Applied Mathematics E-Notes, 23(2023), 424-448 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Eigenvalue Criteria For Existence And Nonexistence Of Bounded And Unbounded Positive Solutions To A Third-Order BVP On The Half Line*

Abdelhamid Benmezaï[†], Salima Mechrouk[‡], El-Djouher Sedkaoui[§]

Received 30 September 2022

Abstract

Under eigenvalue criteria, we establish in this article existence and nonexistence results for positive solutions to the third-order boundary value problem

$$\begin{cases} -u'''(t) + k^2u'(t) = f(t, u(t)),\ t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases}$$

where k is a positive constant and the function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous. The boundedness and the unboundedness of the solution are also discussed.

1 Introduction and Main Results

Because third order ordinary differential equations arise in modeling various physical phenomena, the study of existence of solutions to boundary value problems (bvp for short) related to these, is a rapidly growing branch of applied mathematics. As examples, we start by Danziger and Elemergreen who proposed in [15] (see p. 133) the following third-order linear differential equations

$$\begin{aligned} &\alpha_3y''' + \alpha_2y'' + \alpha_1y' + (1 + k)\,y = kc,\ \theta < c, \text{ and} \\ &\alpha_3y''' + \alpha_2y'' + \alpha_1y' + y = 0,\ \theta > c, \end{aligned} \tag{1}$$

to describe the variation of thyroid hormone with time. Notice that the unown $y = y(t)$ in Equation (1) represents the concentration of thyroid hormone at time t and α_3, α_2, α_2, k and c are constants.

Motivated by the asymptotic behavior of the solutions of Volterra integro-differential equations having the form

$$\begin{cases} y'(t) = \gamma y(t) + \int_0^1 (\lambda + \mu t + \vartheta s)\, y(s)ds, & t \geq 0, \\ y(0) = 1, \end{cases}$$

Jackiewicz et al. have investigated in [20] the third-order differential equations of the type

$$u''' = \gamma u'' + (\lambda + (\mu + \vartheta)\,t)\,u' + (2\mu + \vartheta)\,u, \tag{2}$$

where λ, γ, μ and ϑ are real parameters and $\mu + \vartheta = 0$.

As a simple model exhibiting many of the features of the Hodgkin–Huxley equations, Nagumo proposed (see [27]) third-order differential equation

$$y''' - cy'' + f'(y)y' - \frac{b}{c}y = 0, \tag{3}$$

*Mathematics Subject Classifications: 34B15, 34B18, 34B40.

[†]National High School of Mathematics, Sidi-Abdallah, Algiers, Algeria
[‡]Faculty of Sciences, University Mhamed Bouguera, Boumerdes, Algeria
[§]Faculty of Mathematics, USTHB, Algiers, Algeria

where f is a regular function.

The partial differential equation

$$y_t + y_{xxxx} + y_{xx} + \frac{1}{2}y^2 = 0$$

arises in a large variety of physical phenomena. Commonly known as the Kuramoto-Sivashinsky equation, it was introduced to describe pattern formulation in reaction diffusion systems as well as to model the instability of flame front propagation (see Y. Kuramoto and T. Yamada [23] and D. Michelson [28]). Its traveling wave solutions (i.e. $y(x,t) = y(x - ct)$) are the solutions of the nonlinear third-order differential equation

$$\theta y'''(x) + y'(x) + g(y) = 0, \tag{4}$$

where the parameter θ depends on the constant c and g is an even function.

A three-layer beam is formed by parallel layers of different materials. For an equally loaded beam of this type, Krajcinovic in [22] proved that the deflection u is governed by the third order differential equation

$$-y''' + k^2 y' = a, \tag{5}$$

where the parameters k and a depend on the elasticity of the layers.

Moreover, study of existence of positive solutions for third-order bvps has received a great deal of attention and was the subject of many articles, see, for instance, [13, 14, 16, 17, 18, 26, 30, 32, 33, 34, 35, 36], for third-order bvps posed on finite intervals and [1, 2, 3, 4, 7, 9, 10, 11, 12, 19, 21, 24, 25, 29, 31] for such bvps posed on the half-line.

In this article, we establish under eigenvalue criteria, nonexistence and existence results for positive solutions to the third-order bvp:

$$\begin{cases} -u'''(t) + k^2 u'(t) = f(t, u(t)), \ t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{6}$$

where $k \in (0, +\infty)$, $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function ($\mathbb{R}^+ := [0, +\infty)$) and observe that the form of the differential equation in (6) is more general to those of (1)–(5). The physical constant k will play a crucial role in building an appropriate functional framework for a fixed point formulation to the bvp (6).

In this work we mean by a positive solution to the bvp (6), a function u in $C^3(\mathbb{R}^+, \mathbb{R}^+)$ satisfying $u(t_*) > 0$ for some $t_* > 0$ and all equations in the bvp (6).

When looking for positive solutions by using the fixed point theory in cones, authors often make use of the compression and expansion of a cone principle in a Banach space. This principle states that if P is a cone in a Banach space $(B, \|\cdot\|)$, $T : P_{r,R} \to P$ is a compact mapping where $P_{r,R} = \{u \in P : r \le \|u\| \le R\}$ and one of the following situations a) and b) holds:

a) $\|Tu\| \ge \|u\|$ for all $u \in P$, $\|u\| = r$ and $\|Tu\| \le \|u\|$ for all $u \in P$, $\|u\| = R$,

b) $\|Tu\| \le \|u\|$ for all $u \in P$, $\|u\| = r$ and $\|Tu\| \ge \|u\|$ for all $u \in P$, $\|u\| = R$,

then T has a fixed point w such that $r \le \|w\| \le R$.

This principle has advantage to be applicable on any region of the cone P and it has the flaw that the realization of the inequality $\|Tu\| \ge \|u\|$ requires a specific cone, see, for instance [14, 16, 26, 34, 35].

The main tool in this work consists in the fixed point theory in cones. The operator of our fixed point formulation associated to bvp (6) is defined on the Banach space of continuous functions u satisfying $\lim_{t\to+\infty} \frac{u(t)}{t} = 0$. Notice that this space is imposed by the boundary condition in (6) $\lim_{t\to+\infty} u'(t) = 0$, since by the L'Hopital's rule $\lim_{t\to+\infty} \frac{u(t)}{t} = \lim_{t\to+\infty} u'(t) = 0$. Unfortunately, the cone of nonnegative function lying in the above space does not offer the possibility to realize the inequality $\|Tu\| \ge \|u\|$. To overcome this difficulty we use the approach exposed in Section 3. This approach gives a necessary condition for existence of positive solution (see Proposition 3), and has the advantage to be applicable in any cone. However, it has the disadvantage that the radii r and R must be taken near 0 and $+\infty$ respectively. In other words we lose the localization established in the compression and expansion of a cone principal in a Banach space, $r \leq \|w\| \leq R$.

Since a function u satisfying $\lim_{t\to+\infty} \frac{u(t)}{t} = 0$ may be bounded or unbounded (e.g. $u(t) = \ln(1+t)$), we provide in each existence result established in this paper sufficient conditions for the boundedness or unboundedness of the obtained positive solution. In this paper, we let

$$\Gamma = \left\{q \in C(\mathbb{R}^+, \mathbb{R}^+) : q(s) > 0 \text{ a.e. } s > 0\right\},$$

$$\Gamma_0 = \left\{q \in \Gamma : \sup_{s\geq 0} q(s) < \infty\right\},$$

$$\Gamma_1 = \left\{q \in \Gamma : \lim_{s\to+\infty} q(s) = 0\right\},$$

$$\Gamma_2 = \left\{q \in \Gamma : \lim_{s\to+\infty} q(s) = 0 \text{ and } \int_0^{+\infty} q(s)ds < \infty\right\},$$

$$\Delta_i = \{q \in \Gamma : qp_i \in \Gamma_i\} \text{ for } i = 0, 1, 2,$$

$$\Delta_3 = \{q \in \Gamma : qp_3 \in \Gamma_1\},$$

$$\Delta = \Delta_1 \cup \Delta_2,$$

where

$$p_1(t) = 1 + t, \quad p_0(t) = p_2(t) = 1, \quad p_3(t) = e^{kt}.$$

Notice that $\Gamma_2 \subset \Gamma_1 \subset \Gamma_0$, $\Delta_2 = \Gamma_2$, $\Delta_3 \subset \Delta_1 \cap \Delta_2$, $\Delta_1 \setminus \Delta_2 \neq \emptyset$ and $\Delta_2 \setminus \Delta_1 \neq \emptyset$. Indeed, for

$$q_1(s) = \frac{1}{(1+s)\ln(4+s)}, \quad q_2(s) = \frac{m(s)}{1+s},$$

where

$$m(s) = \begin{cases} 2n^4 s - n(2n^4 - 1) & \text{if } s \in \left[n - \frac{1}{2n^3}, n\right], \\ -2n^4 s + n(2n^4 + 1) & \text{if } s \in \left[n, n + \frac{1}{2n^3}\right], \\ 0 & \text{otherwise,} \end{cases}$$

we have $q_1 \in \Delta_1 \backslash \Delta_2$ and $q_2 \in \Delta_2 \backslash \Delta_1$.

A continuous mapping $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is said to be

- a Γ_i-Caratheodory function for $i = 0, 1, 2$, if for all $r > 0$ there exists a function $\psi_r \in \Gamma_i$ such that

$$|g(t, p_i(t)u)| \leq \psi_r(t) \text{ for all } t \geq 0 \text{ and } u \in [-r, r].$$

- a Γ_{2+i}-Caratheodory function for $i = 1, 2$, if for all $r > 0$ there exists a function $\psi_r \in \Gamma_i$ such that

$$|g(t, p_3(t)u)| \leq \psi_r(t) \text{ for all } t \geq 0 \text{ and } u \in [-r, r].$$

Consider for $q \in \Delta$, the linear eigenvalue problem associated with the bvp (6)

$$\begin{cases} -u'''(t) + k^2 u'(t) = \mu q(t) u(t), & t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{7}$$

where μ is a real parameter.

A positive real number μ_0 is said to be a positive eigenvalue of the bvp (7), if there exists a function $\phi \in C^3(\mathbb{R}^+, \mathbb{R}^+)$ such that $\phi(t_0) > 0$ for some $t_0 > 0$ and the pair (μ_0, ϕ) satisfies all equations in the bvp (7).

The first result of this paper concerns existence of the positive eigenvalue of the bvp (7).

Proposition 1 *For all $q \in \Delta$, the eigenvalue problem (7) admits a unique positive eigenvalue $\mu(q) > 0$ associated with an eigenfunction ϕ. Moreover, if $q \in \Delta_2$ then ϕ is bounded and if not (i.e. $\int_0^{+\infty} q(s)ds = +\infty$), then ϕ is unbounded, i.e. $\lim_{t\to+\infty} \phi(t) = +\infty$.*

Theorem 1 *Assume for $i = 1$ or 2, the nonlinearity f is a Γ_i-Caratheodory function and there exists a function q in Δ_i such that either*

$$\inf\left\{\frac{f(t,p_i(t)u)}{p_i(t)q(t)u} : t,u > 0\right\} > \mu(q) \tag{8}$$

or

$$\sup\left\{\frac{f(t,p_i(t)u)}{p_i(t)q(t)u} : t,u > 0\right\} < \mu(q). \tag{9}$$

Then the bvp (6) admits no positive solution.

The statements of the following existence results need additional notations. For any Γ_i-Caratheodory function $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $q \in \Delta_i$ with $i \in \{0,1,2,3\}$ and $\nu = 0, +\infty$, we set

$$g^+_{i,\nu}(q) = \limsup_{u\to\nu}\left(\max_{t\geq 0}\frac{g(t,p_i(t)u)}{p_i(t)q(t)u}\right)$$

and

$$g^-_{i,\nu}(q) = \liminf_{u\to\nu}\left(\min_{t\geq 0}\frac{g(t,p_i(t)u)}{p_i(t)q(t)u}\right).$$

Theorem 2 *Suppose for $i = 1$ or 2, the function f is Γ_i-Caratheodory and there are two functions q_0 and q_∞ in Δ_i such that either*

$$\frac{f^+_{i,\infty}(q_\infty)}{\mu(q_\infty)} < 1 < \frac{f^-_{i,0}(q_0)}{\mu(q_0)} \leq \frac{f^+_{i,0}(q_0)}{\mu(q_0)} < \infty \tag{10}$$

or

$$\frac{f^+_{i,0}(q_0)}{\mu(q_0)} < 1 < \frac{f^-_{i,+\infty}(q_\infty)}{\mu(q_\infty)} \leq \frac{f^+_{i,\infty}(q_\infty)}{\mu(q_\infty)} < \infty. \tag{11}$$

Then the bvp (6) admits a solution u in K_i. Moreover, if $i = 2$ then u is bounded and if $i = 1$ and

$$\lim_{t\to+\infty}\int_1^t f(s,p_1(s)\lambda)ds = +\infty \text{ uniformly for } \lambda \text{ in compact intervals of } (0,+\infty), \tag{12}$$

then u is unbounded.

In Theorem 2, conditions (10) and (11) impose the nonlinearity f to be sublinear at $+\infty$, that is there is a positive constants d and a function $c \in \Gamma_i$ such that $f(t,u) \leq c(t)\,u$ for all $u \geq d$ and $t \geq 0$. To avoid such a condition, we have been led to look for positive solutions in the largest Banach space. We have obtained then the following result.

Theorem 3 *Suppose that the function f is Γ_3-Caratheodory and there are two functions q_0 and q_∞ in Δ_3 such that either*

$$\frac{f^+_{3,\infty}(q_\infty)}{\mu(q_\infty)} < 1 < \frac{f^-_{3,0}(q_0)}{\mu(q_0)}, \tag{13}$$

or

$$\frac{f^+_{3,0}(q_0)}{\mu(q_0)} < 1 < \frac{f^-_{3,\infty}(q_\infty)}{\mu(q_\infty)}. \tag{14}$$

Then the bvp (6) admits a positive solution u. Moreover, if the nonlinearity f is a Γ_4-Caratheodory function then the solution u is bounded, and if

$$\lim_{t\to+\infty}\int_1^t f(s,p_3(s)\lambda)ds = +\infty \text{ uniformly for } \lambda \text{ in compact intervals of } (0,+\infty), \tag{15}$$

then u is unbounded.

Consider now, the particular version of the bvp (6) where the nonlinearity f takes the form $f(t,u) = q_*(t)h(t,u)$; namely, we consider the bvp

$$\begin{cases} -u'''(t) + k^2u'(t) = q_*(t)h(t,u(t)), & t > 0, \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{16}$$

where $q_* \in \Gamma$ and $h : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function.

If h/p_i is a Γ_0-Caratheodory function for $i = 1, 2$ or 3, then we set for $\nu = 0, +\infty$,

$$h^+_{i,\nu} = h^+_{i,\nu}(1), \quad h^-_{i,\nu} = h^-_{i,\nu}(1).$$

We obtain respectively from Theorems 1, 2 and 3 the following corollaries:

Corollary 1 *Assume for $i = 1$ or 2 that $q_* \in \Delta_i$, the function h/p_i is Γ_0-Caratheodory and either*

$$\inf\left\{\frac{h(t,p_i(t)u)}{p_i(t)u} : t, u > 0\right\} > \mu(q),$$

or

$$\sup\left\{\frac{f(t,p_i(t)u)}{p_i(t)u} : t, u > 0\right\} < \mu(q).$$

Then the bvp (16) has no positive solution.

Corollary 2 *Assume for $i = 1$ or 2 that $q_* \in \Delta_i$, the function h/p_i is Γ_0-Caratheodory and either*

$$h^+_{i,\infty} < \mu(q_*) < h^-_{i,0} \le h^+_{i,0} < \infty,$$

or

$$h^+_{i,0} < \mu(q_*) < h^-_{i,\infty} \le h^+_{i,\infty} < \infty.$$

Then the bvp (16) admits a positive solution. Moreover, if $i = 2$ then u is bounded and if $i = 1$ and

$$\lim_{t\to+\infty} \int_1^t q_*(s)h(s,p_1(s)\lambda)ds = +\infty \text{ uniformly for } \lambda \text{ in compact intervals of } (0,+\infty),$$

then u is unbounded.

Corollary 3 *Suppose that $q_* \in \Delta_3$, the function h/p_3 is Γ_0-Caratheodory and either*

$$h^+_{3,\infty} < \mu(q_*) < h^-_{3,0},$$

or

$$h^+_{3,0} < \mu(q_*) < h^-_{3,\infty}.$$

Then the bvp (16) admits a positive solution. Moreover, if $q_* \in \Delta_2$ then u is bounded and if

$$\lim_{t\to+\infty} \int_1^t q_*(s)h(s,p_3(s)\lambda)ds = +\infty \text{ uniformly for } \lambda \text{ in compact intervals of } (0,+\infty),$$

then u is unbounded.

2 Example

Consider for $i = 1, 2, 3$ the bvp (6) with

$$f(t,u) = F_i(t,u) = Aq_0(t)\frac{p_i(t)u}{(p_i(t))^2 + u^2} + Bq_\infty(t)\frac{u^2}{p_i(t)+u},$$

where A and B are positive real numbers and $q_0, q_\infty \in \Delta_i$.

It is easy to see that F_i is a Γ_i-Caratheodory function and if

$$0 < \inf_{t\geq 0}\frac{q_\infty(t)}{q_0(t)} \leq \sup_{t\geq 0}\frac{q_\infty(t)}{q_0(t)} < \infty,$$

then

$$f_{i,0}^{-}(q_0) = f_{i,0}^{+}(q_0) = A \text{ and } f_{i,\infty}^{-}(q_\infty) = f_{i,\infty}^{+}(q_\infty) = B.$$

We deduce from Theorems 2 and 3 that for such a nonlinearity f, the bvp (6) admits a solution if either

$$A < \mu(q_0) \text{ and } B > \mu(q_\infty)$$

or

$$A > \mu(q_0) \text{ and } B < \mu(q_\infty).$$

Evidently for $i = 2$, the obtained solution u is bounded and for $i = 1$, if $\int_0^{+\infty} q_0 p_1 ds = +\infty$ then u is unbounded. Indeed, for any interval $[a,b] \subset (0,+\infty)$ we have

$$\begin{aligned}\int_1^t f(s, p_2(s)\lambda)ds &\geq A\int_1^t q_0(s)p_1(s)\frac{\lambda}{1+\lambda^2}ds \\ &\geq \frac{Aa}{1+a^2}\int_1^t q_0(s)p_1(s)ds \to +\infty \text{ as } t \to +\infty.\end{aligned}$$

For instance if $q_0(t) = q_\infty(t) = (1+t)^{-2}$ the obtained solution is unbounded.

In the case $i = 3$, if $\int_1^{+\infty} q_0(s)p_3(s)ds < +\infty$ then the solution is bounded and if $\int_1^{+\infty} q_0(s)p_3(s)ds = +\infty$, the same computations as above lead us to u is unbounded. For example, if $q_0(t) = q_\infty(t) = (1+t)^{-1} e^{-kt}$, then the obtained solution is unbounded.

3 Abstract Background

In this section we let $(Z, \|\cdot\|)$ be a real Banach space and by $\mathcal{L}(Z)$ and $r(L)$ we refer respectively to the set of all linear bounded self-mapping defined on Z and the spectral radius of an operator L in $\mathcal{L}(Z)$. We let also C be a cone in Z, that is C is a nonempty closed convex subset of Z such that $C \cap (-C) = \{0_Z\}$ and $tC \subset C$ for all $t \geq 0$. In the reminder of this section, the notation $\preceq$ refers to the partial order induced by the cone C on the Banach space Z. We write for all $u, v \in Z:\ u \preceq v$ (or $v \succeq u$) if $v - u \in C$ and $u \prec v$ (or $v \succ u$) if $v - u \in C \setminus \{0_Z\}$.

Definition 1 *A compact operator L in $\mathcal{L}(Z)$ is said to be*

i) positive, if $L(C) \subset C$,

ii) strongly positive, if $int(C) \neq \emptyset$ and $L(C \setminus \{0_Z\}) \subset int(C)$,

iii) lower bounded on the cone C, if

$$\inf\{\|Lu\| : u \in C \cap \partial B(0_Z, 1)\} > 0.$$

Hereafter we denote by $\mathcal{L}_C(Z)$ the subset of all positive compact operators in $\mathcal{L}(Z)$ and for any operator L in $\mathcal{L}_C(Z)$ we define the sets:

$$\Lambda_L = \{\theta \geq 0 : \exists u \succ 0_Z \text{ such that } Lu \succeq \theta u\} \text{ and}$$
$$\Gamma_L = \{\theta \geq 0 : \exists u \succ 0_Z \text{ such that } Lu \preceq \theta u\}.$$

It is proved in [5] that for all L in $\mathcal{L}_C(Z)$

$$\sup \Lambda_L \geq \inf \Gamma_L. \tag{17}$$

Definition 2 *An operator L in $\mathcal{L}_C(Z)$ is said to have the strongly index-jump property (SIJP for short) at μ, where μ is a positive real number, if*

$$\mu = \sup \Lambda_L = \inf \Gamma_L.$$

Proposition 2 (Proposition 3.16 in [5]) *Suppose that L is an operator in $\mathcal{L}_C(Z)$. If L is strongly positive then L has the SIJP at $r(L)$.*

Theorem 4 (Theorem 3.23 in [5]) *Assume that $L \in \mathcal{L}_C(Z)$ and $(L_n) \subset \mathcal{L}_C(Z)$ are such that (L_n) is increasing, for all integers $n \geq 1$, L_n has the SIJP at μ_n and $L_n \to L$ in operator norm. Then L has the SIJP at $\mu = \lim \mu_n = \sup \mu_n$.*

Remark 1 *From Proposition 3.14 and Proposition 3.15 in [6] we conclude that if $L \in \mathcal{L}_C(Z)$ has the SIJP at μ then μ is the unique positive eigenvalue of L.*

Remark 2 *Observe that if $L \in \mathcal{L}_C(Z)$ has the SIJP at μ and $L(C) \subset P \subset C$ where P is a cone in Z, then $L \in \mathcal{L}_P(Z)$ has the SIJP at μ.*

Our approach in this work is based on a fixed point formulation of the bvp (6). More exactly, we will show that the problem of existence and nonexistence of positive solutions to the bvp (6) is equivalent to that of existence and nonexistence of fixed point for a completely continuous mapping defined on some cone in an appropriate functional space. The following proposition and theorems will be used to prove the main results of this paper.

Let $T : C \to C$ be a completely continuous mapping. We start by the proposition below which provide provide under an eigenvalue criteria a nonexistence result of fixed point to the mapping T.

Proposition 3 *Suppose that there is an operator L in $\mathcal{L}_C(Z)$ having the SIJP at μ such that either*

$$\mu > 1 \text{ and } Tu \succeq Lu \text{ for all } u \in C, \tag{18}$$

or

$$\mu < 1 \text{ and } Tu \preceq Lu \text{ for all } u \in C \tag{19}$$

holds. Then T has no fixed point.

Proof. We prove the proposition in the case where (18) holds, the other case is checked in the same way. To the contrary, suppose that there is $w \succ 0_Z$ such that $Tw = w$. Then we have that $w = Tw \succeq Lw$, that is $1 \in \Gamma_L$ and $\mu = \inf \Gamma_L \leq 1$. This contradicts the condition $\mu > 1$ of Hypothesis (18). ■

The following two theorems are respectively adapted versions of Theorem 3.24 and Theorem 3.25 in [5]. They provide solvability results to the equation $u = Tu$ under eigenvalue criteria.

Theorem 5 *Suppose that C is normal and for $i = 1, 2, 3$ there exists $L_i \in \mathcal{L}_C(Z)$ and $F_i : C \to C$ such that*

$$\begin{cases} L_2 \text{ has the SIJP at } r(L_2), \\ 0 < r(L_2) < 1 < r(L_1) \text{ and} \\ Tv \preceq L_1 v + F_1 v, \\ L_2 v - F_2 v \preceq Tv \preceq L_3 v + F_3 v \text{ for all } v \in C. \end{cases}$$

If either

$$F_1 v = \circ(\|v\|) \ \text{ as } v \to 0 \text{ and } F_i v = \circ(\|v\|) \ \text{ as } v \to \infty \text{ for } i = 2,3 \tag{20}$$

or

$$F_1 v = \circ(\|v\|) \ \text{ as } v \to \infty \text{ and } F_i v = \circ(\|v\|) \ \text{ as } v \to 0 \text{ for } i = 2,3, \tag{21}$$

then T has a fixed point.

Theorem 6 *Suppose that for $i = 1,2$ that there is $L_i \in \mathcal{L}_C(Z)$ and $F_i : C \to C$ such that*

$$\begin{cases} L_1 \text{ has the SIJP at } r(L_1) \\ L_1 \text{ is lower bounded on } C, \\ r(L_2) < 1 < r(L_1) \text{ and} \\ L_1 v - F_1 v \preceq Tv \preceq L_2 v + F_2 v \text{ for all } v \in C. \end{cases}$$

If either

$$F_1 v = \circ(\|v\|) \ \text{ as } v \to \infty \text{ and } F_2 v = \circ(\|v\|) \ \text{ as } v \to 0 \tag{22}$$

or

$$F_1 v = \circ(\|v\|) \ \text{ as } v \to 0 \text{ and } F_2 v = \circ(\|v\|) \ \text{ as } v \to \infty, \tag{23}$$

then T has a positive fixed point.

4 Fixed Point Formulation

In the reminder of this paper we let

$$E_0 = \{u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} u(t) = 0\},$$

$$E_1 = \{u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} \frac{u(t)}{1+t} = 0\},$$

$$E_2 = \{u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} u(t) = l \in \mathbb{R}\},$$

$$E_3 = \{u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} e^{-kt} u(t) = 0\}.$$

Endowed respectively with the norms

$$\|u\|_1 = \sup_{t\geq 0} \frac{|u(t)|}{1+t}, \ \ \|u\|_2 = \sup_{t\geq 0} |u(t)| \ \text{ and } \ \|u\|_3 = \sup_{t\geq 0} e^{-kt} |u(t)|,$$

E_1, E_2 and E_3 become Banach spaces.

We let also, K_1, K_2 and K_3 be respectively the cones in E_1, E_2 and E_3 defined by

$$K_1 = \{u \in E_1 : u(t) \geq 0 \text{ for all } t \geq 0 \text{ and } u \text{ is nondecreasing}\},$$

$$K_2 = \{u \in E_2 : u(t) \geq 0 \text{ for all } t \geq 0\},$$

$$K_3 = \{u \in E_3 : u(t) \geq \gamma(t)||u||_3 \text{ for all } t \geq 0\},$$

where

$$\gamma(t) = \frac{1}{3k}\left(e^{-3kt} - 3e^{-kt} + 2\right).$$

Let $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function given by

$$G(t,s) = \frac{1}{k^2} \begin{cases} e^{-ks}(\cosh(kt) - 1), & \text{if } t \leq s, \\ -e^{-kt}\sinh(ks) + (1 - e^{-ks}), & \text{if } s \leq t. \end{cases}$$

The functions G and $\dfrac{\partial G}{\partial t}$ are continuous and they have the following properties:

$$G(t,s) > 0 \text{ for all } t,s > 0, \tag{24}$$

$$\frac{\partial G}{\partial t}(t,s) > 0 \text{ for all } t,s > 0, \tag{25}$$

$$G(0,s) = \frac{\partial G}{\partial t}(0,s) = 0 \text{ for all } s \in \mathbb{R}^+, \tag{26}$$

$$\lim_{t\to+\infty} G(t,s) = \frac{1}{k^2}(1-e^{-ks}) \text{ for all } s \in \mathbb{R}^+, \tag{27}$$

$$\int_0^{+\infty} G(t,s)ds = \frac{1}{k^2}t - \frac{1}{k^3}(1-e^{-kt}) \text{ for all } t \geq 0, \tag{28}$$

$$\sup_{t\geq 0} \frac{1}{1+t}\int_0^{+\infty} G(t,s)ds = \frac{1}{k^2}, \tag{29}$$

$$\int_0^{+\infty} \left|G(t_2,s) - G(t_1,s)\right| ds \leq \frac{2}{k^2}\left|t_2 - t_1\right| \text{ for all } t_2, t_1 \geq 0. \tag{30}$$

Properties (24)–(28) and (29) are obvious and Property (30) is obtained from Property (28) for each of the cases $t_2 \geq t_1$ and $t_2 \leq t_1$.

Lemma 1 *For all functions v in E_0, $u(t) = \int_0^{+\infty} G(t,s)v(s)ds$ is the unique solution of the bvp*

$$\begin{cases} -u'''(t) + k^2u' = v, \ \text{in } (0,+\infty), \\ u(0) = u'(0) = u'(+\infty) = 0. \end{cases} \tag{31}$$

Moreover u belongs to E_1.

Proof. Let $v \in E_0$. For any $t \geq 0$ we have by Property (28),

$$|u(t)| = \left|\int_0^{+\infty} G(t,s)v(s)ds\right| \leq \|v\|_2 \int_0^{+\infty} G(t,s)ds < \infty.$$

Furthermore, for any $t_1, t_2 \geq 0$, we have by Property (30),

$$\begin{aligned} |u(t_2) - u(t_1)| &= \left|\int_0^{+\infty} G(t_2,s)v(s)ds - \int_0^{+\infty} G(t_1,s)v(s)ds\right| \\ &\leq \int_0^{+\infty} \left|G(t_2,s) - G(t_1,s)\right| ds \, \|v\|_2 \\ &\leq \frac{2\|v\|_2}{k^2}\left|t_2 - t_1\right|. \end{aligned}$$

The above estimates show that u is well defined and u is continuous on $\mathbb{R}^+$.

Differentiating three times in the identity

$$u(t) = -\frac{e^{-kt}}{k^2}\int_0^t \sinh(ks)v(s)\,ds + \frac{1}{k^2}\int_0^t (1-e^{-ks})v(s)\,ds + \frac{\cosh(kt)-1}{k^2}\int_t^{+\infty} e^{-ks}v(s)\,ds,$$

we find

$$u'(t) = \frac{1}{k}\left(e^{-kt}\int_0^t \sinh(ks)v(s)\,ds + \sinh(kt)\int_t^{+\infty} e^{-ks}v(s)\,ds\right),$$

$$u''(t) = -e^{-kt}\int_0^t \sinh(ks)v(s)\,ds + \cosh(kt)\int_t^{+\infty} e^{-ks}v(s)\,ds,$$

$$u'''(t) = k\left(e^{-kt}\int_0^t \sinh(ks)v\left(s\right)ds + \sinh(kt)\int_t^{+\infty} e^{-ks}v\left(s\right)ds\right) - v(t) = k^2u'(t) - v(t).$$

Hence, u satisfies $-u'''(t) + k^2u' = v$. Since (26) gives $u(0) = u'(0) = 0$, it remains to prove that $\lim_{t\to+\infty} u'(t) = \lim_{t\to+\infty} \frac{u(t)}{1+t} = 0$. We have

$$u'(t) = \int_0^{+\infty} \frac{\partial G}{\partial t}(t,s)v(s)ds = \frac{1}{k}e^{-kt}\int_0^t \sinh(ks)v(s)ds + \frac{1}{k}\sinh(kt)\int_t^{+\infty} e^{-ks}v(s)ds.$$

Using L'Hopital's formula, we obtain

$$\lim_{t\to+\infty} e^{-kt}\int_0^t \sinh(ks)v(s)ds = \lim_{t\to+\infty} \frac{\int_0^t \sinh(ks)v(s)ds}{e^{kt}} = \lim_{t\to+\infty} \frac{\sinh\left(kt\right)}{ke^{kt}}v(t) = 0$$

and

$$\begin{aligned}\lim_{t\to+\infty}\left(\sinh(kt)\int_t^{+\infty} e^{-ks}v(s)ds\right) &= \lim_{t\to+\infty} \frac{\sinh(kt)}{e^{kt}}\frac{\int_t^{+\infty} e^{-ks}v(s)ds}{e^{-kt}} \\ &= \lim_{t\to+\infty} \frac{\int_t^{+\infty} e^{-ks}v(s)ds}{e^{-kt}} = \lim_{t\to+\infty}\frac{v(t)}{k} = 0.\end{aligned}$$

This completes the proof. ■

Lemma 2 *Assume for $i = 1$ or 2 the function $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a Γ_i-Caratheodory. Then the operator $T_g^i : E_i \to E_i$ where for $u \in E_i$,*

$$T_g^i u(t) = \int_0^{+\infty} G(t,s)g(s,u(s))ds,$$

is well defined and if $g(t,x) \geq 0$ for all $t, x \geq 0$ then $T_g^i\left(K_i\right) \subset K_i$. Moreover, if $u \in E_i$ is a fixed point of T_g^i then u is a solution to the bvp

$$\begin{cases} -u'''(t) + k^2u' = g(t,u), \text{ in } (0,+\infty), \\ u(0) = u'(0) = u'(+\infty) = 0. \end{cases} \tag{32}$$

Proof. Since $\Gamma_2 \subset \Gamma_1$, in both the cases $i = 1$ or 2, g is a Γ_1-Caratheodory function. Hence for any $u \in E_i$, $g(t,u)$ belongs to E_0 and $T_g^i u$ belongs to E_1 and satisfies the bvp (31) within $v = g(t,u)$. In the case $i = 2$, for $u \in E_2$ we have $g(t,u)$ belongs to Γ_2 (i.e. $\int_0^{+\infty} g(s,u(s))ds < \infty$). Therefore, Lebesgue convergence theorem and Property (27) lead to

$$\lim_{t\to+\infty} T_g^2 u(t) = \frac{1}{k^2}\int_0^{+\infty}\left(1 - e^{-ks}\right)g(s,u(s))ds \leq \frac{1}{k^2}\int_0^{+\infty} g(s,u(s))ds < \infty.$$

This shows that T_g^2 is well defined.

At the end, we conclude by Lemma 1 that any fixed point of T_g^i in E_i is a solution to the bvp (32) and it is easy to see that if g is nonnegative then $T_g^i(K_i) \subset K_i$ for $i = 1, 2$. ■

Lemma 3 *Assume for $i = 1$ or 2 the function $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a Γ_3-Caratheodory. Then the operator $T_g^3 : E_3 \to E_3$ where for $u \in E_3$,*

$$T_g^3 u(t) = \int_0^{+\infty} G(t,s)g(s,u(s))ds,$$

is well defined and if $g(t,x) \geq 0$ for all $t, x \geq 0$ then $T_g^3\left(K_3\right) \subset K_3$. Moreover, if $u \in E_3$ is a fixed point of T_g^3 then u is a solution to the bvp (32).

Proof. Since g is a Γ_3-Caratheodory function, for any $u \in E_3$ we have $|g(t,u)|$ belongs to Γ_1 (i.e. $\lim_{s\to+\infty} g(s,u(s)) = 0$). Hence Lemma 1 guarantees that $T_g^3 u \in E_1$ and and satisfies the bvp (31) within $v = g(t,u)$. Furthermore, for any $u \in E_3$ we have

$$e^{-kt}\left|T_g^3 u(t)\right| \leq \sup_{s\geq 0}|g(s,u(s))|\left(e^{-kt}\int_0^{+\infty} G(t,s)ds\right) \to 0 \text{ as } t \to +\infty.$$

This shows that T_g^3 is well defined.

Clearly, if $u \in E_3$ is a fixed point of T_g^3 then u is a solution to the bvp (32). So let us prove that if g is nonnegative then $T_g^3(K_3) \subset K_3$.

Let $u \in E_3$, taking in consideration Lemma 2.3 in [12], we obtain

$$\begin{aligned}
T_g^3 u(t) &= \int_0^t \frac{dT_g^3 u}{dt}(\xi)d\xi = \int_0^t \int_0^{+\infty} \frac{\partial G}{\partial t}(\xi,s)g(s,u(s)dsd\xi \\
&= \int_0^t e^{k\xi}\int_0^{+\infty} e^{-k\xi}\frac{\partial G}{\partial t}(\xi,s)g(s,u(s)dsd\xi \\
&\geq \int_0^t \int_0^{+\infty} e^{k\xi}\widetilde{\gamma}(\xi)e^{-k\tau}\frac{\partial G}{\partial t}(\tau,s)g(s,u(s)dsd\xi \\
&\geq \left(\int_0^t e^{k\xi}\widetilde{\gamma}(\xi)d\xi\right)\left(e^{-k\tau}\int_0^{+\infty}\frac{\partial G}{\partial t}(\tau,s)g(s,u(s)ds\right),
\end{aligned}$$

where $\widetilde{\gamma}(\xi) = \left(e^{2k\xi}-1\right)e^{-4k\xi}$. This leads to

$$T_g^3 u(t) \geq \left(\int_0^t e^{k\xi}\widetilde{\gamma}(\xi)d\xi\right)\left\|\frac{dT_g^3 u}{dt}\right\|_3. \tag{33}$$

Because $\frac{dT_g^3 u}{dt} \in E_3$, we have

$$\begin{aligned}
T_g^3 u(t) &= \int_0^t \frac{dT_g^3 u}{dt}(\xi)d\xi = \int_0^t e^{k\xi}\left(e^{-k\xi}\frac{dT_g^3 u}{dt}(\xi)\right)d\xi \leq \int_0^t e^{k\xi}d\xi\left\|\frac{dT_g^3 u}{dt}\right\|_3 \\
&\leq \frac{\left(e^{kt}-1\right)}{k}\left\|\frac{dT_g^3 u}{dt}\right\|_3 \leq \frac{e^{kt}}{k}\left\|\frac{dT_g^3 u}{dt}\right\|_3,
\end{aligned}$$

which yields

$$\left\|\frac{dT_g^3 u}{dt}\right\|_3 \geq k\left\|T_g^3 u\right\|_3. \tag{34}$$

Combining (33) with (34), we obtain

$$T_g^3 u(t) \geq \gamma(t)\left\|T_g^3 u\right\|_3.$$

Ending the proof. ■

As usual, the use of the fixed point approach needs a compactness criterion. The following result provides a compactness criterion for a subset in the Banach space E_i, $i = 1,2$ or 3. In fact this result is just is a version of Corduneanu's compactness criterion ([8], p. 62) adapted to the space E_i. It will be used in this work to prove that the operator associated with the fixed point formulation of the bvp (6) is completely continuous.

Lemma 4 *Let M be a nonempty subset of E_i, $i = 1,2,3$. If the following conditions hold:*

(a) M is bounded in E_i,

(b) *the set* $\left\{u : u(t) = \frac{x(t)}{p_i(t)},\ x \in M\right\}$ *is locally equicontinuous on* $[0, +\infty)$, *and*

(c) *the set* $\left\{u : u(t) = \frac{x(t)}{p_i(t)},\ x \in M\right\}$ *is equiconvergent at* $+\infty$,

then the subset M *is relatively compact in* E_i.

Lemma 5 *Assume that the function* $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is* Γ_1*-Caratheodory. Then the operator* T_g^1 *is completely continuous.*

Proof. First we prove that the operator T_g^1 is continuous. To this aim let (u_n) be a sequence in E_1 with $\lim u_n = u$ in E_1, and let $R > 0$ and $\psi_R \in \Gamma_2 \subset \Gamma_0$ be such that $\|u_n\|_1 \le R$ for all $n \ge 1$ and

$$\left|g\left(t, p_1(t)\left(\frac{u}{p_1(t)}\right)\right)\right| \le \psi_R(t) \text{ for all } t \ge 0 \text{ and } u \in [-R, R].$$

We have then

$$\left\|T_g^1 u_n - T_g^1 u\right\|_1 = \sup_{t \ge 0} \frac{\left|T_g^1 u_n(t) - T_g^1 u(t)\right|}{p_1(t)} \le \sup_{t \ge 0} \Phi_n(t),$$

where

$$\begin{aligned}
\Phi_n(t) &= \frac{1}{p_1(t)} \int_0^{+\infty} G(t,s) \left|g(s, u_n(s)) - g(s, u(s))\right| ds \\
&= \frac{1}{1+t} \int_0^{+\infty} G(t,s) \left|g\left(s, p_1(s)\left(\frac{u_n(s)}{p_1(s)}\right)\right) - g\left(s, p_1(s)\left(\frac{u(s)}{p_1(s)}\right)\right)\right| ds \\
&\le \frac{2}{p_1(t)} \int_0^{+\infty} G(t,s)\psi_R(s) ds \\
&\le \|\psi_R\|_2 \sup_{t \ge 0} \left(\frac{2}{p_1(t)} \int_0^{+\infty} G(t,s) ds\right) = \frac{2\|\psi_R\|_2}{k^2}.
\end{aligned}$$

Let (t_n) be such that $\Phi_n(t_n) = \sup_{t \ge 0} \Phi_n(t)$ and let (t_{n_l}) be such that $\lim \Phi_{n_l}(t_{n_l}) = \limsup \Phi_n(t_n)$. Therefore, we have to prove that $\lim \Phi_{n_l}(t_{n_l}) = 0$. We distinguish then two cases:

i) (t_{n_l}) is bounded by $c > 0$: In this case we have

$$\begin{aligned}
\Phi_{n_l}(t_{n_l}) &= \left(\frac{1}{p_1(t_{n_l})} \int_0^{+\infty} G(t_{n_l}, s) \left|g(s, u_{n_l}(s)) - g(s, u(s))\right| ds\right) \\
&\le \int_0^{+\infty} G(c, s) \left|g(s, u_{n_l}(s)) - g(s, u(s))\right| ds,
\end{aligned}$$

$$\lim_{n \to +\infty} G(c,s) \left|g(s, u_n(s)) - g(s, u(s))\right| = 0,$$

$$\left|g(s, u_n(s)) - g(s, u(s))\right| = \left|g\left(t, p_1(s)\left(\frac{u_n(s)}{p_1(s)}\right)\right) - g\left(t, p_1(s)\left(\frac{u(s)}{p_1(s)}\right)\right)\right| \le 2\psi_R(s),$$

for all $s > 0$ and by (28) $\int_0^{+\infty} G(c,s)\psi_R(s) ds < \infty$. Hence the dominated convergence theorem leads to $\lim \Phi_{n_l}(t_{n_l}) = \limsup \Phi_n(t_n) = 0$.

ii) $\lim t_{n_l} = +\infty$ (up to a subsequence): In this case we have from Lemma 2,

$$\begin{aligned}
\Phi_{n_l}(t_{n_l}) &= \left(\frac{1}{p_1(t_{n_l})} \int_0^{+\infty} G(t_{n_l}, s) \left|g(s, u_{n_l}(s)) - g(s, u(s))\right| ds\right) \\
&\le \frac{2}{p_1(t_{n_l})} \int_0^{+\infty} G(t_{n_l}, s)\psi_R(s) ds \to 0 \text{ as } l \to \infty.
\end{aligned}$$

Thus, we have proved that $\lim T_g^1 u_{n_l} = T_g^1 u$ in E_1 and T_g^1 is continuous.

Now we prove by means of Lemma 4 that T_g^1 maps bounded sets of E_1 into relatively compact sets of E_1. To this aim, let Ω be a subset of E_1 bounded by $R > 0$ and let $\psi_R \in \Gamma_1$ be such that

$$|g(s, p_1(s)u)| \leq \psi_R(s) \text{ for all } s \geq 0 \text{ and all } u \in [-R, R].$$

For any $u \in \Omega$ we have by Property (29),

$$\begin{aligned} \left\|T_g^1 u\right\|_1 &= \sup_{t\geq 0}\left|\frac{T_g^1 u\,(t)}{p_1(t)}\right| = \sup_{t\geq 0}\left(\frac{1}{p_1(t)}\int_0^{+\infty} G(t,s)\left|g\left(s, p_1(s)\left(\frac{u(s)}{p_1(s)}\right)\right)\right| ds\right) \\ &\leq \sup_{t\geq 0}\left(\frac{1}{p_1(t)}\int_0^{+\infty} G(t,s)\psi_R(s)ds\right) \\ &\leq \sup_{t\geq 0}\left(\frac{1}{p_1(t)}\int_0^{+\infty} G(t,s)ds\right)\|\psi_R\|_1 = \frac{1}{k^2}\|\psi_R\|_1 . \end{aligned}$$

Hence $T_g^1(\Omega)$ is bounded in E_1.

Let $t_1, t_2 \in [\eta, \zeta] \subset \mathbb{R}^+$ with $t_1 \leq t_2$. For all $u \in \Omega$ we have

$$\begin{aligned} \left|\frac{T_g^1 u\,(t_2)}{p_1(t_2)} - \frac{T_g^1 u\,(t_1)}{p_1(t_1)}\right| \leq{} & \int_0^{t_1}\left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right|\psi_R(s)ds + \int_{t_1}^{t_2}\left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right|\psi_R(s)ds \\ & + \int_{t_2}^{+\infty}\left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right|\psi_R(s)ds, \end{aligned}$$

$$\begin{aligned} \int_0^{t_1}\left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right|\psi_R(s)ds &\leq \frac{1}{k^2}\left(\frac{e^{-kt_1}}{p_1(t_1)} - \frac{e^{-kt_2}}{p_1(t_2)}\right)\int_0^{\zeta}\sinh(ks)\psi_R(s)ds \\ &\quad + \frac{1}{k^2}\left(\frac{1}{p_1(t_1)} - \frac{1}{p_1(t_2)}\right)\int_0^{\zeta}(1-e^{-ks})\psi_R(s)ds \\ &\leq \frac{C_1(k)}{k^2}\left(\int_0^{\zeta}\psi_R(s)ds\right)(t_2 - t_1), \end{aligned}$$

$$\begin{aligned} \int_{t_1}^{t_2}\left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right|\psi_R(s)ds &\leq \frac{1}{k^2}\int_{t_1}^{t_2}\left(\frac{e^{-kt_2}}{p_1(t_2)}\sinh(ks) + \frac{1-e^{-ks}}{p_1(t_2)} + \frac{\cosh(kt_1)-1}{p_1(t_1)}e^{-ks}\right)\psi_R(s)ds \\ &\leq \frac{C_2(k)}{k^2}(t_2 - t_1) \end{aligned}$$

and

$$\begin{aligned} \int_{t_2}^{+\infty}\left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right|\psi_R(s)ds &\leq \frac{1}{k^2}\left|\frac{\cosh(kt_2)-1}{p_1(t_2)} - \frac{\cosh(kt_1)-1}{p_1(t_1)}\right|\int_{\eta}^{+\infty} e^{-ks}\psi_R(s)e^{-ks}ds \\ &\leq \frac{C_3(k)}{k^2}(t_2 - t_1), \end{aligned}$$

where

$$C_1(k) = (k+1)\sinh(k\zeta) + 1,$$

$$C_2(k) = \left(\frac{\sinh(k\zeta)e^{-k\eta}}{1+\eta} + 1 + \frac{\cosh(k\zeta)-1}{1+\zeta}\right)\sup_{s\in[\eta,\zeta]}\psi_R(s),$$

$$C_3(k) = \sup_{t\in[\eta,\zeta]}\left(\frac{\cosh(kt)-1}{1+t}\right)'.$$

We obtain from the above computations that

$$\left|\frac{T_g^1 u\left(t_2\right)}{p_1(t_2)}-\frac{T_g^1 u\left(t_1\right)}{p_1(t_1)}\right| \leq \frac{C_1(k)+C_2(k)+C_3(k)}{k^2}\left(t_2-t_1\right).$$

Hence $T_g^1(\Omega)$ is equicontinuous on compact intervals of $\mathbb{R}^+$.

We have for all $u \in \Omega$ and $t \geq 0$

$$\frac{\left|T_g^1 u(t)\right|}{1+t} \leq \int_0^{+\infty} \frac{G(t,s)}{1+t}\left|g(s,u(s))\right| ds \leq \frac{1}{1+t} \int_0^{+\infty} G(t,s)\psi_R(s) ds := \widetilde{H}(t).$$

Since Lemma 2 guarantees that $\lim_{t\to+\infty} \widetilde{H}(t)=0$, we conclude that $T_g^1(\Omega)$ is equiconvergent at $+\infty$. This ends the proof. ■

Lemma 6 *Let $g: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a Γ_2-Caratheodory function. Then the operator T_g^2 is completely continuous.*

Proof. First, let us prove that T_g^2 is continuous. To this aim let (u_n) be a sequence in E_2 with $\lim u_n = u$ in E_2, and let $R>0$ and ψ_R be such that $\|u_n\|_2 \leq R$ for all $n \geq 1$ and $|g(t,p_2(t)u)| \leq \psi_R(t)$ for all $t \geq 0$ and $u \in [-R,R]$. Hence we have

$$\left\|T_g^2 u_n - T_g^2 u\right\|_2 = \sup_{t\geq 0}\left|T_g^2 u_n(t) - T_g^2 u(t)\right| \leq \int_0^{+\infty} G(\infty,s)\left|g(s,u_n(s))-g(s,u(s))\right| ds$$

with

$$\lim_{n\to+\infty}\left|g(s,u_n(s))-g(s,u(s))\right| = 0$$

and

$$\left|g(s,u_n(s))-g(s,u(s))\right| = \left|g(s,p_2(s)u_n(s))-g(s,p_2(s)u(s))\right| \leq 2\psi_R(s).$$

for all $s>0$. Since $\psi_R \in L^1(\mathbb{R}^+)$, we conclude by means of the dominated convergence theorem that $\lim T_g^2 u_n = T_g^2 u$ in E_2, proving the continuity of T_g^2.

Now we prove by means of Lemma 4 that T_g^2 maps bounded sets of E_2 into relatively compact sets of E_2. To this aim, let Ω be a subset of E_2 bounded by a constant $R>0$ and let $\psi_R \in \Gamma_2$ be such that

$$|g(s,p_2(s)u)| \leq \psi_R(s) \text{ for all } s \geq 0 \text{ and all } u \in [-R,R].$$

Hence for all $u \in \Omega$, we have by Property (25) and (27)

$$\begin{aligned}\left\|T_g^2 u\right\|_2 &\leq \sup_{t\geq 0}\int_0^{+\infty} G(t,s)\left|g(s,u(s))\right| ds = \sup_{t\geq 0}\int_0^{+\infty} G(t,s)\left|g(s,p_2(s)u(s))\right| ds \\ &\leq \int_0^{+\infty} G(\infty,s)\psi_R(s) ds < \infty.\end{aligned}$$

This estimate proves that $T_g^2(\Omega)$ is bounded in E_2.

Let $t_1, t_2 \in [\eta,\zeta] \subset \mathbb{R}^+$ and $u \in \Omega$. By Property (30) of the function G, we obtain

$$\left|T_g^2 u\left(t_2\right) - T_g^2 u\left(t_1\right)\right| \leq \int_0^{+\infty}\left|G(t_2,s)-G(t_1,s)\right| ds \left\|\psi_R\right\|_1 \leq \frac{2\left\|\psi_R\right\|_1}{k^2}\left|t_2-t_1\right|.$$

Proving that $T_g^2(\Omega)$ is equicontinuous on compact intervals of $\mathbb{R}^+$.

We have for all $u \in \Omega$ and $t \geq 0$

$$\left|T_g^2 u(\infty) - T_g^2 u(t)\right| \leq \int_0^{+\infty}\left(G(\infty,s)-G(t,s)\right)\psi_R(s) ds := H(t).$$

Taking in account Property (27) and the fact that

$$(G(\infty, s) - G(t, s))\, \psi_R(s) \leq \frac{1}{k^2}\psi_R(s) \text{ for all } s > 0,$$

where $\psi_R \in L^1(\mathbb{R}^+)$, we obtain by the dominated convergence theorem that $\lim_{t\to+\infty} H(t) = 0$. Thus $T_g^2(\Omega)$ is equiconvergent at $+\infty$ and the proof is complete. ■

Lemma 7 *Assume the function $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is Γ_3-Caratheodory with $i = 1$ or 2. Then the operator T_g^3 is completely continuous.*

Proof. Observe that since g is Γ_3-Caratheodory, for all $u \in E_3$ we have $T_g^3 u \in E_1$. Therefore considering the operator $T_g^{1,3} : E_3 \to E_1$ with $T_g^{1,3}u(t) = T_g^3 u(t)$ and arguing as in the proofs of Lemmas 5, we obtain that $T_g^{1,3}$ is completely continuous. Since $T_g^3 = I_1 \circ T_g^{1,3}$, where I_1 is the continuous embedding of E_1 in E_3, we have that T_g^3 is completely continuous. ■

We obtain from Lemmas 5, 6 and 7 the following fixed point formulation for the bvp (6).

Corollary 4 *Suppose that the function f is Γ_i-Caratheodory for some $i \in \{1, 2, 3\}$. Then $u_i \in E_i$ is a positive solution to the bvp (6) if and only if u_i is a fixed point of T_f^i where $T_f^i : K_i \to K_i$ is completely continuous.*

5 Proofs of Main Results

5.1 Auxiliary Results

Let for $q \in \Delta_i$ with $i = 1, 2, 3$, $L_q^i : E_i \to E_i$ be the linear operator defined by

$$L_q^i u(t) = \int_0^{+\infty} G(t, s)q(s)u(s)ds \quad \text{for } u \in E_i.$$

We have from Lemmas 5, 6 and 7 that for $i = 1, 2, 3$, the linear operator L_q^i is compact. The main goal of this subsection is to prove that for $i = 1, 2, 3$, the operator L_q^i has the SIJP at its spectral radius $r(L_q^i)$ and in particular, L_q^3 is lower bounded on K_3. These results are requirement of Proposition 3, Theorem 5 and Theorem 6, and so are needed for the proofs of the main results of this article. We start by introducing some notations.

Let for $T > 0$, $G_T : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$G_T(t, s) = \begin{cases} G(t, s), & \text{if } t \leq T, \\ G(T, s), & \text{if } t \geq T. \end{cases}$$

and for $i = 1, 2$,

$$E_T = \left\{u \in C\left(\mathbb{R}^+\right) : u(0) = 0 \text{ and } u(t) = u(T) \text{ for } t \geq T\right\},$$

$$X_T = \left\{u \in E_T \cap C^2[0, T] : u'(0) = 0\right\},$$

$$Y_T = X_T \cap C^3[0, T].$$

Equipped respectively with the norms

$$\|u\|_T = \sup_{t\in[0,T]} |u(t)| \text{ for all } u \in E_T,$$

$$\|u\|_X = \max(\|u\|_T, \|u'\|_T, \|u''\|_T) \text{ for all } u \in X_T$$

and

$$\|u\|_Y = \max(\|u\|_X, \|u'''\|_T) \text{ for all } u \in Y_T,$$

E_T, X_T and Y_T become Banach spaces.

In what follows E_T^+ and X_T^+ denote respectively the cones of nonnegative functions in the Banach spaces E_T and X_T. For $q \in \Delta$ and $T > 0$, let $L_{q,T}^i : E_i \to E_i$, $L_{q,T} : E_T \to E_T$, $A_{q,T} : X_T \to X_T$, $\widetilde{L}_{q,T} : E_T \to Y_T$, and $\widetilde{A}_{q,T} : X_T \to Y_T$ be the linear bounded operators defined by

$$L_{q,T}^i u(t) = \int_0^{+\infty} G_T(t,s)q(s)u(s)ds \ \text{ for } u \in E_i,$$

$$\widetilde{L}_{q,T} u = L_{q,T} u = L_{q,T}^i u \ \text{ for } u \in E_T$$

and

$$A_{q,T} u(t) = \widetilde{A}_{q,T} u = L_{q,T} u \text{ for } u \in X_T.$$

Let I, J be respectively the compact embedding of Y_T into E_T and Y_T into X_T. Since $L_{q,T} = I \circ \widetilde{L}_{q,T}$ and $A_{q,T} = J \circ \widetilde{A}_{q,T}$, we have that $L_{q,T}$ and $A_{q,T}$ are compact operators. Moreover, arguing as in the proofs of Lemmas 5 and 6, we obtain that for $i = 1, 2$, $L_{q,T}^i$ is a compact operator.

Lemma 8 *The set O_T defined by*

$$O_T = \left\{ u \in X_T : u' > 0 \ \textit{in} \ (0,T] \ \textit{and} \ u''(0) > 0 \right\},$$

is open in the Banach space X_T.

Proof. We have $O_T^c = F_1 \cup F_2$ where

$$F_1 = \{u \in X_T : \ u'(t_0) \leq 0 \text{ for some } t_0 \in (0,T]\},$$

$$F_2 = \{u \in X_T : u''(0) \leq 0\}.$$

Since F_2 is a closed set in X_T, we have to show that $\overline{F_1} \subset F_1 \cup F_2$. To this aim, let $(u_n) \subset F_1$ with $\lim u_n = u$ and let $(x_n) \subset (0,T]$ be such $u'(x_n) \leq 0$ and $\lim x_n = \overline{x}$. We distinguish the following two cases:

Case 1. $\overline{x} \in (0,T]$: In this case we have

$$u'(\overline{x}) = \lim u_n'(x_n) \leq 0,$$

proving that $u \in F_1$.

Case 2. $\overline{x} = 0$: In this case we have

$$u''(0) = \lim_{n \to \infty} \frac{u_n'(x_n)}{x_n} \leq 0,$$

proving that $u \in F_2$. ■

Lemma 9 *For $i = 1$ or 2, q in Δ_i and $T > 0$, the operator $L_{q,T}^i$ has the SIJP at its spectral radius $r(L_{q,T}^i)$.*

Proof. First, we show that the linear mapping $A_{q,T}$ is strongly positive. Let $u \in X_T^+ \setminus \{0\}$ and $v = A_{q,T} u$, we have from Property (25) of the function G that

$$v'(t) = \int_0^T \frac{\partial G_T}{\partial t}(t,s)q(s)u(s)ds > 0 \text{ for all } t \ \in (0,T). \tag{35}$$

Moreover, we have

$$v''(0) = \int_0^T \frac{\partial^2 G_T}{\partial t^2}(t,s)q(s)u(s)ds > 0. \tag{36}$$

Clearly, (35) and (36) show that $v = A_{q,T} u \in O_T \subset int\left(X_T^+\right)$, proving that

$$A_{q,T}\left(X_T^+ \setminus \{0\}\right) \subset O_T \subset int\left(X_T^+\right) \ \text{ and } \ A_{q,T} \text{ is strongly positive.}$$

Therefore, we conclude from Proposition 2 that the operator $A_{q,T}$ has the SIJP at $r(A_{q,T})$.

Now, we are able to prove that the operator $L_{q,T}$ has the SIJP at $r(L_{q,T})$. Let $\mu_0 > 0$ and $u \in E_T^+ \setminus \{0\}$ such that $L_{q,T}u \geq \mu_0 u$, then $U = L_{q,T}u \in X_T^+ \setminus \{0\}$ and satisfies $L_{q,T}U = A_{q,T}U \geq \mu_0 U$. Hence, we have that $\mu_0 \in \Lambda_{A_{q,T}}$ and $\mu_0 \leq \sup \Lambda_{A_{q,T}} = r(A_{q,T})$.

Similarly if $\eta_0 \geq 0$ and $v \in E_T^+ \setminus \{0\}$ are such that $L_{q,T}v \leq \eta_0 v$, then $V = L_{q,T}v \in X_T^+ \setminus \{0\}$ and satisfies $L_{q,T}V = A_{q,T}V \leq \eta_0 V$. Therefore, we have that

$$\eta_0 \in \Gamma_{A_{q,T}} \quad \text{and} \quad \eta_0 \geq \inf \Gamma_{A_{q,T}} = r(A_{q,T}).$$

Therefore, we have proved that

$$\sup \Lambda_{L_{q,T}} \leq r\left(A_{q,T}\right) = \inf \Gamma_{A_{q,T}} = \sup \Lambda_{A_{q,T}} \leq \inf \Gamma_{L_{q,T}}$$

and this combined with (17) leads to $\inf \Gamma_{L_{q,T}} = \sup \Lambda_{L_{q,T}} = r(A_{q,T})$ and $L_{q,T}$ has the SIJP at $r(A_{q,T})$. Since the cone E_T^+ is total in the Banach space E_T, we have that $r\left(L_{q,T}\right)$ is a positive eigenvalue. Hence taking in consideration Remark 1, we obtain that $r\left(L_{q,T}\right) = r\left(A_{q,T}\right)$ and $L_{q,T}$ has the SIJP at $r(L_{q,T})$.

Noticing that for all $u \in K_i \setminus \{0\}$,

$$U = L_{q,T}^i u \in E_T^+ \setminus \{0\} \quad \text{and} \quad L_{q,T}^i U = L_{q,T}U,$$

then arguing as above we obtain that $L_{q,T}^i$ has the SIJP at $r(L_{q,T}^i)$. Ending the proof. ■

Theorem 7 *For $i = 1$ or 2 and q in Δ_i the operator L_q^i has the SIJP at its spectral radius $r(L_q^i)$.*

Proof. In order to make use of Theorem 4 we prove that for a function q in Δ_i, $T \to L_{q,T}^i$ is increasing and $\lim_{T\to+\infty} L_{q,T}^i = L_q^i$. Let q in Δ_i and T_1, T_2 be such that $0 < T_1 < T_2 < \infty$. For $u \in K_i$ we have

$$L_{q,T_2}^i u(t) - L_{q,T_1}^1 u(t) = \begin{cases} \int_0^{+\infty} \left(G(t,s) - G(t,s)\right) q(s)u(s)ds = 0, & \text{if } t \leq T_1, \\ \int_0^{T_1} \left(G(t,s) - G(T_1,s)\right) q(s)u(s)ds \geq 0, & \text{if } T_1 < t \leq T_2, \\ \int_0^{T_1} \left(G(T_2,s) - G_{T_1}(T_1,s)\right) q(s)u(s)ds \geq 0, & \text{if } T_2 < t, \end{cases}$$

proving that $L_{q,T_2}^i u - L_{q,T_1}^i u \in K_i$ and $L_{q,T_1}^i \leq L_{q,T_2}^i$.

If $i = 1$, for $u \in E_1$ with $\|u\|_1 = 1$, we have

$$\begin{aligned} \left|\tfrac{L_q^1 u(t) - L_{q,T}^1 u(t)}{p_1(t)}\right| &\leq \tfrac{1}{1+t} \int_0^{+\infty} \left(G(t,s) - G_T(t,s)\right) q(s)ds \\ &= \begin{cases} 0, \text{ if } t \leq T, \\ \tfrac{1}{1+t} \int_0^{+\infty} \left(G(t,s) - G(T,s)\right) q(s)ds, \text{ if } t \geq T. \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{t\geq 0} \left|\frac{L_q^1 u(t) - L_{q,T}^1 u(t)}{1+t}\right| &= \sup_{t\geq T} \left(\frac{1}{1+t} \int_0^{+\infty} \left(G(t,s) - G(T,s)\right) q(s)ds\right) \\ &\leq \sup_{t\geq T} \left(\frac{1}{1+t} \int_0^{+\infty} G(t,s)q(s)ds\right). \end{aligned}$$

Since

$$\lim_{t\to+\infty} \left(\tfrac{1}{1+t} \int_0^{+\infty} G(t,s)q(s)ds\right) = 0,$$

we have

$$\begin{aligned} \lim_{T\to+\infty} \left(\sup_{\|u\|_2=1} \left\|L_q^1 u - L_{q,T}^1 u\right\|_1\right) &= \lim_{T\to+\infty} \left(\sup_{\|u\|_1=1} \left(\sup_{t\geq 0} \left|\tfrac{L_q^1 u(t) - L_{q,T}^1 u(t)}{1+t}\right|\right)\right) \\ &\leq \lim_{T\to+\infty} \left(\sup_{t\geq T} \left(\tfrac{1}{1+t} \int_0^{+\infty} G(t,s)q(s)ds\right)\right) = 0. \end{aligned}$$

Hence we obtain by Theorem 4 that the operator L_q^1 has the SIJP at its spectral radius $r(L_q^1)$.
If $i=2$, for $u \in E_2$ with $\|u\|_2 = 1$ we have

$$\begin{aligned} \left|L_q^2 u(t) - L_{q,T}^2 u(t)\right| &\leq \int_0^{+\infty} \left(G(t,s) - G_T(t,s)\right) q(s) ds \\ &= \begin{cases} 0, \text{ if } t \leq T, \\ \int_0^{+\infty} \left(G(t,s) - G(T,s)\right) q(s) ds, \text{ if } t \geq T. \end{cases} \end{aligned}$$

Hence we have

$$\left\|L_q^2 - L_{q,T}^2\right\| = \sup_{\|u\|_2 = 1} \left\|L_q^2 u - L_{q,T}^2 u\right\|_2 \leq \int_0^{+\infty} \left(G(t,s) - G(T,s)\right) q(s) ds,$$

then by Lebesgue dominated convergence theorem we conclude that $L_{q,T}^2 \to L_q^2$ as $T \to +\infty$. By Theorem 4, we obtain that the operator L_q^2 has the SIJP at its spectral radius $r(L_q^2)$. ■

Theorem 8 *For $i = 1$ or 2 and q in Δ_3 the operator L_q^3 has the SIJP at its spectral radius $r(L_q^3)$ and L_q^3 is bounded on the cone K_3 from below.*

Proof. Notice first that for all $u \in K_3$, $L_q^3 u \in K_1$. Indeed, we have for $u \in K_3$ and for all $t > 0$

$$\frac{L_q^3 u(t)}{1+t} \leq \frac{||u||_3}{1+t} \int_0^{+\infty} G(t,s) \left(e^{ks} q(s)\right) ds \to 0 \text{ as } t \to +\infty,$$

since $\lim_{s \to +\infty} e^{ks} q(s) = 0$, and

$$\left(L_q^3 u\right)'(t) = \int_0^{+\infty} \frac{\partial G}{\partial t}(t,s) q(s) u(s) ds > 0.$$

Let now, $\lambda_0 > 0$ and $u \in K_3 \setminus \{0\}$ be such that $L_q^3 u \leq \lambda_0 u$. Then $U = L_q^3 u$ satisfies $L_q^1 U = L_q^3 U \leq \lambda_0 U$ and we have $\lambda_0 \geq \inf \Gamma_{L_q^1} = r(L_q^1)$. Similarly if $\theta_0 > 0$ and $u \in K_3 \setminus \{0\}$ are such that $L_q^3 u \geq \theta_0 u$ then $U = L_q^3 u \in K_1 \setminus \{0\}$ and satisfies $L_q^1 U = L_q^3 U \geq \theta_0 U$ and we have $\theta_0 \leq \sup \Lambda_{L_q^1} = r(L_q^1)$.

The above leads to $r(L_q^1) = \inf \Gamma_{L_q^1} = \sup \Lambda_{L_q^1}$and the operator L_q^3 has the SIJP at $r(L_q^1)$. Since the cone K_3 is total in the Banach space E_3 and Remark 1 claims that $r\left(L_q^1\right)$ is the unique positive eigenvalue of the positive operator L_q^3, we have that $r\left(L_q^3\right) = r\left(L_q^1\right)$ and L_q^3 has the SIJP at $r(L_q^3)$.

It remains to show that L_q^3 is lower bounded on K_3. Let $u \in K_3$, with $\|u\|_3 = 1$, we have then for all $t \geq 0$,

$$L_q^3 u(t) = \int_0^{+\infty} G(t,s) q(s) u(s) ds \geq \int_0^{+\infty} G(t,s) q(s) \gamma(s) ds,$$

leading to

$$\inf \left\{ \left\|L_q^3 u\right\|_3 : u \in K_3 \cap \partial B(0_{E_3}, 1) \right\} \geq \sup_{t \geq 0} e^{-kt} \int_0^{+\infty} G(t,s) q(s) \gamma(s) ds > 0$$

and the operator L_q^3 is lower bounded on the cone K_3 from below. This ends the proof. ■

5.2 Proof of Proposition 1

Let $q \in \Delta$, we have from Lemma 2 that μ is a positive eigenvalue of the linear eigenvalue problem (7) if and only if μ^{-1} is a positive eigenvalue of the compact operator L_q^i for $i = 1$ or 2. Since Theorem 7 claims that L_q^i has the SIJP at $r(L_q^i)$, we have from Remark 1 that $r(L_q^i)$ is the unique positive eigenvalue of L_q^i. Therefore, we have that $\mu(q) = 1/r(L_q^i)$ is the unique positive eigenvalue of the linear eigenvalue problem (7).

Now, let ϕ be the eigenfunction associated with $\mu(q)$. Clearly if $q \in \Delta_2$ then ϕ is bounded and if not then ϕ satisfies

$$\begin{aligned}
\phi(t) &= \int_0^{+\infty} G(t,s)q(s)\phi(s)ds \geq \frac{1}{k^2}\int_1^t \left(-e^{-kt}\sinh(ks) + (1-e^{-ks})\right) q(s)\phi(s)ds \\
&\geq \frac{\left(1-e^{-k}\right)^2}{2k^2}\int_1^t q(s)\phi(s)ds \\
&\geq \frac{\left(1-e^{-k}\right)^2}{2k^2}\phi(1)\int_1^t q(s)ds. \qquad (37)
\end{aligned}$$

Thus, suppose to the contrary that ϕ is bounded, then passing to the limits in (37), we obtain the contradiction

$$+\infty > \lim_{t\to+\infty}\phi(t) = \lim_{t\to+\infty}\frac{\left(1-e^{-k}\right)^2}{2k^2}\phi(1)\int_1^t q(s)ds = +\infty.$$

Ending the proof.

5.3 Proof of Theorem 1

Assume that Hypothesis (8) holds true (the case where (9) holds is checked similarly). Let $\epsilon > 0$ be so small such that for $i = 1, 2$,

$$\inf\left\{\frac{f(t,p_i(t)u)}{p_i(t)q(t)u} : t,u > 0\right\} \geq (\mu(q)+\epsilon).$$

Hence for all $u \in K_i$, we have

$$\begin{aligned}
T_f^i u(t) &= \int_0^{+\infty} G(t,s)f(s,u(s))ds \\
&= \int_0^{+\infty} G(t,s)f(s,p_i(s)\frac{u(s)}{p_i(s)})ds \\
&\geq (\mu(q)+\epsilon)\int_0^{+\infty} G(t,s)q(s)u(s)ds \\
&= (\mu(q)+\epsilon)\, L_q^i u(t) := \widehat{L}_q^i u(t)
\end{aligned}$$

and

$$r(\widehat{L}_q^i) = \frac{\mu(q)+\epsilon}{\mu\,(q)} > 1.$$

Since Theorems 7 and 8 state that the operator $\widehat{L}_q^i$ has the SIJP at $r(\widehat{L}_q^i)$, Hypothesis (18) holds and Proposition 3 guarantees that the operator T_f^i has no fixed point in K_i. Thus, we conclude by Corollary 4 that the bvp (6) has no positive solution.

5.4 Proof of Theorem 2

Step 1. Existence in the case where (10) is satisfied

Let $\epsilon \in \left(0, \mu(q_\infty) - f_{i,+\infty}^{+}(q_\infty)\right)$ there is R such that

$$f(t,p_i(t)u) \leq (\mu\,(q_\infty)-\epsilon)\,p_i(t)q_\infty(t)u \text{ for all } t \geq 0 \text{ and } u \geq R.$$

Since the function f is Γ_i-Caratheodory, there is $\psi_R \in \Gamma_i$ such that

$$f(t,p_i(t)u) \leq (\mu\,(q_\infty)-\epsilon)\,p_i(t)q_\infty(t)u + \psi_R\,(t) \text{ for all } t,u \geq 0,$$

and this leads to

$$f(t,u) \leq (\mu(q_\infty) - \epsilon) q_\infty(t)u + \psi_R(t) \text{ for all } t, u \geq 0. \tag{38}$$

Let $\varepsilon \in \left(0, f_{i,0}^{-}(q_0) - \mu(q_\infty)\right)$ there is $r > 0$ such that for all $t \geq 0$ and $u \in [0, r]$

$$\left(f_{i,0}^{-}(q_0) + \varepsilon\right) p_i(t) q_0(t) u \geq f(t, p_i(t)u) \geq (\mu(q_\infty) + \varepsilon) p_{1i}(t) q_0(t) u,$$

leading to

$$\left(f_{i,0}^{-}(q_0) + \varepsilon\right) q_0(t) u \geq f(t,u) \geq (\mu(q_\infty) + \varepsilon) q_0(t) u \text{ for all } t \geq 0 \text{ and } u \in [0, r].$$

Therefore, for all $t, u \geq 0$ we have

$$\left(f_{i,0}^{-}(q_0) + \varepsilon\right) q_0(t) u + \widehat{f}(t,u) \geq f(t,u) \geq (\mu(q_0) + \varepsilon) q_0(t) u - \widetilde{f}(t,u), \tag{39}$$

where

$$\widetilde{f}(t,u) = \sup\left(0, (\mu(q_\infty) + \varepsilon) q_0(t) u - f(t,u)\right),$$
$$\widehat{f}(t,u) = \sup\left(0, f(t,u) - \left(f_{i,0}^{-}(q_0) + \varepsilon\right) q_0(t) u\right).$$

Therefore, we obtain from (38) and (39) that

$$T_f^i u \leq L_{q_\infty}^i u + F_\infty u \text{ for all } u \in K_i,$$

and

$$L_{q_0}^i u - F_0 u \leq T_f^i u \leq L_{q_0}^i u + \widehat{F}_0 u \text{ for all } u \in K_i$$

where

$$F_0 u(t) = \int_0^{+\infty} G(t,s) \widetilde{f}(t, u(s)) ds,$$

$$\widehat{F}_0 u(t) = \int_0^{+\infty} G(t,s) \widehat{f}(t, u(s)) ds,$$

$$F_\infty u(t) = \int_0^{+\infty} G(t,s) \psi_R(s)\, ds,$$

$$r\left(L_{q_\infty}^i\right) = \frac{(\mu(q_\infty) - \epsilon)}{\mu(q_\infty)} < 1 < r\left(L_{q_0}^i\right) = \frac{(\mu(q_0) + \varepsilon)}{\mu(q_0)}.$$

We conclude from Theorem 7, Theorem 5 and Corollary 4 that the bvp (6) admits a positive solution $u \in K_i$.

Step 2. Existence in the case where (11) is satisfied

Let $\epsilon \in \left(0, \mu_i(q_0) - f_{i,0}^{+}(q_0)\right)$ there is $r > 0$ small such that

$$f(t, p_i(t)u) \leq (\mu(q_\infty) - \epsilon) p_i(t) q_\infty(t) u \text{ for all } t \geq 0 \text{ and } u \leq r,$$

leading to

$$f(t,u) \leq (\mu(q_0) - \epsilon) q_0(t) u \text{ for all } t \geq 0 \text{ and } u \leq r.$$

Therefore, for all $t, u \geq 0$ we have

$$f(t,u) \leq (\mu(q_0) - \epsilon) q_0(t) u + \widehat{f}(t,u), \tag{40}$$

with

$$\widehat{f}(t,u) = \sup\left(0, f(t,u) - (\mu(q_0) - \epsilon) q_0(t) u\right).$$

Let $\varepsilon \in \left(0, f_{i,\infty}^{-}(q_\infty) - \mu_i(q_\infty)\right)$ there is $R > 0$ such that for all $t \geq 0$ and $u \geq R$,

$$(\mu(q_\infty) + \varepsilon) p_i(t) q_\infty(t) u \leq f(t, p_i(t)u) \leq \left(f_{i,\infty}^{+}(q_\infty) + \varepsilon\right) p_i(t) q_\infty(t) u,$$

Since the nonlinearity f is a Γ_i-Caratheodory function, there is $\psi_R \in \Gamma_i$ such that

$$f(t,u) \leq \left(f^{+}_{i,\infty}(q_\infty) + \varepsilon\right) q_\infty(t) p_i(t) u + \psi_R(t) \text{ for all } t, u \geq 0 .$$

Therefore, for all $t, u \geq 0$ we have

$$\left(\mu_i\left(q_\infty\right) + \varepsilon\right) q_\infty(t) u - \widetilde{f}(t,u) \ \leq f(t,u) \leq \left(f^{+}_{i,\infty}(q_\infty) + \varepsilon\right) q_\infty(t) u + \psi_R(t), \tag{41}$$

where

$$\widetilde{f}(t,u) = \sup\left(0, \left(\mu\left(q_\infty\right) + \varepsilon\right) q_\infty(t) u - f(t,u)\right).$$

Therefore, we obtain from (40) and (41) that

$$T^i_f u \leq L^i_{q_0} u + F_0 u \text{ for all } u \in K_i$$

and

$$L^i_{q_\infty} u - F_\infty u \leq T^i_f u \leq L^i_{q_\infty} u + \widehat{F}_\infty u \text{ for all } u \in K_i,$$

where

$$F_0 u(t) = \int_0^{+\infty} G(t,s) \widehat{f}(t, u(s)) ds,$$

$$\widehat{F}_\infty u(t) = \int_0^{+\infty} G(t,s) \psi_R(s)\, ds,$$

$$F_\infty u(t) = \int_0^{+\infty} G(t,s) \widetilde{f}(t, u(s)) ds,$$

$$r\left(L^i_{q_0}\right) = \frac{\left(\mu\left(q_\infty\right) - \epsilon\right)}{\mu\left(q_\infty\right)} < 1 < r\left(L^i_{q_\infty}\right) = \frac{\left(\mu\left(q_0\right) + \varepsilon\right)}{\mu\left(q_0\right)}.$$

We conclude from Theorem 7, Theorem 5 and Corollary 4 that the bvp (6) admits a positive solution $u \in K_i$.

Step 3. Boundedness and unboundedness of the solution

Evidently, if $i = 1$ the solution u is bounded. If $i = 2$ and Hypothesis (12) is fulfilled, then the solution u satisfies

$$u(t) = \int_0^{+\infty} G(t,s) f(s, u(s)) ds \geq \frac{\left(1 - e^{-k}\right)^2}{2k^2} \int_1^t f(s, u(s)) ds = \frac{\left(1 - e^{-k}\right)^2}{2k^2} \int_1^t f(s, p_1(s) \left(\frac{u(s)}{p_1(s)}\right)) ds. \tag{42}$$

Thus, suppose to the contrary that the solution u is bounded, then passing to the limits in (42), we obtain the contradiction

$$+\infty > \lim_{t \to +\infty} u(t) = \lim_{t \to +\infty} \frac{\left(1 - e^{-k}\right)^2}{2k^2} \int_1^t f(s, p_1(s) \left(\frac{u(s)}{p_1(s)}\right)) ds = +\infty.$$

Ending the proof.

5.5 Proof of Theorem 3

Step 1. Existence in the case where (13) is satisfied

Let $\epsilon \in \left(0, \mu(q_\infty) - f^{+}_{i,3,\infty}(q_\infty)\right)$, there is R such that

$$f(t, p_3(t) u) \leq \left(\mu_1\left(q_\infty\right) - \epsilon\right) p_3(t) q_\infty(t) u \text{ for all } t \geq 0 \text{ and } u \geq R.$$

Since the nonlinearity f is a Γ_3-Caratheodory function, there is $\psi_R \in \Gamma_1$ such that

$$f(t, p_3(t) u) \leq \left(\mu\left(q_\infty\right) - \epsilon\right) p_3(t) q_\infty(t) u + \psi_R(t) \text{ for all } t, u \geq 0,$$

and this leads to

$$f(t,u) \leq (\mu(q_\infty) - \epsilon) q_\infty(t)u + \psi_R(t) \text{ for all } t,u \geq 0. \tag{43}$$

Also, we have from $f^-_{3,0}(q_0) > \mu(q_0)$ that for $\varepsilon \in \left(0, f^-_{3,0}(q_0) - \mu(q_\infty)\right)$ there is $r > 0$ such that

$$f(t, p_3(t)u) \geq (\mu(q_\infty) + \varepsilon) p_3(t)q_0(t)u \text{ for all } t \geq 0 \text{ and } u \in [0,r],$$

leading to

$$f(t,u) \geq (\mu(q_\infty) + \varepsilon) q_0(t)u \text{ for all } t \geq 0 \text{ and } u \in [0,r].$$

Therefore we have

$$f(t,u) \geq (\mu(q_0) + \varepsilon) q_0(t)u - \widetilde{f}(t,u) \text{ for all } t,u \geq 0, \tag{44}$$

where

$$\widetilde{f}(t,u) = \sup(0, (\mu(q_\infty) + \varepsilon) q_0(t)u - f(t,u)).$$

Hence, we obtain from (43) and (44) that

$$L^3_{q_0}u - F_0u \leq T^3_f u \leq L^3_{q_\infty}u + F_\infty u \text{ for all } u \in K_3,$$

where

$$F_0u(t) = \int_0^{+\infty} G(t,s)\widetilde{f}(t,u(s))ds,$$

$$F_\infty u(t) = \int_0^{+\infty} G(t,s)\psi_R(s)\, ds,$$

$$r\left(L^3_{q_\infty}\right) = \frac{(\mu(q_\infty) - \epsilon)}{\mu(q_\infty)} < 1 < r\left(L^3_{q_0}\right) = \frac{(\mu(q_0) + \varepsilon)}{\mu(q_0)}.$$

We conclude from Theorem 8, Theorem 6 and Corollary 4 that the bvp (6) admits a positive solution.

Step 2. Existence in the case where (14) is satisfied

Let $\epsilon \in \left(0, \mu(q_0) - f^+_{3,0}(q_0)\right)$, there is $r > 0$ such that

$$f(t, p_3(t)u) \leq (\mu(q_0) - \epsilon) p_3(t)q_0(t)u \text{ for all } t \geq 0 \text{ and } u \leq r.$$

Hence for all $t, u \geq 0$ we have

$$f(t,u) \leq (\mu(q_0) - \epsilon) q_0(t)u + \widetilde{f}(t,u), \tag{45}$$

where

$$\widetilde{f}(t,u) = \sup(0, (f(t,u) - (\mu(q_0) - \epsilon) q_0(t)u).$$

Let $\varepsilon \in \left(0, f^-_{3,\infty}(q_0) - \mu(q_\infty)\right)$ there is $R > 0$ such that

$$f(t, p_3(t)u) \geq (\mu(q_\infty) + \varepsilon) p_3(t)q_\infty(t)u \text{ for all } t \geq 0 \text{ and } u \geq R,$$

leading to

$$f(t,u) \geq (\mu(q_\infty) + \varepsilon) q_\infty(t)u \text{ for all } t \geq 0 \text{ and } u \geq R.$$

Therefore, we have

$$f(t,u) \geq (\mu(q_\infty) + \varepsilon) q_\infty(t)u - \widehat{f}(t,u) \text{ for all } t,u \geq 0, \tag{46}$$

where

$$\widehat{f}(t,u) = \sup(0, (\mu(q_\infty) + \varepsilon) q_\infty(t)u - f(t,u)).$$

Hence, we obtain from (45) and (46) that

$$L^3_{q_\infty}u - F_\infty u \leq T^3_f u \leq L^3_{q_0}u + F_0u \text{ for all } u \in K_3,$$

where

$$F_0 u(t) = \int_0^{+\infty} G(t,s)\widetilde{f}(t,u(s))ds,$$

$$F_\infty u(t) = \int_0^{+\infty} G(t,s)\widehat{f}(t,u(s))ds,$$

$$r\left(L^3_{q_0}\right) = \frac{(\mu(q_0) - \epsilon)}{\mu(q_0)} < 1 < r\left(L^3_{q_\infty}\right) = \frac{(\mu(q_\infty) + \varepsilon)}{\mu(q_\infty)}.$$

We conclude from Theorem 8, Theorem 6 and Corollary 4 that the bvp (6) admits a positive solution.

Step 3. Boundedness and unboundedness of the solution

Evidently, if f is a Γ_4-Caratheodory function the solution u is bounded. If Hypothesis (15) is fulfilled, then the solution u satisfies

$$u(t) = \int_0^{+\infty} G(t,s)f(s,u(s))ds \geq \frac{\left(1-e^{-k}\right)^2}{2k^2}\int_1^t f(s,u(s)d = \frac{\left(1-e^{-k}\right)^2}{2k^2}\int_1^t f(s,p_3(s)\left(\frac{u(s)}{p_3(s)}\right))ds. \quad (47)$$

Thus, by the contrary if the solution u is bounded then passing to the limits in (47) we obtain the contradiction

$$\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty}\frac{\left(1-e^{-k}\right)^2}{2k^2}\int_1^t f(s,p_3(s)\left(\frac{u(s)}{p_3(s)}\right))ds = +\infty.$$

Ending the proof.

Acknowledgment. The authors are thankful to the anonymous referee for his deep and careful reading of the manuscript and for all his comments and suggestions, which led to a substantial improvement of the original manuscript.

References

[1] R. P. Agarwal and D. O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Academic Publisher, Dordrecht, 2001.

[2] C. Bai and C. Li, Unbounded upper and lower solution method for third-order boundary value problem on the half-line, Electron. J. Differential Equations, 2009(2009), 1–12.

[3] Y. Bao, L. Wang and M. Pei, Existence of positive solutions for a singular third-order two point boundary value problem on the half-line, Bound. Value Probl., (2022), 11 pp.

[4] Z. Benbaziz and S. Djebali, On a singular multi-point third-order boundary value problem on the half-line, Math. Bohem., 145(2020), 305–324.

[5] A. Benmezaï, Fixed point theorems in cones under local conditions, Fixed Point Theory, 18(2017), 107–126.

[6] A. Benmezaï, B. Boucheneb, J. Henderson and S. Mechrouk, The index jump property for 1-homogeneous positive maps and fixed point theorems in cones, J. Nonlinear Funct. Anal., 2017(2017), Article ID 6.

[7] F. Bernis and L. A. Petelier, Two problems from draining flows involving third-order ordinary differential equations, SIAM J. Math. Anal., 27(1996), 515–527.

[8] C. Corduneanu, Integral Equations and Stability of Feedback Systems, Academic Press, New York, 1973.

[9] S. Djebali and O. Saifi, Singular ϕ-Laplacian third-order BVPs with derivative dependence, Arch. Math., 52(2016), 35–48.

[10] S. Djebali and O. Saifi, Upper and lower solution for ϕ-Laplacian third-order BVPs on the half-line, Cubo, 16(2014), 105–116.

[11] S. Djebali and O. Saifi, Third order BVPs with ϕ-Laplacian operators on $[0, +\infty)$, Afr. Diaspora J. Math., 16(2013), 1–17.

[12] S. Djebali and O. Saifi, Positive solutions for singular BVPs with sign changing and derivative depending nonlinearity on the half-line, Acta Appl. Math., 110(2010), 639–665.

[13] X. Feng, H. Feng and H. Tan, Existence and iteration of positive solutions for third-order Sturm-Liouville boundary value problem, Appl. Math. Comput., 266 (2015), 634–641.

[14] Y. Feng, On the existence and multiplicity of positive periodic solutions of a nonlinear third-order equation, Appl. Math. Lett., 22(2009), 1220–1224.

[15] N. Finizio and G. Ladas, Ordinary Differential Equations with Modern Applications, Third Edition, Wadsworth Pub. Co., Belmont, 1988.

[16] D. Fu and W. Ding, Existence of positive solutions of third-order boundary value problems with integral boundary conditions in Banach spaces, Adv. Difference Equ., 65(2013), 12 pp.

[17] J. R. Graef, L. Kong and B. Yong, Positive solutions for third-order multi-point singular boundary value problems, Czechoslovak Math. J., 60(2010), 173–182.

[18] Y. Guo, Y. Liu and Y. Liang, Positive solutions for the third-order boundary value problems with the second derivatives, Bound. Value Probl., (2012), 9 pp.

[19] S. A. Iyase, On a third-order three point boundary value problem at resonance on the half-line, Arab. J. Math., 8(2019), 43–53.

[20] Z. Jackiewicz, M. Klaus and C. O'Cinneide, Asymptotic behaviour of solutions to Volterra integro-differential equations, J. Integral Equations Appl., 1(1988), 501–516.

[21] D. Jiang and R. P. Agarwal, Uniqueness and existence theorem for a singular third-order Boundary value problem on $[0, +\infty)$, Appl. Math. Lett., 15(2002), 445–451.

[22] D. Krajcinovic, Sandwich Beam Analysis, Appl. Mech., 1(1972), 773–778.

[23] Y. Kuramoto and T. Yamada, Turbulent State in Chemical Reaction, Progress of Theoretical Physics, 56(1976), 679–681.

[24] H. Lian and J. Zhao, Existence of unbounded solution for a third-order boundary value problem on infinite intervals, Discrete Dyn. Nat. Soc., (2012), 14 pp.

[25] S. Liang and J. Zhang, Positive solutions for singular third-order boundary-value problem with dependence on the first order derivative on the half-line, Acta. Appl. Math., 111(2010), 27–43.

[26] Z. Liu, H. Chen and C. Liu, Positive solutions for singular third-order nonhomogeneous boundary value problems, J. Appl. Math. Comput., 38(2012), 161–172.

[27] H. P. McKean, Nagumo's equation, Advances in Mathematics, 4(1970), 209–223.

[28] D. Michelson, Steady solutions of the Kuramoto-Sivashinsky equation, Physica D, 19(1986), 89–111.

[29] P. K. Palamides and R. P. Agarwal, An existence result for a singular third-order boundary value problem on $[0, +\infty)$, Appl. Math. Lett., 21(2008), 1254–1259.

[30] H. Pang, W. Xie and L. Cao, Successive iteration and positive solutions for a third-order boundary value problem involving integral conditions, Bound. Value Probl., 139(2015), 10 pp.

[31] H. Shi, M. Pei and L. Wang, Solvability of a $t$hird-order three point boundary value problem on a half-line, Bull. Malays. Math. Sci. Soc., 38(2015), 909–926.

[32] Y. Sun, Triple positive solutions for a class of third-order p-Laplacian singular boundary value problems, J. Appl. Math. Comput., 37(2011), 587–599.

[33] Z. Wei, Some necessary and sufficient conditions for existence of positive solutions for third-order singular sublinear multi-point boundary value problems, Acta Math. Sin., 34 B, 6(2014), 1795–1810.

[34] Z. Wei, Some necessary and sufficient conditions for existence of positive solutions for third-order singular super-linear multi-point boundary value problems, J. Appl. Math. Comput., 46(2014), 407–422.

[35] Y. Wu and Z. Zhao, Positive solutions for a third-order boundary value problems with change of signs, Appl. Math. Comput., 218(2011), 2744–2749.

[36] J. Zhang, Z. Wei and W. Dong, The method of lower and upper solutions for third-order singular four-point boundary value problems, J. Appl. Math. Comput., 36(2011), 275–289.