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On Turan's Inequality Concerning Polynomials*

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Abstract

In this paper, we prove some inequalities that relate the uniform norm of the derivative of a complex polynomial having μ-fold zero at origin and the uniform norm of the polynomial itself. We further extend the obtained result to the polar derivative of a polynomial.

1 Introduction

Let $P(z)$ be a polynomial of degree n and $P'(z)$ its derivative. The comparison of the norm of $P(z)$ and that of $P'(z)$ on the unit circle is given by Turan's inequality [9] which states that, if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1}$$

Equality in (1) holds for polynomials having all zeros on $|z| = 1$. As a generalisation of inequality (1) to the polynomials having all their zeros in $|z| \leq k$ where $k \geq 1$, Govil [3] proved if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{2}$$

Inequality (2) is sharp and equality holds for the polynomial $P(z) = z^n + k^n$. Dubinin [2] established the refinement of inequality (1) by introducing the extreme coefficients of the polynomial involved. He proved that if $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ has all zeros in the disc $|z| \leq 1$, then for each z on $|z| = 1$ for which $P'(z) \neq 0$, the following inequality holds

$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{2} + \frac{|a_n| - |a_0|}{2(|a_n| + |a_0|)}.$$

Taking into consideration the size of each zero of $P(z)$, Aziz [1] established the following generalisation of inequality (2) for the class of polynomials having all their zeros in $|z| \leq k$, where $k \geq 1$ by proving that if $P(z) = a_n \prod_{\nu=1}^{n} (z - z_\nu)$ is a complex polynomial of degree n with $|z_\nu| \leq k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{2}{1+k^n} \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \max_{|z|=1} |P(z)|. \tag{3}$$

However the bound in inequality (3) was recently improved by Kumar [5] by involving the modulus of each zero and some of the coefficients of the underlying polynomial. In fact, he proved that if $P(z) = a_n \prod_{\nu=1}^{n} (z - z_\nu)$ is a polynomial of degree n with $|z_\nu| \leq k,\ 1 \leq \nu \leq n$ and $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \left\{\frac{2}{1+k^n} + \frac{(|a_n|k^n - |a_0|)(k-1)}{(1+k^n)(|a_n|k^n + |a_0|k)}\right\} \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \max_{|z|=1} |P(z)|. \tag{4}$$

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Very recently, Milovanovic and Mir [6] strengthened inequality (4) by involving the minimum value of $|P(z)|$ on $|z|=k$. They proved that if $P(z)=\prod_{\nu=1}^{n}(z-z_\nu)$ has all its zeros in $|z|\le k,\ k\ge 1$, then for each t with $0\le t\le 1$, the following inequality holds

$$\begin{aligned}\max_{|z|=1}|P'(z)| \quad &\geq \quad \sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\Bigg[\left\{\frac{2}{1+k^n}+\frac{(k^n-|a_0|-tm)(k-1)}{(1+k^n)(k^n+k|a_0|-tm)}\right\}\max_{|z|=1}|P(z)| \\ &\quad +\left\{\frac{k^n-1}{k^n(1+k^n)}-\frac{(k^n-|a_0|-tm)(k-1)}{(1+k^n)(k^n+k|a_0|-tm)}\right\}tm\Bigg], \qquad (5)\end{aligned}$$

where $m=\min_{|z|=k}|P(z)|$.

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial of degree n with respect to a real or complex number α, then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

The polar derivative $D_\alpha P(z)$ is a polynomial of degree at most $n-1$. Furthermore, it generalises the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha}=P'(z)$$

uniformly with respect to z for $|z|\le R,\ R>0$.

For more information about the polar derivative of a polynomial one can refer monographs by Rahman and Schmeisser or Milovanovic et al. [7].

Over the last few decades many different authors produced a large number of interesting versions and generalizations of the above inequalities by introducing restrictions on the multiplicity of zero at $z=0$, the modulus of largest root of $P(z)$, restrictions on coefficients, etc. Many of these generalizations involve the comparison of polar derivative $D_\alpha P(z)$ with various choices of $P(z),\alpha$ and other parameters. For the latest research and development pertaining to this topic see ([8], [10]). For the class of polynomials having all their zeros in $|z|\le k,\ k\ge 1$, Govil and Kumar [4] recently proved that if $P(z)=z^s(a_0+a_1z+\cdots+a_{n-s}z^{n-s})$, $0\le s\le n$ is a polynomial of degree n having all its zeros in $|z|\le k,\ k\ge 1$, then for any complex number α with $|\alpha|\ge k$,

$$\max_{|z|=1}|D_\alpha P(z)|\ge(|\alpha|-k)\left(\frac{n+s}{1+k^n}-\frac{|a_{n-s}|k^{n-s}-|a_0|}{(1+k^n)(|a_{n-s}|k^{n-s}+|a_0|)}\right)\max_{|z|=1}|P(z)|.$$

Milovanovic and Mir [6] also generalised inequality (5) to the polar derivative of a polynomial by establishing that if $P(z)=\prod_{\nu=1}^{n}(z-z_\nu)$ has all its zeros in $|z|\le k,\ k\ge 1$, then for any complex number α with $|\alpha|\ge k$ and $0\le t\le 1$, the following inequality holds

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \quad &\geq \quad \sum_{\nu=1}^{n}\frac{k(|\alpha|-k)}{k+|z_\nu|}\Bigg[\left\{\frac{2}{1+k^n}+\frac{(k^n-|a_0|-tm)(k-1)}{(1+k^n)(k^n+k|a_0|-tm)}\right\}\max_{|z|=1}|P(z)| \\ &\quad +\left\{\frac{k^n-1}{k^n(1+k^n)}-\frac{(k^n-|a_0|-tm)(k-1)}{(1+k^n)(k^n+k|a_0|-tm)}\right\}tm\Bigg], \qquad (6)\end{aligned}$$

where $m=\min_{|z|=k}|P(z)|$.

In this paper, we extend inequality (5) to the class of polynomials having μ-fold zero at origin. In fact, we prove

Theorem 1 *If $P(z)=z^\mu(a_0+a_1z+...+a_{n-\mu}z^{n-\mu})=a_{n-\mu}z^\mu\prod_{j=1}^{n-\mu}(z-z_j),\ 0\le\mu\le n$ with $z_j\ne 0$ for $1\le j\le n-\mu$ is a polynomial of degree n which has all its zeros in $|z|\le k$ with $k\ge 1$, then for $0\le t\le 1$, we have*

$$\begin{aligned}\max_{|z|=1}|P'(z)| \quad &\geq \quad \left(\frac{\mu}{1+k^{n-\mu}}+\sum_{j=1}^{n-\mu}\frac{k}{(1+k^{n-\mu})(k+|z_j|)}\right) \\ &\quad \times\left[(2+X(k,t,m))\max_{|z|=1}|P(z)|+\left(\frac{k^{n-\mu}-1}{k^n}+\frac{X(k,t,m)}{k^n}\right)tm\right], \qquad (7)\end{aligned}$$

where $m = \min_{|z|=k} |P(z)|$ *and*

$$X(k,t,m) = \frac{(k-1)(|a_{n-\mu}|k^n - |a_0|k^\mu - tm)}{|a_{n-\mu}|k^n + |a_0|k^{\mu+1} - tm}.$$

Remark 1 *If we take* $\mu = 0$ *and* $a_{n-\mu} = 1$ *in Theorem 1, we obtain inequality (5) due to Milovanovic and A. Mir.*

If we take $k = 1$ in Theorem 1, we obtain the following refinement of inequality (4) for the polynomials having $\mu-$fold zero at origin.

Corollary 1 *If* $P(z) = z^\mu(a_0 + a_1 z + ... + a_{n-\mu}z^{n-\mu}) = a_{n-\mu}z^\mu \prod_{j=1}^{n-\mu}(z - z_j),\ 0 \le \mu \le n$ *with* $z_j \neq 0$ *for* $1 \le j \le n-\mu$ *is a polynomial of degree* n *which has all its zeros in* $|z| \le 1$*, then*

$$\max_{|z|=1} |P'(z)| \ge \left(\mu + \sum_{j=1}^{n-\mu} \frac{1}{1+|z_j|}\right) \max_{|z|=1} |P(z)| \ge \frac{n+\mu}{2} \max_{|z|=1} |P(z)|. \quad (8)$$

The lower bound given by inequality (8) is sharp and provides the stronger information than the inequality (1) due to Turan. For instance if we take a polynomial $P(z) = z^{10}(z^2+1)$, then $P(z)$ has all its zeros in $|z| \le 1$ with $n = 20,\ \mu = 10$ and

$$\max_{|z|=1} |P'(z)| = 30 \ \text{ and } \ \frac{n}{2} \max_{|z|=1} |P(z)| = 20 \textit{ while as } \frac{n+\mu}{2} \max_{|z|=1} |P(z)| = 30.$$

We next prove the following extension of Theorem 1 to the polar derivative of a polynomial having μ-fold zero at origin.

Theorem 2 *If* $P(z) = z^\mu(a_0 + a_1 z + ... + a_{n-\mu}z^{n-\mu}) = a_{n-\mu}z^\mu \prod_{j=1}^{n-\mu}(z - z_j),\ 0 \le \mu \le n$ *with* $z_j \neq 0$ *for* $1 \le j \le n-\mu$ *is a polynomial of degree* n *which has all its zeros in* $|z| \le k$*, then for any complex number* α *with* $|\alpha| \ge k,\ \ k \ge 1$ *and* $0 \le t \le 1$*, we have*

$$\begin{aligned} \max_{|z|=1} |D_\alpha P(z)| \ \ &\ge \ \ \left(\frac{\mu(|\alpha|-k)}{1+k^{n-\mu}} + \sum_{j=1}^{n-\mu} \frac{k(|\alpha|-k)}{(1+k^{n-\mu})(k+|z_j|)}\right) \\ &\times \left[\left(2 + \frac{X(k,t,m)}{k^n}\right) \max_{|z|=1} |P(z)| + \left(\frac{k^{n-\mu}-1}{k^n} + \frac{X(k,t,m)}{k^n}\right) tm\right], \end{aligned} \quad (9)$$

where $m = \min_{|z|=k} |P(z)|$ *and*

$$X(k,t,m) = \frac{(k-1)(|a_{n-\mu}|k^n - |a_0|k^\mu - tm)}{|a_{n-\mu}|k^n + |a_0|k^{\mu+1} - tm}.$$

Remark 2 *If we divide both sides to inequality (9) by* $|\alpha|$ *and let* $|\alpha| \to \infty$*, we get inequality (7) as a special case of Theorem 2.*

2 Lemmas

Following lemma is due to Mir et al. [8].

Lemma 1 *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* $n \ge 2$ *with no zeros in* $|z| < 1$*, then for any* $\rho > 1$ *and* $0 \le t \le 1$,

$$\max_{|z|=\rho} |P(z)| \le \left(\frac{(1+\rho^n)(\rho|a_n| + |a_0| - tm)}{(1+\rho)(|a_n| + |a_0| - tm)}\right) \max_{|z|=1} |P(z)| - \left(\frac{(1+\rho^n)(\rho|a_n| + |a_0| - tm)}{(1+\rho)(|a_n| + |a_0| - tm)} - 1\right) tm,$$

where $m = \min_{|z|=1} |P(z)|$.

Lemma 2 *If $P(z)=z^{\mu}(a_0+a_1z+...+a_{n-\mu}z^{n-\mu})$, $0\le\mu\le n$ is a polynomial of degree n which has all its zeros in $|z|\le k$ with $k\ge 1$, then for $0\le t\le 1$, we have*

$$\begin{aligned}\max_{|z|=k}|P(z)| \quad &\ge \quad \Bigg[\left(\frac{2k^n}{1+k^{n-\mu}}+\frac{(k-1)k^n(|a_{n-\mu}|k^n-|a_0|k^{\mu}-tm)}{(k^{n-\mu}+1)(|a_{n-\mu}|k^n+|a_0|k^{\mu+1}-tm)}\right)\max_{|z|=1}|P(z)| \\ &+\left(\frac{k^{n-\mu}-1}{k^{n-\mu}+1}+\frac{(k-1)(|a_{n-\mu}|k^n-|a_0|k^{\mu}-tm)}{k^n(k^{n-\mu}+1)(|a_{n-\mu}|k^n+|a_0|k^{\mu+1}-tm)}\right)tm\Bigg],\end{aligned}$$

where $m=\min_{|z|=k}|P(z)|$. The result is sharp and the extremal polynomial is $P(z)=z^{\mu}(z^{n-\mu}+k^{n-\mu})$.

Proof. Since all the zeros of $P(z)$ lie in $|z|\le k$, $k\ge 1$, the polynomial $T(z)=P(kz)$ has all zeros in $|z|\le 1$. Therefore the $(n-\mu)th$ degree polynomial $H(z)=z^nT(1/z)$ does not vanish in $|z|<1$. Hence applying Lemma 1 to the polynomial $H(z)$ with $\rho=k\ge 1$, we get

$$\begin{aligned}\max_{|z|=k}|H(z)| \quad &\le \quad \Bigg[\left(\frac{(1+k^{n-\mu})(|a_{n-\mu}|k^n+|a_0|k^{\mu+1}-tm^*)}{(1+k)(|a_{n-\mu}|k^n+|a_0|k^{\mu}-tm^*)}\right)\max_{|z|=1}|H(z)| \\ &-\left(\frac{(1+k^{n-\mu})(|a_{n-\mu}|k^n+|a_0|k^{\mu+1}-tm^*)}{(1+k)(|a_{n-\mu}|k^n+|a_0|k^{\mu}-tm^*)}-1\right)tm\Bigg],\end{aligned}$$

where

$$m^*=\min_{|z|=1}|H(z)|=\min_{|z|=1}|z^nP(k/z)|=\min_{|z|=1}|P(k/z)|=\min_{|z|=k}|P(z)|=m$$

and

$$\max_{|z|=1}|H(z)|=\max_{|z|=1}|z^nP(k/z)|=\max_{|z|=k}|P(z)|.$$

Using these observations in (11), we get

$$\begin{aligned}\max_{|z|=k}|P(z)| \quad &\ge \quad \frac{k^n(1+k)(|a_{n-\mu}|k^n+|a_0|k^{\mu}-tm)}{(1+k^{n-\mu})(|a_{n-\mu}|k^n+|a_0|k^{\mu+1}-tm)}\max_{|z|=1}|H(z)| \\ &+\left(1-\frac{(1+k)(|a_{n-\mu}|k^n+|a_0|k^{\mu}-tm)}{(1+k^{n-\mu})(|a_{n-\mu}|k^n+|a_0|k^{\mu+1}-tm)}\right)tm,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\max_{|z|=k}|P(z)| \quad &\ge \quad \left\{\frac{2k^n}{1+k^{n-\mu}}+\frac{k^n(k-1)(|a_{n-\mu}|k^n-|a_0|k^{\mu}-tm)}{(1+k^{n-\mu})(|a_{n-\mu}|k^n+|a_0|k^{\mu+1}-tm)}\right\}\max_{|z|=1}|P(z)| \\ &+\left\{\frac{k^{n-\mu}-1}{k^{n-\mu}+1}-\frac{(k-1)(|a_{n-\mu}|k^n-|a_0|k^{\mu}-tm)}{(1+k^{n-\mu})(|a_{n-\mu}|k^n+|a_0|k^{\mu+1}-tm)}\right\}tm.\end{aligned}$$

This completes the proof of Lemma 2. ■

Next lemma is a simple deduction from maximum modulus principle, see ([7]).

Lemma 3 *If $P(z)$ is a polynomial of degree n, then for $R\ge 1$*

$$\max_{|z|=R}|P(z)|\le R^n\max_{|z|=1}|P(z)|.$$

3 Proofs of Theorems

Proof of Theorem 1. Since $P(z)=a_{n-\mu}z^{n-\mu}\prod_{j=1}^{n-\mu}(z-z_j)$, $0\le\mu\le n$ has all its zeros in $|z|\le k$, $k\ge 1$, the polynomial $T(z)=P(kz)=k^na_{n-\mu}z^{\mu}\prod_{j=1}^{n-\mu}(z-z_j/k)$ has all its zeros in $|z|\le 1$. Hence for all z on $|z|=1$ for which $G(z)\ne 0$, we have

$$\frac{zG'(z)}{G(z)}=\mu+\sum_{j=1}^{n-s}\frac{z}{z-\frac{z_j}{k}}.$$

This gives for all z on $|z| = 1$ for which $G(z) \neq 0$

$$Re\left(\frac{zG'(z)}{G(z)}\right) = \mu + Re\left(\sum_{j=1}^{n-\mu} \frac{z}{z - z_j/k}\right) \geq \mu + \sum_{j=1}^{n-\mu} \frac{k}{k + |z_j|},$$

which implies

$$|G'(z)| \geq \left(\mu + \sum_{j=1}^{n-\mu} \frac{k}{k + |z_j|}\right) |G(z)| \tag{10}$$

for all z on $|z| = 1$ for which $G(z) \neq 0$. Since the inequality (10) is already true for the points z for which $G(z) = 0$. It follows that

$$\max_{|z|=1} |G'(z)| \geq \left(\mu + \sum_{j=1}^{n-\mu} \frac{k}{k + |z_j|}\right) \max_{|z|=1} |G(z)| \tag{11}$$

or equivalently

$$k \max_{|z|=1} |P'(kz)| \geq \left(\mu + \sum_{j=1}^{n-\mu} \frac{k}{k + |z_j|}\right) \max_{|z|=1} |P(kz)|. \tag{12}$$

Using Lemma 2 and the fact that $k^{n-1} \max_{|z|=1} |P'(z)| \geq |P'(kz)|$, we obtain from (12)

$$\begin{aligned} k^n \max_{|z|=1} |P'(z)| \quad &\geq \quad \left(\frac{\mu}{1 + k^{n-\mu}} + \sum_{j=1}^{n-\mu} \frac{k}{(1 + k^{n-\mu})(k + |z_j|)}\right) \\ &\times \left[(2k^n + k^n X(k,t,m)) \max_{|z|=1} |P(z)| + \left(k^{n-\mu} - 1 + X(k,t,m)\right) tm \right], \end{aligned}$$

which is equivalent to

$$\begin{aligned} \max_{|z|=1} |P'(z)| \quad &\geq \quad \left(\frac{\mu}{1 + k^{n-\mu}} + \sum_{j=1}^{n-\mu} \frac{k}{(1 + k^{n-\mu})(k + |z_j|)}\right) \\ &\times \left[(2 + X(k,t,m)) \max_{|z|=1} |P(z)| + \left(\frac{k^{n-\mu} - 1}{k^n} + \frac{X(k,t,m)}{k^n}\right) tm \right]. \end{aligned}$$

This completes the proof of Theorem 1. ■

Proof of Theorem 2. Since $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, the polynomial $G(z) = P(kz)$ has all its zeros in $|z| \leq 1$. Therefore for $|\alpha|/k \geq 1$, it can be easily seen that

$$\max_{|z|=1} |D_{\alpha/k} G(z)| \geq \frac{(|\alpha| - k)}{k} \max_{|z|=1} |G'(z)|$$

or

$$\max_{|z|=k} |D_\alpha P(z)| \geq \frac{(|\alpha| - k)}{k} \max_{|z|=1} |G'(z)|.$$

Using inequality (12), we have

$$\max_{|z|=k} |D_\alpha P(z)| \geq \frac{(|\alpha| - k)}{k} \left(\mu + \sum_{j=1}^{n-\mu} \frac{k}{k + |z_j|}\right) \max_{|z|=1} |G(z)|,$$

which is equivalent to

$$\max_{|z|=k}|D_\alpha P(z)| \geq \frac{(|\alpha|-k)}{k}\left(\mu+\sum_{j=1}^{n-\mu}\frac{k}{k+|z_j|}\right)\max_{|z|=k}|P(z)|.$$

Now applying Lemma 2 in the right hand side of above inequality, we get

$$\begin{aligned} k\max_{|z|=k}|D_\alpha P(z)| \quad &\geq \quad \left(\frac{\mu(|\alpha|-k)}{1+k^{n-\mu}}+\sum_{j=1}^{n-\mu}\frac{k(|\alpha|-k)}{(1+k^{n-\mu})(k+|z_j|)}\right) \\ &\times\Big[(2k^n+k^nX(k,t,m))\max_{|z|=1}|P(z)|+\big(k^{n-\mu}-1+X(k,t,m)\big)\,tm\Big] \end{aligned} \tag{13}$$

where

$$X(k,t,m)=\frac{(k-1)(|a_{n-\mu}|k^n-|a_0|k^\mu-tm)}{|a_{n-\mu}|k^n+|a_0|k^{\mu+1}-tm}.$$

Since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, so that by Lemma 3, we get

$$\max_{|z|=k}|D_\alpha P(z)| \leq k^{n-1}\max_{|z|=1}|D_\alpha P(z)|.$$

Using this observation in (13), we obtain

$$\begin{aligned} \max_{|z|=k}|D_\alpha P(z)| \quad &\geq \quad \left(\frac{\mu(|\alpha|-k)}{1+k^{n-\mu}}+\sum_{j=1}^{n-\mu}\frac{k(|\alpha|-k)}{(1+k^{n-\mu})(k+|z_j|)}\right) \\ &\times\left[(2+X(k,t,m))\max_{|z|=1}|P(z)|+\left(\frac{k^{n-\mu}-1}{k^n}+\frac{X(k,t,m)}{k^n}\right)tm\right]. \end{aligned}$$

This is the desired inequality and hence completes the proof of Theorem 2. ■

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