(u_k^{S_{ij}*}\right)$ and $\left(v_k^{S_{ij}*}\right)$ such that*

$$J\left(u^{S_{ij}*}, v^{S_{ij}*}\right) = \left\{ \max J_{ij}\left(u^{S_{ij}}, v^{S_{ij}}\right) / u^{S_{ij}} \in \mathcal{U}, u^{S_{ij}} \in \mathcal{V} \right\}.$$

Proof. See Dabbs, K [38, Theorem 1]. ■

The finite dimensional structure of this system ensure the existence of an optimal control. The discrete version of the Pontryagin's maximum principle [39] is used to characterize the necessary conditions that an optimal control and corresponding states must assure. The adjoint variables are used to connect the difference equations to our minimization problem. As in optimal control of ordinary differential equations, we can obtain the necessary conditions from the Hamiltonian $\mathcal{H}$. In the discrete case, at each time $k < N$, the Hamiltonian is formed from the terms in the objective functional (at time k) and the adjoint variables (at time $k+1$) multiplying the corresponding right-hand side of the difference equations. Thus

$$\begin{aligned}
\mathcal{H}(\Psi) \quad = \quad & \left(\zeta_{1,N}^{S_{ij}} x_k^{S_{ij}} + \zeta_{2,N}^{S_{ij}} y_k^{S_{ij}} - \frac{A_1}{2}\left(u_k^{S_{ij}}\right)^2 - \frac{A_2}{2}\left(v_k^{S_{ij}}\right)^2 \right) \\
& + \zeta_{1,i+1}^{S_{ij}} \left[r_1^{S_{ij}} x_k^{S_{ij}} \left(1 - \frac{x_k^{S_{ij}}}{K_1^{S_{ij}}}\right) - \left(d_1^{S_{ij}} + u_k^{S_{ij}} \delta_1^{S_{ij}}\right) x_k^{S_{ij}} - \alpha_{12}^{S_{ij}} x_k^{S_{ij}} y_k^{S_{ij}} - E_1^{S_{ij}} q_1^{S_{ij}} x_k^{S_{ij}} \right] \\
& + \zeta_{2,i+1}^{S_{ij}} \left[r_2^{S_{ij}} y_k^{S_{ij}} \left(1 - \frac{y_k^{S_{ij}}}{K_2^{S_{ij}}}\right) - \left(d_2^{S_{ij}} + v_k^{S_{ij}} \delta_2^{S_{ij}}\right) y_k^{S_{ij}} + \alpha_{21}^{S_{ij}} x_k^{S_{ij}} y_k^{S_{ij}} - E_2^{S_{ij}} q_2^{S_{ij}} y_k^{S_{ij}} \right]
\end{aligned}$$

where $\zeta_{l,k}^{S_{ij}}$, $k = 1 \ldots N$, $l = 1, 2$ represent the adjoint variables associated to $x_k^{S_{ij}}$ and $y_k^{S_{ij}}$, respectively.

Theorem 4 (Necessary Conditions) *Given optimal controls* $\left(u_k^{S_{ij}*}\right)$ *and* $\left(v_k^{S_{ij}*}\right)$ *and solutions* $x_k^{S_{ij}*}$ *and* $y_k^{S_{ij}*}$, *there exist* $\zeta_{l,k}^{S_{ij}}$ *and* $\zeta_{l,k}^{S_{ij}}$, $k = 1...N$, *adjoint variables satisfying the following system:*

$$\begin{cases} \Delta\zeta_{1,k}^{S_{ij}} = & -\Bigg[\zeta_{1,N}^{S_{ij}} + \zeta_{1,k+1}^{S_{ij}}\left(r_1^{S_{ij}}\left(1-\frac{x_k^{S_{ij}}}{K_1^{S_{ij}}}\right) - r_1^{S_{ij}}\frac{x_k^{S_{ij}}}{K_1^{S_{ij}}} - \alpha_{12}^{S_{ij}}y_k^{S_{ij}}\right) \\ & -\zeta_{1,k+1}^{S_{ij}}\left(\left(d_1^{S_{ij}} + u_k^{S_{ij}}\delta_1^{S_{ij}}\right)x_k^{S_{ij}} + E_1^{S_{ij}}q_1^{S_{ij}}\right) + \zeta_{2,k+1}^{S_{ij}}\alpha_{21}^{S_{ij}}y_k^{S_{ij}}\Bigg], \\ \Delta\zeta_{2,k}^{S_{ij}} = & -\Bigg[\zeta_{2,N}^{S_{ij}} + \zeta_{2,k+1}^{S_{ij}}\left(r_2^{S_{ij}}\left(1-\frac{y_k^{S_{ij}}}{K_2^{S_{ij}}}\right) - r_2^{S_{ij}}\frac{y_k^{S_{ij}}}{K_2^{S_{ij}}} + \alpha_{21}^{S_{ij}}x_k^{S_{ij}}\right) \\ & -\zeta_{1,k+1}^{S_{ij}}\left(\left(d_2^{S_{ij}} + v_k^{S_{ij}}\delta_2^{S_{ij}}\right)y_k^{S_{ij}} + E_2^{S_{ij}}q_2^{S_{ij}}\right) - \zeta_{1,k+1}^{S_{ij}}\alpha_{12}^{S_{ij}}x_k^{S_{ij}}\Bigg], \end{cases}$$

where

$$u_k^{S_{ij}*} = \min\left\{\max\left\{u^{\min}, \frac{\zeta_{1,k+1}^{S_{ij}}\delta_1^{S_{ij}}x_k^{S_{ij}}}{W_1}\right\}, u^{\max}\right\},\ k = 1,...,n,$$

$$v_k^{S_{ij}*} = \min\left\{\max\left\{v^{\min}, \frac{\zeta_{2,k+1}^{S_{ij}}\delta_2^{S_{ij}}y_k^{S_{ij}}}{W_2}\right\}, v^{\max}\right\},\ k = 1,...,n.$$

Proof. Let $x_k^{S_{ij}} = x_k^{S_{ij}*}$, $y_k^{S_{ij}} = y_k^{S_{ij}*}$, $u_k^{S_{ij}} = u_k^{S_{ij}*}$, $v_k^{S_{ij}} = v_k^{S_{ij}*}$ and denoting the transversality conditions $\zeta_{1,N}^{S_{ij}}$ and $\zeta_{2,N}^{S_{ij}}$ by Γ_1 and Γ_2. Based on the Pontryagin's Maximum Principle [36] we have the following adjoint equations:

$$\begin{cases} \Delta\zeta_{1,k}^{S_{ij}} = \dfrac{-\partial\mathcal{H}}{\partial x_k^{S_{ij}}} = & -\Bigg[\Gamma_1 + \zeta_{1,k+1}^{S_{ij}}\left(r_1^{S_{ij}}\left(1-\frac{x_k^{S_{ij}}}{K_1^{S_{ij}}}\right) - r_1^{S_{ij}}\frac{x_k^{S_{ij}}}{K_1^{S_{ij}}} - \alpha_{12}^{S_{ij}}y_k^{S_{ij}}\right) \\ & -\zeta_{1,k+1}^{S_{ij}}\left(\left(d_1^{S_{ij}} + u_k^{S_{ij}}\delta_1^{S_{ij}}\right)x_k^{S_{ij}} + E_1^{S_{ij}}q_1^{S_{ij}}\right) + \zeta_{2,k+1}^{S_{ij}}\alpha_{21}^{S_{ij}}y_k^{S_{ij}}\Bigg], \\ \Delta\zeta_{2,k}^{S_{ij}} = \dfrac{-\partial\mathcal{H}}{\partial y_k^{S_{ij}}} = & -\Bigg[\Gamma_2 + \zeta_{2,k+1}^{S_{ij}}\left(r_2^{S_{ij}}\left(1-\frac{y_k^{S_{ij}}}{K_2^{S_{ij}}}\right) - r_2^{S_{ij}}\frac{y_k^{S_{ij}}}{K_2^{S_{ij}}} + \alpha_{21}^{S_{ij}}x_k^{S_{ij}}\right) \\ & -\zeta_{2,k+1}^{S_{ij}}\left(\left(d_2^{S_{ij}} + v_k^{S_{ij}}\delta_2^{S_{ij}}\right)y_k^{S_{ij}} + E_2^{S_{ij}}q_2^{S_{ij}}\right) - \zeta_{1,k+1}^{S_{ij}}\alpha_{12}^{S_{ij}}x_k^{S_{ij}}\Bigg]. \end{cases}$$

Therefore

$$\begin{cases} \zeta_{1,k}^{Z_{ij}} & = \Gamma_1 + \zeta_{1,k+1}^{S_{ij}}\Bigg(1 + r_1^{S_{ij}}\left(1-\frac{x_k^{S_{ij}}}{K_1^{S_{ij}}}\right) - r_1^{S_{ij}}\frac{x_k^{S_{ij}}}{K_1^{S_{ij}}} - \left(d_1^{S_{ij}} + u_k^{S_{ij}}\delta_1^{S_{ij}}\right)x_k^{S_{ij}} \\ & -\alpha_{12}^{S_{ij}}y_k^{S_{ij}} - E_1^{S_{ij}}q_1^{S_{ij}}\Bigg) + \zeta_{2,k+1}^{S_{ij}}\alpha_{21}^{S_{ij}}y_k^{S_{ij}}, \\ \zeta_{2,k}^{Z_{ij}} & = \Gamma_2 + \zeta_{2,k+1}^{S_{ij}}\Bigg(1 + r_2^{S_{ij}}\left(1-\frac{y_k^{S_{ij}}}{K_2^{S_{ij}}}\right) - r_2^{S_{ij}}\frac{y_k^{S_{ij}}}{K_2^{S_{ij}}} - \left(d_2^{S_{ij}} + v_k^{S_{ij}}\delta_2^{S_{ij}}\right)y_k^{S_{ij}} \\ & +\alpha_{21}^{S_{ij}}x_k^{S_{ij}} - E_2^{S_{ij}}q_2^{S_{ij}}\Bigg) - \zeta_{1,k+1}^{S_{ij}}\alpha_{12}^{S_{ij}}x_k^{S_{ij}}. \end{cases}$$

Taking the variation with respect to controls $u_k^{S_{ij}}$ and $v_k^{S_{ij}}$ equal to zero we obtain the following optimally conditions

$$\frac{\partial\mathcal{H}}{\partial u_k^{S_{ij}}} = W_1u_k^{S_{ij}} - \zeta_{1,k+1}^{S_{ij}}\delta_1^{S_{ij}}x_k^{S_{ij}} = 0,$$

$$\frac{\partial\mathcal{H}}{\partial v_k^{S_{ij}}} = W_2v_k^{Z_{ij}} - \zeta_{2,k+1}^{S_{ij}}\delta_2^{S_{ij}}y_k^{S_{ij}} = 0.$$

Then, the optimal controls are given by

$$u_k^{S_{ij}} = \frac{\zeta_{1,k+1}^{S_{ij}}\delta_1^{S_{ij}}x_k^{S_{ij}}}{W_1} \ \text{ and } v_k^{S_{ij}} = \frac{\zeta_{2,k+1}^{S_{ij}}\delta_2^{S_{ij}}y_k^{S_{ij}}}{W_2}.$$

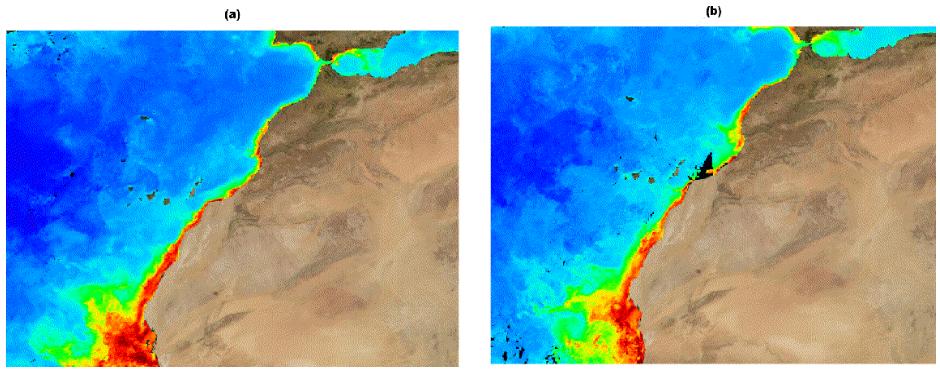


Figure 2: Distribution of the *Chlorophyll a* in Moroccan maritime areas throughout the month of May 2019 **(a)** and 2020 **(b)**

Following the boundedness of the controls, it is easy to obtain in $\mathcal{U}$ and $\mathcal{V}$, the form of $(u_k^{S_{ij}*})$ and $(v_k^{S_{ij}*})$

$$u_k^{S_{ij}*} = \min\left\{\max\left\{u^{\min}, \frac{\zeta_{1,k+1}^{S_{ij}}\delta_1^{S_{ij}} x_k^{S_{ij}}}{W_1}\right\}, u^{\max}\right\},\ k = 1, ..., n,$$

$$v_k^{S_{ij}*} = \min\left\{\max\left\{v^{\min}, \frac{\zeta_{2,k+1}^{S_{ij}}\delta_2^{S_{ij}} y_k^{S_{ij}}}{W_2}\right\}, v^{\max}\right\},\ k = 1, ..., n.$$

■

5 Numerical Simulation

In this section, we carry out some numerical simulations and we discuss the results. The raster maps (a) and (b) of Figure 2 shows the distribution of the *Chlorophyll-a* in the different fishing zones of Morocco during the month of May of last year (2019) and this year (2020). As we mention in the introduction the *chlorophyll-a* provide the best index of phytoplankton biomass abundance which mean the existence of the other species. These maps are obtained using the **Sea-Das** software. We have chosen these two maps precisely because we have noticed that during the international lockdown period (caused by Covid-19 pandemic) a significant number of marine resources have come to light. The appearance and abundance of this marine species are due to the reduction in vessels level, which implies the reduction of the pollution rate caused by fishing and pleasure boats. The main objective of this section is to show the influence of the optimal control of the pollution coefficient on the biomass level of the two populations' kind, in other words, on the level of *chlorophyll-a*. We note the value 4.92 of the pollution rates led to the death of 322 million tons of marine populations, demonstrating the importance of controlling this rate.

The parameter values adopting in this section are listed in the Table (1). From an iterative discrete schema, we formulate a code in Matlab that converges following an appropriate test identical to the one related to the forward backward swept method. We add that the solution of the optimally system is given based on an iterative method. In this method, the state system with an initial guess is solved forward in time and then the adjoint system is solved backward in time because of the transversality conditions.

In the process described, after obtaining the values of the state and costate variables from the previous steps, the next step is to update the values of the optimal controls. This involves adjusting the control inputs or actions in order to optimize the desired outcome.

The process of updating the optimal controls is performed iteratively, repeatedly going through the steps and adjusting the controls based on the current values of the state and costate variables. This iteration continues until a tolerance criterion is met, indicating that the desired level of accuracy or convergence has been achieved.

Once the numerical calculations are completed, the results are presented in spatial space with three dimensions. This spatial representation allows for a visual understanding and analysis of the optimized system behavior. The two cases mentioned suggest that the results are presented for two different scenarios or sets of conditions, providing a comparative analysis or exploration of the system's performance under different circumstances.

Overall, this iterative process of updating the optimal controls based on the values of the state and costate variables aims to find the best possible controls that maximize the desired objective while satisfying the given constraints. The presentation of numerical results in spatial space facilitates the interpretation and evaluation of the system's behavior in a visually meaningful way.

Table 1. Parameter values of marines species

x_0	$r_1^{S_{ij}}$	$K_1^{S_{ij}}$	$\alpha_{12}^{S_{ij}}$	$d_1^{S_{ij}}$	$\delta_1^{S_{ij}}$	$E_1^{S_{ij}}$	$q_1^{S_{ij}}$
125	0.37	225	0.21	0.001	0.2	30	0.13
y_0	$r_2^{S_{ij}}$	$K_2^{S_{ij}}$	$\alpha_{21}^{S_{ij}}$	$d_2^{S_{ij}}$	$\delta_2^{S_{ij}}$	$E_2^{S_{ij}}$	$q_2^{S_{ij}}$
350	0.5	700	0.19	0.0015	0.3	50	0.18

Case 1: without control of the pollution rate: Where the pollution rate is not controlled, the impact on the ecosystem is evident. Specifically, the biomass level of planktonic organisms is observed to decrease. This decline can be attributed to the uncontrolled pollution rate, which negatively affects the survival and growth of these organisms. The decreasing trend in biomass is depicted in Figure 3.

Moreover, the uncontrolled pollution rate also has consequences for the predator population. As the prey population diminishes due to the adverse effects of pollution, the predator population experiences a decline in biomass as well. The availability of prey plays a crucial role in sustaining the predator population, and the decrease in prey biomass leads to a reduction in the predator population's biomass. This trend is illustrated in Figure 4.

The results from Case 1 highlight the importance of controlling pollution rates in order to maintain a healthy and balanced ecosystem. Uncontrolled pollution not only directly affects the target organisms but also has cascading effects on other species within the ecosystem. By examining the decline in biomass for both planktonic organisms and predators, it becomes evident that the uncontrolled pollution rate poses a significant threat to the overall ecological stability.

These findings emphasize the need for effective pollution control measures and environmental management practices to safeguard the health and sustainability of ecosystems. Implementing measures to reduce pollution rates can help mitigate the adverse impacts on planktonic organisms and their predators, thereby promoting a more stable and resilient ecosystem.

Case 2: with control of the pollution rate: A different scenario unfolds. By maximizing the optimal controls associated with the evolution of the two marine species, notable improvements are observed.

Figure 5 illustrates the effect of controlling the pollution rate on the level of phytoplankton organism biomass. The biomass shows a steady decrease, reaching a minimum value that ensures its abundance over an extended period. This controlled decrease is justified by the presence of predation, as the predator population relies on the phytoplankton organisms as a food source. By maintaining a minimum biomass level, the phytoplankton organisms can sustain their population and promote their long-term abundance.

Figure 5 presents the impact of the optimal control value on the biomass of the predator population. The optimal control strategy used in this case leads to the maximization and continuous increase of the predator population's biomass over time. This indicates that the controlled pollution rate, along with the predation dynamics, promotes a favorable environment for the growth and sustainability of the predator population.

Through this study, we have discovered the optimal control values that ensure the long-term sustainability of the marine populations. In this particular case, the optimal control values are determined to be $0,52$ and $0,73$. These values represent the effective pollution control measures needed to maintain a balanced ecosystem and support the growth and survival of both phytoplankton organisms and predators.

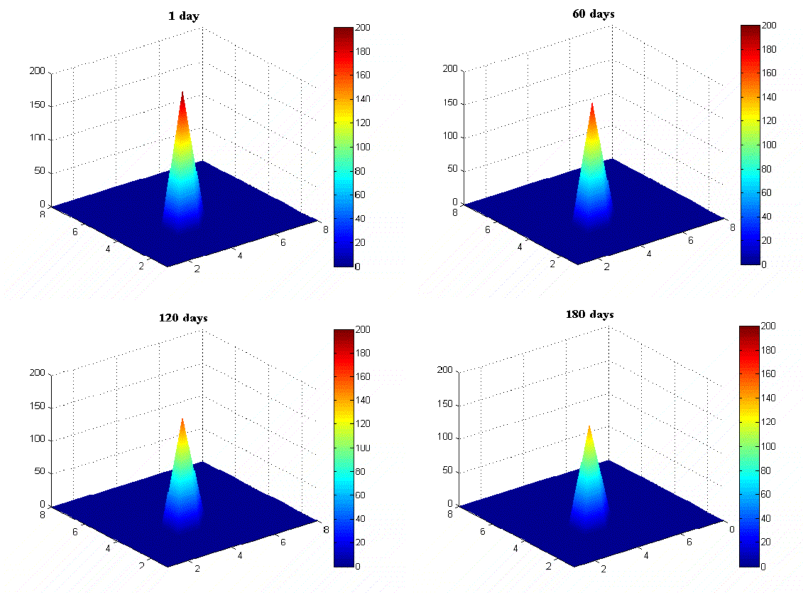


Figure 3: *Evolution of* $x^{S_{ij}}$ *without controls along time*

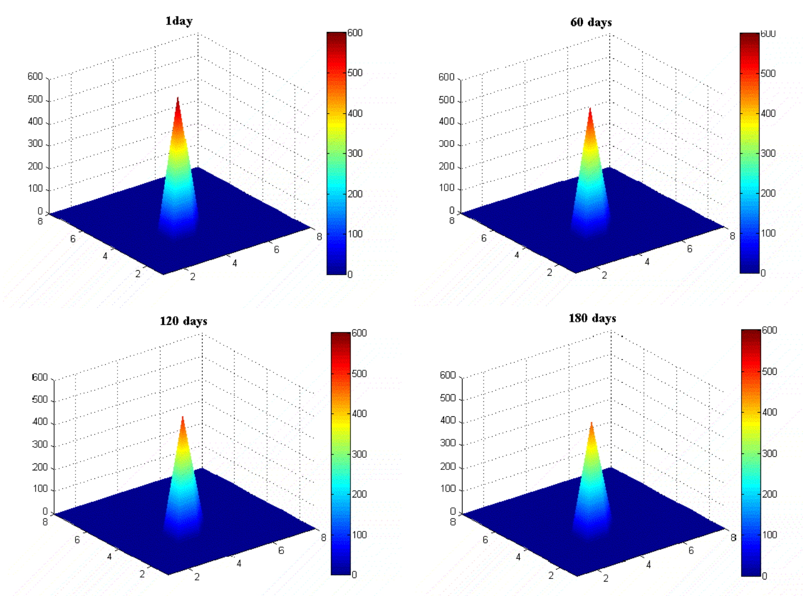
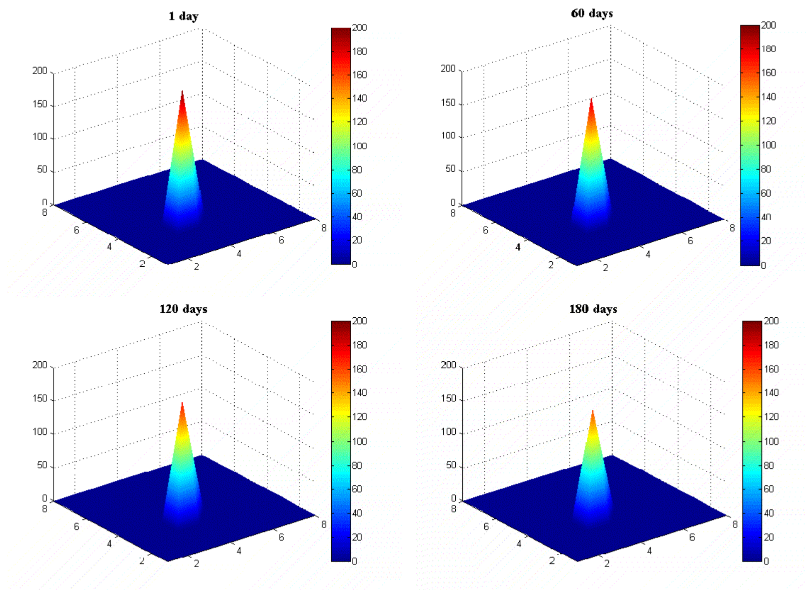
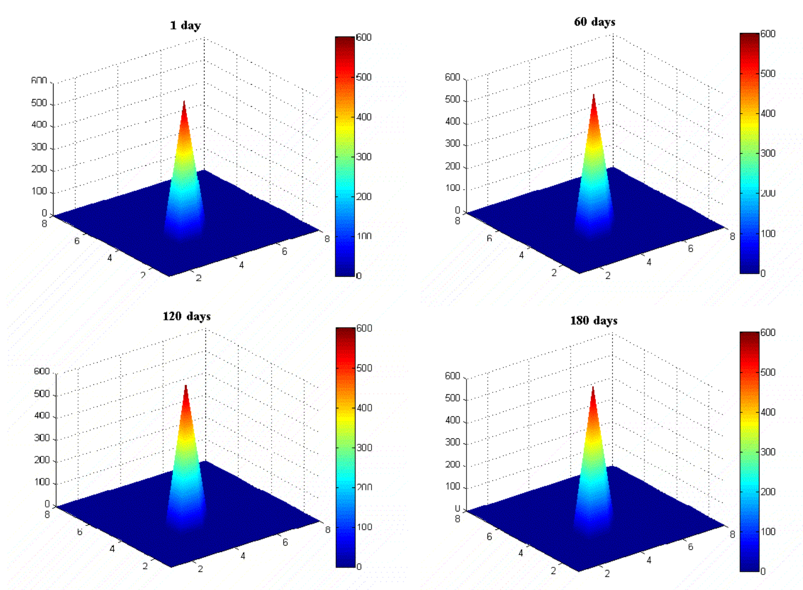


Figure 4: *Evolution of* $y^{S_{ij}}$ *without controls along time.*

Overall, Case 2 demonstrates the positive outcomes of implementing pollution control measures. By carefully managing the pollution rate, we can create conditions that favor the long-term abundance and sustainability of marine populations. These findings emphasize the significance of responsible environmental practices and highlight the potential for maintaining a thriving ecosystem through effective control strategies.



Evolution of $x^{S_{ij}}$ *with controls along time.*



Evolution of $y^{S_{ij}}$ *with controls along time.*

6 Conclusion

In this paper, we showed the impact of the proposed optimal control of pollution rates into the biomass levels of planktonic organisms. We studied the dynamics of interaction between phytoplankton and zooplankton in Moroccan maritime areas by analyzing a multi-areas prey-predator model associated in the continuous and discrete time. We proved the existence of the optimal controls and we determined the characterization of controls in terms of states and adjoint functions. We solved the optimally system by using the forward-backward sweep method (FBSM). In numerical simulations of the resulting optimality system we found, as a principal result, that the optimal control of mortality pollution rate may help us to give more realistic interpretations for ensuring the sustainability of marines resources.

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