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A Study Of The Left Local General Truncated M-Fractional Derivative*

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Abstract

We introduce a new type of conformable fractional derivative, which generalizes the standard properties and results of the classical integer order calculus viz. the Rolle's theorem, the Mean Value Theorems, the inverse property, the fundamental theorem of calculus, the theorem of integration by parts and the Taylor's theorem with integral remainder. After this, the extant conformable fractional derivatives are shown as the special cases of the new one. At the end, the well known Bernoulli's differential equation is generalized in terms of our newly defined fractional derivative. Also, some well known physical problems like Newton's law of cooling and Kirchoff's current law are generalized and solved in terms of the conformable fractional sense and the importance of this newly defined operator with respect to the flexibility in the parametric values is described via the comparison of the solutions in the graphs using MATLAB software. At last, the image processing has been done with the aid of our newly defined fractional derivative operator.

1 Introduction

The study of non-integer order calculus was discovered in 1695 by L'Hospital and Leibniz [13]. Due to its vast applications in the fields like engineering, sciences etc., it has become more popular and interesting among the researchers. The various types of fractional derivatives and integrals have been defined and investigated through the unification of the classical integration and differentiation. Some important works have been also carried forward in this direction as in [2, 4, 5, 6, 27, 17, 23, 25, 26].

Many varieties of fractional derivatives and integrals have been introduced, amongst which the Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard, Grünwald-Letnikov, Riesz [18, 19] are worth mentioning. Most of them have the background of the corresponding fractional integral in the Riemann-Liouville sense. But they are non local and they do not have the fundamental assets of the ordinary differentiation.

To overcome this, Khalil et al. [12], Katugampola [11], Sousa and Oliveira [30] and Anastassiou [3] have worked in this direction and gave the following fractional derivatives in terms of the conformable sense which encompasses the classical properties of integer order calculus. Khalil et al. [12] defined the conformable fractional derivative of order α as

Definition 1 *Let $f : [0, \infty) \to \mathbf{R}$. Then the conformable fractional derivative of order α is given by*

$$T^{(\alpha)} f(t) = \lim_{\xi \to 0} \frac{f(t + \xi t^{1-\alpha}) - f(t)}{\xi}, \tag{1}$$

for all $t > 0$ and $\alpha \in (0, 1)$.

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Many of the researchers have studied the conformable fractional derivative with various applications [1, 10]. Moreover, in 2014, Katugampola [11] has proposed a new fractional derivative with classical properties similar to the conformable fractional derivative as

Definition 2 *Let* $f : [0, \infty) \to \mathbb{R}$*. Then the alternative fractional derivative of order* α *is defined as*

$$\mathbf{D}^{\alpha} f(t) = \lim_{\xi \to 0} \frac{f(t\ e^{\xi t^{-\alpha}}) - f(t)}{\xi}, \tag{2}$$

for all $t > 0$ *and* $\alpha \in (0, 1)$*.*

In 2017, Sousa and Oliveira [30] have defined a generalization of the usual definition of a derivative as follows:

Definition 3 *Let* $f : [0, \infty) \to \mathbb{R}$*. Then for all* $t > 0$ *and* $\alpha \in (0, 1)$*, the local* M*-derivative of order* α *of* f *is defined as*

$$\mathbb{D}_M^{\alpha,\beta} f(t) = \lim_{\xi \to 0} \frac{f\left(t\ \mathbb{E}_{\beta}\left(\xi t^{-\alpha}\right)\right) - f(t)}{\xi}, \tag{3}$$

where $\mathbb{E}_{\beta}(\cdot),\ \beta > 0$ *is the Mittag-Leffler function with one parameter [15, 16].*

Sousa and Oliveira [28, 29] have defined the truncated M-fractional derivative with the aid of the truncated Mittag-Leffler function of one parameter defined by

$${}_{i}\mathbb{E}_{\beta}(z) = \sum_{k=0}^{i} \frac{z^k}{\Gamma(\beta k + 1)}, \tag{4}$$

with $\beta > 0$ and $z \in \mathbb{C}$ as follows:

Definition 4 *Let* $f : [0, \infty) \to \mathbb{R}$*. Then for all* $t > 0$ *and* $\alpha \in (0, 1)$*, a truncated* M*-fractional derivative of order* α *of* f *is defined as*

$${}_{i}\mathcal{D}_M^{\alpha,\beta} f(t) = \lim_{\xi \to 0} \frac{f\left(t\ {}_{i}\mathbb{E}_{\beta}\left(\xi t^{-\alpha}\right)\right) - f(t)}{\xi}, \tag{5}$$

where ${}_{i}\mathbb{E}_{\beta}(\cdot),\ \beta > 0$ *is the truncated Mittag-Leffler function with one parameter.*

As a generalization of the truncated M-fractional derivative, Sousa and Oliveira have defined the truncated ν-fractional derivative [28]. In 2019, Anastassiou [3] has defined the left local general M-fractional derivative as

Definition 5 *Let* $f : [a, \infty) \to \mathbb{R}$ *and* $t > a,\ a \in \mathbb{R}$*. For* $\alpha \in (0, 1]$*, left local general* M*-fractional derivative of order* α *of* f *is defined as*

$$\mathfrak{D}_{M,a}^{\alpha,\beta} f(t) = \lim_{\xi \to 0} \frac{f\left(t\ \mathbb{E}_{\beta}\left(\xi (t-a)^{-\alpha}\right)\right) - f(t)}{\xi}, \tag{6}$$

where $\mathbb{E}_{\beta}(\cdot),\ \beta > 0$ *is the Mittag-Leffler function with one parameter.*

By focusing on all these definitions, we now generalize the left local general M-fractional derivative given in (6) by adding a flavor of the truncated Mittag-leffler function (4).

2 Main Results

As described in Section 1, the well known fractional derivatives are non-local and they do not preserve some classical properties of ordinary calculus. So to inculcate these assets, we have already generalized the conformable derivative by taking argument as the truncated Mittag-Leffler function in right local sense [7]. Now, we will define the operator using again truncated Mittag-Leffler function in left local sense. With the aid of this newly defined operator, various results having likeness to the results of classical calculus are obtained. As an important aspects of these results, three physical problems have been extended, solved and faster convergence rate has been observed through the graphs. Also image enhancement has been performed for the betterment of the 4D ultrasound image of a fetal using our newly defined operator.

Now, we begin with the following definition, which is the generalization of (6).

Definition 6 *Let $f : [a, \infty) \to \mathbb{R}$ and $t > a,\ a \in \mathbb{R}$. For $0 < \alpha \leq 1$, we define the left local general truncated M-fractional derivative of order α of f (α-LLGT M-fractional derivative) as*

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t) := \lim_{\xi \to 0} \frac{f\left(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})\right) - f(t)}{\xi}, \tag{7} $$

where ${}_i\mathbb{E}_\beta(\cdot)$ is the truncated Mittag-Leffler function of one parameter as defined in (4).

Now, if f is differentiable in some open interval $(a, \delta), \delta \in \mathbb{R}$ and $\lim_{t \to a^+}\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t)$ exists then we have

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(a) = \lim_{t \to a^+}\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t). $$

Remark 1 *Note that,*

$$ \begin{aligned} t\ {}_i\mathbb{E}_\beta\left(\xi(t-a)^{-\alpha}\right) &= t \sum_{k=0}^{i} \frac{\left(\xi(t-a)^{-\alpha}\right)^k}{\Gamma(\beta k + 1)} \\ &= t + \frac{t\ \xi(t-a)^{-\alpha}}{\Gamma(\beta+1)} + \frac{t\left(\xi(t-a)^{-\alpha}\right)^2}{\Gamma(2\beta+1)} + \frac{t\left(\xi(t-a)^{-\alpha}\right)^3}{\Gamma(3\beta+1)} \\ &\quad + \cdots + \frac{t\left(\xi(t-a)^{-\alpha}\right)^i}{\Gamma(i\beta+1)}. \end{aligned} \tag{8} $$

Now, applying limit $\xi \to 0$ on both sides of (8), we get

$$ \lim_{\xi \to 0} t\ {}_i\mathbb{E}_\beta\left(\xi(t-a)^{-\alpha}\right) = t. $$

Next, we try to establish the generalization of the result "Every differentiable function is continuous." in the following theorem in context of the α-LLGT M-fractional derivative.

Theorem 1 *If a function $f : [a, \infty) \to \mathbb{R}$ is α-LLGT M-fractional differentiable at $t_0,\ t_0 > a$ with $\beta > 0$, then f is continuous at t_0.*

Proof. For $\xi \neq 0$, consider

$$ f\left(t_0\ {}_i\mathbb{E}_\beta(\xi(t_0-a)^{-\alpha})\right) - f(t_0) = \left(\frac{f\left(t_0\ {}_i\mathbb{E}_\beta(\xi(t_0-a)^{-\alpha})\right) - f(t_0)}{\xi}\right)\xi. \tag{9} $$

Now, applying the limit $\xi \to 0$ on both sides of (9), we have

$$ \lim_{\xi \to 0}\left(f\left(t_0\ {}_i\mathbb{E}_\beta(\xi(t_0-a)^{-\alpha})\right) - f(t_0)\right) = \lim_{\xi \to 0}\left(\frac{f\left(t_0\ {}_i\mathbb{E}_\beta(\xi(t_0-a)^{-\alpha})\right) - f(t_0)}{\xi}\right) \times \lim_{\xi \to 0} \xi $$

$$= {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t_0) \lim_{\xi\to 0} \xi$$
$$= 0.$$

Hence, f is continuous at t_0. ■

In the next theorem, a relation between the α-LLGT M-fractional derivative and classical derivative is obtained.

Theorem 2 *If $f : [a, \infty) \to \mathbb{R}$ has the α-LLGT M-fractional derivative at t, $t > a$ with $\beta > 0$, then*

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t) = \frac{t(t-a)^{-\alpha}}{\Gamma(\beta+1)} f'(t). \tag{10}$$

Proof. For $t > a$ and from Remark 1, we have

$$\begin{aligned} t\ {}_i\mathbb{E}_\beta\left(\xi(t-a)^{-\alpha}\right) &= t + \frac{t\ \xi(t-a)^{-\alpha}}{\Gamma(\beta+1)} + \frac{t\left(\xi(t-a)^{-\alpha}\right)^2}{\Gamma(2\beta+1)} + \frac{t\left(\xi(t-a)^{-\alpha}\right)^3}{\Gamma(3\beta+1)} \\ &\quad + \ldots + \frac{t\left(\xi(t-a)^{-\alpha}\right)^i}{\Gamma(i\beta+1)} \\ &= t + \frac{t\ \xi(t-a)^{-\alpha}}{\Gamma(\beta+1)} + \mathcal{O}(\xi^2). \end{aligned} \tag{11}$$

Let

$$h := \xi t(t-a)^{-\alpha}\left(\frac{1}{\Gamma(\beta+1)} + \mathcal{O}(\xi^2)\right). \tag{12}$$

Then

$$\xi = \frac{h}{t(t-a)^{-\alpha}\left(\frac{1}{\Gamma(\beta+1)} + \mathcal{O}(\xi^2)\right)} = \frac{h\ (t-a)^\alpha\ \Gamma(\beta+1)}{t\left(1+\Gamma(\beta+1)\mathcal{O}(\xi^2)\right)}. \tag{13}$$

Therefore from the Definition 6 and (11), we have

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t) = \lim_{\xi\to 0} \frac{f\left(t + \frac{\xi t(t-a)^{-\alpha}}{\Gamma(\beta+1)} + \mathcal{O}(\xi^2)\right) - f(t)}{\xi}.$$

Then from (12), we have

$$\begin{aligned} {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t) &= \lim_{\xi\to 0} \frac{f(t+h) - f(t)}{\xi} \\ &= \lim_{\xi\to 0} \frac{(f(t+h) - f(t))\ t\left(1+\Gamma(\beta+1)\mathcal{O}(\xi^2)\right)}{h(t-a)^\alpha\Gamma(\beta+1)} \\ &= \frac{t(t-a)^{-\alpha}}{\Gamma(\beta+1)}\left[\lim_{h\to 0}\left(\frac{f(t+h)-f(t)}{h}\right)\lim_{\xi\to 0}\left(1+\Gamma(\beta+1)\mathcal{O}(\xi^2)\right)\right] \\ &= \frac{t(t-a)^{-\alpha}}{\Gamma(\beta+1)} f'(t) \quad \text{as if } \xi\to 0 \text{ then } h\to 0. \end{aligned}$$

■

Remark 2 *From Theorem 2, if $f(t) = c$, where c is any constant, then ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t) = 0$ as $f'(t) = 0$, for $t \in [a, \infty)$.*

Remark 3 *For $\alpha = 1, a = 0$ and $\beta = 0$ or 1, (10) becomes ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t) = f'(t)$.*

Now, we will derive the theorem that encompasses the classical properties of integer order derivatives.

Theorem 3 *Let $f_1, f_2 : [a,\infty) \to \mathbb{R}$ be α-LLGT M-fractional differentiable at t, $t > a$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\beta > 0$. Then*

1. ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(\mu_1 f_1 + \mu_2 f_2)(t) = \mu_1\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(t) + \mu_2\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(t)$.
2. ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(f_1 \cdot f_2)(t) = f_1(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(t) + f_2(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(t)$.
3. ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}\left(\frac{f_1}{f_2}\right)(t) = \frac{f_2(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(t) - f_1(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(t)}{[f_2(t)]^2}$.
4. ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(k) = 0$, *where k is a constant.*
5. *If $f_1(t)$ is differentiable at $f_2(t)$, then* ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(f_1 o f_2)(t) = f_1'(f_2(t))\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(t)$.

Proof.

1. From Definition 6, we have

$$\begin{aligned}
&{}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(\mu_1 f_1 + \mu_2 f_2)(t)\\
&= \lim_{\xi\to 0} \frac{(\mu_1 f_1 + \mu_2 f_2)(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - (\mu_1 f_1 + \mu_2 f_2)(t)}{\xi}\\
&= \lim_{\xi\to 0} \frac{\mu_1\ f_1(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) + \mu_2\ f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - \mu_1 f_1(t) - \mu_2 f_2(t)}{\xi}\\
&= \lim_{\xi\to 0} \frac{\mu_1\ f_1(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - \mu_1 f_1(t)}{\xi}\\
&\quad + \lim_{\xi\to 0} \frac{\mu_2\ f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - \mu_2 f_2(t)}{\xi}\\
&= \mu_1\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(t) + \mu_2\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(t).
\end{aligned}$$

2. From Definition 6, we have

$$\begin{aligned}
&{}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(f_1 \cdot f_2)(t)\\
&= \lim_{\xi\to 0} \frac{f_1(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) \cdot f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_1(t)\cdot f_2(t)}{\xi}\\
&= \lim_{\xi\to 0} \left\{ \begin{array}{c} f_1(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) \cdot f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_1(t)\cdot f_2(t) \\ +f_1(t) f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_1(t) f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) \end{array} \right\} / \xi\\
&= \lim_{\xi\to 0} \left(\frac{f_1(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_1(t)}{\xi} \right) \lim_{\xi\to 0} f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))\\
&\quad + \lim_{\xi\to 0} \left(\frac{f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_2(t)}{\xi} \right) \lim_{\xi\to 0} f_1(t).
\end{aligned}$$

Using Theorem 1 and now applying Definition 6, we get

$$\begin{aligned}
{}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(f_1 \cdot f_2)(t) &= {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(t)\ f_2(t) +_i \mathfrak{D}^{\alpha,\beta}_{M,a} f_2(t)\ f_1(t)\\
&= f_1(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(t) + f_2(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(t).
\end{aligned}$$

3. Again with the aid of the Definition 6, we have

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}\left(\frac{f_1}{f_2}\right)(t)$$

$$
\begin{aligned}
&= \lim_{\xi\to 0} \frac{\frac{f_1(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))}{f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))} - \frac{f_1(t)}{f_2(t)}}{\xi} \\
&= \lim_{\xi\to 0} \frac{f_2(t)\ f_1(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_1(t)\ f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))}{\xi f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))\ f_2(t)} \\
&\quad + \lim_{\xi\to 0} \frac{f_1(t)f_2(t)}{\xi\ f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))\ f_2(t)} - \lim_{\xi\to 0} \frac{f_1(t)f_2(t)}{\xi\ f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))\ f_2(t)} \\
&= \frac{\lim_{\xi\to 0} \frac{f_2(t)\left(f_1(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))-f_1(t)\right)}{\xi} - \lim_{\xi\to 0} \frac{f_1(t)\left(f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))-f_2(t)\right)}{\xi}}{\lim_{\xi\to 0} f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))f_2(t)} \\
&= \frac{f_2(t)\left(\lim_{\xi\to 0} \frac{\left(f_1(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))-f_1(t)\right)}{\xi}\right) - f_1(t)\left(\lim_{\xi\to 0} \frac{\left(f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))-f_2(t)\right)}{\xi}\right)}{f_2(t)\ \lim_{\xi\to 0} f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))} \\
&= \frac{f_2(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(t) - f_1(t)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(t)}{[f_2(t)]^2},
\end{aligned}
$$

as $\lim_{\xi\to 0} f_2(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) = f_2(t)$.

4. In this case, the proof directly follows from Remark 2.

5. This result is proved in two cases: (I) f_2 is constant and (II) f_2 is non-constant.
Case-I: Let $f_2(t) = b$, where b is any constant.
Then from Remark 2, we have

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}\ (f_1 o f_2)(b) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}\ f_1(f_2(t)) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}\ f_1(b) = 0. $$

Case-II: f_2 is not a constant in a neighborhood of b.
Since f_2 is continuous at b, for ξ to be small enough, we have

$$
\begin{aligned}
&{}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(f_1 o f_2)(b) \\
&= \lim_{\xi\to 0} \frac{f_1(f_2(b\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))) - f_1(f_2(b))}{\xi} \\
&= \lim_{\xi\to 0} \frac{f_1(f_2(b\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))) - f_1(f_2(b))}{\xi}\ \frac{f_2(b\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_2(b)}{f_2(b\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_2(b)} \\
&= \lim_{\xi\to 0} \frac{f_1(f_2(b\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha}))) - f_1(f_2(b))}{f_2(b\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_2(b)} \times \lim_{\xi\to 0} \frac{f_2(b\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_2(b)}{\xi}.
\end{aligned}
$$

Now, let

$$ \xi_1 = f_2(b\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_2(b). $$

Then

$$ f_2(b\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) = \xi_1 + f_2(b). $$

Also, it is observed that if $\xi \to 0$ then $\xi_1 \to 0$. Therefore,

$$
\begin{aligned}
&{}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(f_1 o f_2)(b) \\
&= \lim_{\xi_1\to 0} \frac{f_1(f_2(b) + \xi_1) - f_1(f_2(b))}{\xi_1}\ \lim_{\xi\to 0} \frac{f_2(b\ {}_i\mathbb{E}_\beta(\xi(t-a)^{-\alpha})) - f_2(b)}{\xi} \\
&= f_1^{'}(f_2(b))\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(b), \quad b > 0.
\end{aligned}
$$

Hence,

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(f_1 o f_2)(t) = f_1^{'}(f_2(t))\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(t). $$

■

Now, using Theorem 2, we have the following α-LLGT M-fractional derivatives of the various functions.

Theorem 4 *Let $\mu \in \mathbb{R},\ \beta > 0,\ \alpha \in (0,1]$ and $t > a$. Then*

1. ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(1) = 0$.
2. ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(e^{\mu t}) = \dfrac{t(t-a)^{-\alpha}}{\Gamma(\beta+1)}\mu\ e^{\mu t}$.
3. ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(\sin \mu t) = \dfrac{t(t-a)^{-\alpha}}{\Gamma(\beta+1)}\mu \cos \mu t$.
4. ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(\cos \mu t) = -\dfrac{t(t-a)^{-\alpha}}{\Gamma(\beta+1)}\mu \sin \mu t$.
5. ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(t^{\mu}) = \dfrac{t(t-a)^{-\alpha}}{\Gamma(\beta+1)}\mu t^{\mu-1} = \dfrac{(t-a)^{-\alpha}\mu\ t^{\mu}}{\Gamma(\beta+1)}$.

Proof. We omit the proof here as it follows from Theorem 2. ■

2.1 Generalization of Fundamental Results of Calculus

Further, we have observed that the α-LLGT M-fractional derivative also has various important theorems similar to the classical integer order calculus. We have derived the Rolle's theorem, the Mean Value Theorem and its extension using this newly defined fractional derivative in the next three theorems.

Theorem 5 *Let $f : [\gamma, \rho] \to \mathbb{R}$, where $\gamma > a$. If*

1. *f is continuous on $[\gamma, \rho]$,*
2. *f is α-LLGT M-fractional differentiable on (γ, ρ), and*
3. *$f(\gamma) = f(\rho)$,*

then there exists $c \in (\gamma, \rho)$ such that ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}f(c) = 0,\ \beta > 0$.

Proof. We will prove this theorem in three cases:

Case-I: If $f(x) = k$ on $[\gamma, \rho]$ where k is any constant, then from Remark 2, ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}f(x) = 0$ for all $x \in [\gamma, \rho]$. In other words, we can say that there exists $c \in (\gamma, \rho)$ such that

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}f(c) = 0. $$

Case-II: Let f be non-constant. In this case, suppose that there is some d in (γ, ρ) such that $f(d) > f(\gamma)$. Since f is continuous on $[\gamma, \rho]$, by the extreme value theorem [22], $f(x)$ has maximum in $[\gamma, \rho]$. Also, as $f(\gamma) = f(\rho)$ and $f(d) > f(\gamma)$, we have the maximum value of f at some c in (γ, ρ). Here, c occurs in the interior of the interval means that $f(x)$ has relative maximum at $x = c$ and by the second hypothesis, we have ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}f(x)$ exists. Therefore, ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}f(c) = 0$.

Case-III: Let f be non-constant, but in this case, suppose that there is some d in (γ, ρ) such that $f(d) < f(\gamma)$. Now, similar to Case-II, by extreme value theorem [22], $f(x)$ has minimum in $[\gamma, \rho]$. Also, as $f(\gamma) = f(\rho)$ and $f(d) < f(\gamma)$, we have the minimum value of f at some c in (γ, ρ). Hence, ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}f(c) = 0$. ■

Theorem 6 *Let $f : [\gamma, \rho] \to \mathbb{R}$, where $\gamma > a,\ 0 \notin [\gamma, \rho]$. If*

1. *f is continuous on $[\gamma, \rho]$, and*

2. *f is α-LLGT M-fractional differentiable on (γ, ρ),*

then there exists $c \in (\gamma, \rho)$ such that

$$f(\rho) - f(\gamma) = \left({}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(c)\right) \frac{\Gamma(\beta+1)(c-a)^{\alpha}}{c}(\rho - \gamma).$$

Proof. For $x \in [\gamma, \rho]$, consider

$$g(x) := f(x) - f(\gamma) - \left(\frac{f(\rho) - f(\gamma)}{\rho - \gamma}\right)(x - \gamma). \tag{14}$$

Since f is continuous on $[\rho, \gamma]$, g is continuous on $[\rho, \gamma]$ too. Also, it can be easily verified that $g(\gamma) = 0 = g(\rho)$. Therefore from Theorem 3, we can say that f is the α-LLGT M-fractional differentiable on (γ, ρ).

Now, from Theorem 5, there exists $c \in (\gamma, \rho)$ such that

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} g(c) = 0. \tag{15}$$

Taking ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}$ on both sides of (14), we get

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} g(x) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(x) - {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(\gamma) - \left(\frac{f(\rho) - f(\gamma)}{\rho - \gamma}\right) {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(x - \gamma).$$

Applying Theorem 2 by taking f to be linear function, we obtain

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} g(x) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(x) - {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(\gamma) - \left(\frac{f(\rho) - f(\gamma)}{\rho - \gamma}\right) \frac{x(x-a)^{-\alpha}}{\Gamma(\beta+1)}.$$

Whence at $x = c$,

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} g(c) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(c) - {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(\gamma) - \left(\frac{f(\rho) - f(\gamma)}{\rho - \gamma}\right) \frac{c(c-a)^{-\alpha}}{\Gamma(\beta+1)}.$$

Then using (15), we get

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(c) - 0 - \left(\frac{f(\rho) - f(\gamma)}{\rho - \gamma}\right) \frac{c(c-a)^{-\alpha}}{\Gamma(\beta+1)} = 0.$$

Hence,

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(c) = \left(\frac{f(\rho) - f(\gamma)}{\rho - \gamma}\right) \frac{c(c-a)^{-\alpha}}{\Gamma(\beta+1)}.$$

■

Theorem 7 *Let $\gamma > a$, $0 \notin [\gamma, \rho]$ and $f_1, f_2 : [\gamma, \rho] \to \mathbb{R}$. If*

1. *f_1, f_2 are continuous on $[\gamma, \rho]$ and $f_2(\gamma) \neq f_2(\rho)$, and*
2. *f is the α-LLGT M-fractional differentiable on (γ, ρ),*

then there exists $c \in (\gamma, \rho)$ such that

$$\frac{{}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(c)}{{}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(c)} = \frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)} \quad \textit{with } \beta > 0.$$

Proof. For $x \in [\gamma, \rho]$, consider

$$G(x) := f_1(x) - f_2(\gamma) - \left(\frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)}\right)(f_2(x) - f_2(\gamma)). \tag{16}$$

Since f_1, f_2 are continuous on $[\rho, \gamma]$, G is continuous on $[\rho, \gamma]$ too. Also, it can be easily seen that $G(\gamma) = 0 = G(\rho)$. Therefore from Theorem 3, we can say that f_1, f_2 are α-LLGT M-fractional differentiable functions on (γ, ρ).

Now, from Theorem 5, there exists $c \in (\gamma, \rho)$ such that

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} G(c) = 0. \tag{17}$$

Taking ${}_i\mathfrak{D}^{\alpha,\beta}_{M,a}$ on both sides of (16), we get

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} G(x) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(x) - {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(\gamma) - \left(\frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)}\right) {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}(f_2(x) - f_2(\gamma)).$$

Applying Remark 2 and then writing the expression at $x = c$, we obtain

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} G(c) = {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(c) - 0 - \left(\frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)}\right) {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(c) - 0,$$

which implies from (17),

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(c) - \left(\frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)}\right) {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(c) = 0.$$

Therefore,

$$\frac{{}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_1(c)}{{}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f_2(c)} = \frac{f_1(\rho) - f_1(\gamma)}{f_2(\rho) - f_2(\gamma)}.$$

■

Definition 7 *Let f be analytic at t, $t > a$, $\beta > 0$, and $\alpha \in (n, n+1]$, $n \in \mathbb{N} \cup \{0\}$. Then the general form of the α-LLGT M-fractional derivative of function f at t is defined by*

$${}_i\mathfrak{D}^{\alpha,\beta;n}_{M,a} f(t) := \lim_{\xi \to 0} \frac{f^{(n)}\left(t\ {}_i\mathbb{E}_\beta(\xi(t-a)^{n-\alpha})\right) - f^{(n)}(t)}{\xi}, \tag{18}$$

if the limit exists.

Now, from the above definition, Theorem 2 and by the principle of mathematical induction on n, we have for $t > a$

$${}_i\mathfrak{D}^{\alpha,\beta;n}_{M,a} f(t) = \frac{t(t-a)^{n-\alpha}}{\Gamma(\beta+1)} f^{(n+1)}(t),$$

for f to be an analytic function. Further, this α-LLGT M-fractional derivative has a corresponding left M-integral, which is defined below:

Definition 8 *Let $t \geq a$, f be a continuous function defined on $(a, t]$, $0 \notin (a, t]$ and $\alpha \in (0, 1]$. Then the left M-integral of order α of f is defined as*

$$\mathfrak{I}^{\alpha,\beta}_{M,a} f(t) = \Gamma(\beta+1) \int_a^t \frac{f(x)}{x(x-a)^{-\alpha}}\, dx, \quad \textit{with } \beta > 0.$$

In connection with the above definition, we have generalized the inverse property, the fundamental theorem of calculus and the theorem of integration by parts in the upcoming theorems.

Theorem 8 *Let $a \in \mathbb{R},\ \alpha \in (0,1]$. If f is a continuous function at $t,\ 0 \neq t > a$ then*

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}\ \mathfrak{I}^{\alpha,\beta}_{M,a} f(t) = f(t),\ \beta > 0. \tag{19} $$

Proof. From Theorem 2, we have

$$ \begin{aligned} {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}\left(\mathfrak{I}^{\alpha,\beta}_{M,a} f(t)\right) &= \frac{t(t-a)^{-\alpha}}{\Gamma(\beta+1)}\frac{d}{dt}\left(\mathfrak{I}^{\alpha,\beta}_{M,a} f(t)\right) \\ &= \frac{t(t-a)^{-\alpha}}{\Gamma(\beta+1)}\frac{d}{dt}\left(\Gamma(\beta+1)\int_a^t \frac{f(x)}{x(x-a)^{-\alpha}}\ dx\right) \\ &= \frac{t(t-a)^{-\alpha}}{\Gamma(\beta+1)}\ \Gamma(\beta+1)\ \frac{f(t)}{t(t-a)^{-\alpha}} = f(t). \end{aligned} $$

■

Theorem 9 *Let $f : [a,\infty) \to \mathbb{R}$ be the α-LLGT M-fractional differentiable function and $\alpha \in (0,1]$. Then for all $t > a$,*

$$ \mathfrak{I}^{\alpha,\beta}_{M,a}\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t) = f(t) - f(a),\ \textit{with } \beta > 0. \tag{20} $$

Proof. From Definition 8 and then applying Theorem 2, we have

$$ \begin{aligned} \mathfrak{I}^{\alpha,\beta}_{M,a}\left({}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t)\right) &= \Gamma(\beta+1)\int_a^t \frac{{}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(x)}{x(x-a)^{-\alpha}}\ dx \\ &= \Gamma(\beta+1)\int_a^t \frac{1}{x(x-a)^{-\alpha}}\ \frac{x(x-a)^{-\alpha}}{\Gamma(\beta+1)}\ f'(x)\ dx \\ &= \int_a^t f'(x)\ dx = f(t) - f(a), \end{aligned} $$

by the classical fundamental theorem of calculus. ■

It can be easily observed that, if $f(a) = 0$, then by (20) for all $t > a$, we have

$$ \mathfrak{I}^{\alpha,\beta}_{M,a}\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a} f(t) = f(t). $$

Now, for the sake of brevity, we denote

$$ \mathfrak{I}^{\alpha,\beta}_{M,a} f(t) = \int_a^t f(x)\ d_{\alpha,\beta}\ x, \quad \text{where } d_{\alpha,\beta}\ x = \frac{\Gamma(\beta+1)}{x(x-a)^{-\alpha}}\ dx. $$

In this notation, we derive the generalization of the integration by parts in the following theorem for the left M-integral.

Theorem 10 *Let $f_1, f_2 : [c,d] \to \mathbb{R}$ be continuously differentiable and $\alpha \in (0,1]$. Then for $\beta > 0$*

$$ \int_c^d f_1(x)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}\ f_2(x)\ d_{\alpha,\beta}\ x = [f_1(x)f_2(x)]_c^d - \int_c^d f_2(x)\ {}_i\mathfrak{D}^{\alpha,\beta}_{M,a}\ f_1(x)\ d_{\alpha,\beta}\ x. $$

Proof. In the stated notations,

$$\int_c^d f_1(x)\ {}_i\mathfrak{D}_{M,a}^{\alpha,\beta}\ f_2(x)\ d_{\alpha,\beta}\ x = \int_c^d f_1(x)\ {}_i\mathfrak{D}_{M,a}^{\alpha,\beta}\ f_2(x)\ \frac{\Gamma(\beta+1)}{x(x-a)^{-\alpha}}\ dx$$
$$= \int_c^d f_1(x)\ \frac{x(x-a)^{-\alpha}}{\Gamma(\beta+1)}\ f_2'(x)\ \frac{\Gamma(\beta+1)}{x(x-a)^{-\alpha}}\ dx,$$

by Theorem 2. Now, applying the classical integration by parts, we obtain

$$\int_c^d f_1(x)\ {}_i\mathfrak{D}_{M,a}^{\alpha,\beta}\ f_2(x)\ d_{\alpha,\beta}\ x = \int_c^d f_1(x)\ f_2'(x)\, dx$$
$$= [f_1(x)f_2(x)]_c^d - \int_c^d f_1'(x)\ f_2(x)\, dx$$
$$= [f_(x)f_2(x)]_c^d - \int_c^d f_2(x)\ \frac{x(x-a)^{-\alpha}}{\Gamma(\beta+1)}\ f_1'(x)\ \frac{\Gamma(\beta+1)}{x(x-a)^{-\alpha}}\ dx$$
$$= [f_1(x)f_2(x)]_c^d - \int_c^d f_2(x)\ {}_i\mathfrak{D}_{M,a}^{\alpha,\beta}\ f_1(x)\ d_{\alpha,\beta}\ x,$$

by using Theorem 2 again . ■

The general form of the left M-integral is given by

Definition 9 *Let $t \geq a$ and f be a function defined in $(a,t]$ and $\alpha \in (n, n+1],\ n \in \mathbb{N} \cup \{0\}$. Then the general form of the left M-integral of order α of f is defined as*

$$\mathfrak{I}_{M,a}^{\alpha,\beta;n} f(t) = \frac{\Gamma(\beta+1)}{n!} \int_a^t \frac{(t-x)^n}{x(x-a)^{n-\alpha}} f(x)\ dx. \tag{21}$$

Clearly, for $n = 0$, $\mathfrak{I}_{M,a}^{\alpha,\beta;0} f(t) = \mathfrak{I}_{M,a}^{\alpha,\beta} f(t)$.

Next, we derive a left fractional Taylor's theorem with integral remainder associated to the above definition.

Theorem 11 *Let $f : [a,\infty) \to \mathbb{R}$ be $(n+1)$ times continuously differentiable for $t,\ t > a$ with $\beta > 0$ and $\alpha \in (n, n+1],\ n \in \mathbb{N} \cup \{0\}$. Then for all $t > a$,*

$$\mathfrak{I}_{M,a}^{\alpha,\beta;n} \left({}_i\mathfrak{D}_{M,a}^{\alpha,\beta;n} f(t)\right) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}. \tag{22}$$

Proof. From Definition 9, we have

$$\mathfrak{I}_{M,a}^{\alpha,\beta;n} \left({}_i\mathfrak{D}_{M,a}^{\alpha,\beta;n} f(t)\right) = \frac{\Gamma(\beta+1)}{n!} \int_a^t \frac{(t-x)^n}{x(x-a)^{n-\alpha}}\ {}_i\mathfrak{D}_{M,a}^{\alpha,\beta;n} f(x)\ dx$$
$$= \frac{\Gamma(\beta+1)}{n!} \int_a^t \frac{(t-x)^n}{x(x-a)^{n-\alpha}}\ \frac{x(x-a)^{n-\alpha}}{\Gamma(\beta+1)} f^{(n+1)}(x)\ dx$$

$$= \frac{1}{n!} \int_a^t (t-x)^n \ f^{(n+1)}(x) \ dx.$$

Now, taking one by one integer order integration, we get

$$\mathfrak{I}_{M,a}^{\alpha,\beta;n} \left({}_i\mathfrak{D}_{M,a}^{\alpha,\beta;n} f(t) \right) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}.$$

■

3 Relation with Other Fractional Derivatives

Here we will show the particular cases of our newly defined α-LLGT M-fractional derivative with the various fractional derivatives. Taking $\beta = 1,\ a = 0$ and $i = 1$ in (10), we get

$${}_1\mathfrak{D}_{M,0}^{\alpha,1} f(t) = \lim_{\xi \to 0} \frac{f\left(t\ {}_1\mathbb{E}_1(\xi t^{-\alpha})\right) - f(t)}{\xi}.$$

But note that

$${}_1\mathbb{E}_1(\xi t^{-\alpha}) = \sum_{k=0}^{1} \frac{(\xi t^{-\alpha})^k}{\Gamma(k+1)} = 1 + \xi t^{-\alpha}.$$

Therefore,

$${}_1\mathfrak{D}_{M,0}^{\alpha,1} f(t) = \lim_{\xi \to 0} \frac{f\left(t\ (1+\xi t^{-\alpha})\right) - f(t)}{\xi} = \lim_{\xi \to 0} \frac{f\left(t + \xi t^{1-\alpha}\right) - f(t)}{\xi} = T^{(\alpha)}(t),$$

which is the conformable fractional derivative given in (1) . Now, taking $\beta = 1,\ a = 0$ and applying the limit $i \to \infty$ on both sides of (10), we get

$${}_\infty\mathfrak{D}_{M,0}^{\alpha,1} f(t) = \lim_{\xi \to 0} \frac{f\left(t\ {}_\infty\mathbb{E}_1(\xi t^{-\alpha})\right) - f(t)}{\xi}. \tag{23}$$

But by (4)

$${}_\infty\mathbb{E}_1(\xi t^{-\alpha}) = \sum_{k=0}^{\infty} \frac{(\xi t^{-\alpha})^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{(\xi t^{-\alpha})^k}{k!} = e^{\xi t^{-\alpha}}. \tag{24}$$

Thus, from (23) and (24)

$${}_\infty\mathfrak{D}_{M,0}^{\alpha,1} f(t) = \lim_{\xi \to 0} \frac{f\left(t\ e^{\xi t^{-\alpha}}\right) - f(t)}{\xi} = \mathbf{D}^\alpha f(t),$$

which is the alternative fractional derivative given in (2).

Now, Taking $\beta = 1$ and $a = 0$ in (10), we get

$${}_i\mathfrak{D}_{M,0}^{\alpha,1} f(t) = \lim_{\xi \to 0} \frac{f\left(t\ {}_i\mathbb{E}_1(\xi t^{-\alpha})\right) - f(t)}{\xi}.$$

Again (4) yields

$${}_i\mathbb{E}_1(\xi t^{-\alpha}) = \sum_{k=0}^{i} \frac{(\xi t^{-\alpha})^k}{\Gamma(k+1)} = e_i^{\xi t^{-\alpha}},$$

where e_i denotes the truncated exponential function. Therefore, we have

$$ {}_i\mathfrak{D}^{\alpha,1}_{M,0}f(t) = \lim_{\xi\to 0}\frac{f\left(t\ e_i^{\xi t^{-\alpha}}\right) - f(t)}{\xi} = \mathbb{D}_i^{\alpha}f(t), $$

which is the generalized fractional derivative [11].

Now, applying the limit $i \to \infty$ on both sides of (10) and taking $a = 0$, we get

$$ {}_\infty\mathfrak{D}^{\alpha,\beta}_{M,0}f(t) = \lim_{\xi\to 0}\frac{f\left(t\ {}_\infty\mathbb{E}_\beta(\xi t^{-\alpha})\right) - f(t)}{\xi}, \tag{25} $$

where from (4)

$$ {}_\infty\mathbb{E}_\beta(\xi t^{-\alpha}) = \sum_{k=0}^{\infty}\frac{(\xi t^{-\alpha})^k}{\Gamma(\beta k+1)} = \mathbb{E}_\beta\left(\xi t^{-\alpha}\right). \tag{26} $$

Thus, by (25) and (26)

$$ {}_\infty\mathfrak{D}^{\alpha,\beta}_{M,0}f(t) = \lim_{\xi\to 0}\frac{f\left(t\ \mathbb{E}_\beta(\xi t^{-\alpha})\right) - f(t)}{\xi} = \mathbb{D}_M^{\alpha,\beta}f(t), $$

which is the local M-fractional derivative given in (3). Taking $a = 0$ in (10), we get

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,0}f(t) = \lim_{\xi\to 0}\frac{f\left(t\ {}_i\mathbb{E}_\beta(\xi t^{-\alpha})\right) - f(t)}{\xi} = {}_i\mathcal{D}_M^{\alpha,\beta}f(t), $$

which is the truncated M-fractional derivative as in (5).

Lastly, taking the limit $i \to \infty$ on both sides of (10), we get

$$ {}_\infty\mathfrak{D}^{\alpha,\beta}_{M,a}f(t) = \lim_{\xi\to 0}\frac{f\left(t\ {}_\infty\mathbb{E}_\beta(\xi(t-a)^{-\alpha})\right) - f(t)}{\xi}. $$

But from (4)

$$ {}_\infty\mathbb{E}_\beta\left(\xi(t-a)^{-\alpha}\right) = \sum_{k=0}^{\infty}\frac{\left(\xi(t-a)^{-\alpha}\right)^k}{\Gamma(\beta k+1)} = \mathbb{E}_\beta(\xi(t-a)^{-\alpha}). $$

Thus, we conclude that

$$ {}_\infty\mathfrak{D}^{\alpha,\beta}_{M,a}f(t) = \lim_{\xi\to 0}\frac{f\left(t\ \mathbb{E}_\beta(\xi(t-a)^{-\alpha})\right) - f(t)}{\xi} = \mathfrak{D}^{\alpha,\beta}_{M,a}f(t), $$

which is the left local general M-fractional derivative given in (6). So, from this section, we can say that this newly defined fractional derivative generalizes the cited conformable fractional derivatives.

4 Applications

4.1 Role of α-LLGT M-Fractional Derivative in Physical Problems

In this section, we have generalized some well known physical problems using this newly defined conformable fractional derivative, α-LLGT M-fractional derivative.

1. First, we obtain the general solution of a differential equation with the help of α-LLGT M-fractional derivative which is represented by

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,0}u(t) + P(t)\ u = Q(t)\ u^n, \tag{27} $$

where $P(t), Q(t)$ are α-differentiable functions, $u(t)$ is an unknown function to be determined and $n \in \mathbb{N} \cup \{0\}$. Now with the use of Theorem 2, (27) becomes

$$\frac{t^{1-\alpha}}{\Gamma(\beta+1)} \frac{du}{dt} + P(t)\ u = Q(t)\ u^n,$$

which is Bernoulli's equation whose solution is given by

$$u^{1-n} = e^{-(1-n)\ \mathfrak{I}^{\alpha,\beta}_{M,0} P(t)} \left(\mathfrak{I}^{\alpha,\beta}_{M,0} \left(Q(t)\ e^{(1-n)\ \mathfrak{I}^{\alpha,\beta}_{M,0} P(t)} \right) \right) + C,$$

where C is an arbitrary constant.

Now, we select some particular cases for this example as follows: Take $P(t) = \mu, \mu \in \mathbb{R},\ Q(t) = 0,\ u(0) = u_0,\ a = 0,\ 0 < \alpha \leq 1,\ \beta > 0$ and $n = 0$. Then (27) becomes

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,0} u(t) + \mu u = 0. \tag{28}$$

Using Theorem 2 the equation (28) can be written as

$$\frac{t^{1-\alpha}}{\Gamma(\beta+1)} \frac{du}{dt} + \mu u = 0,$$

which is a linear differential equation. By taking $u(0) = u_0$, we get the solution of (28) as

$$u(t) = u_0\ \mathbb{E}_1 \left(-\frac{\mu}{\alpha} \Gamma(\beta+1) t^{\alpha} \right).$$

It can be observed from Theorem 2, if we restrict the parameters $\alpha = 1, a = 0$ and $\beta = 1$ of the α-LLGT M-fractional derivative, then it reduces to the classical derivative operator and for this restricted parametric values, the reduced equation of (28) becomes

$$\frac{du}{dt} + \mu u = 0,\ u(0) = u_0,$$

whose solution is given by $u(t) = u_0 e^{-\mu t}$. The comparison of the α-LLGT M-fractional derivative with the classical integer order derivative has been carried out in the following graphs in which the solid line represents the classical solution whereas the other lines show the solution corresponds to the α-LLGT M-fractional derivative with different values of α as shown in the Figures 1, 2 and 3.

MATLAB Code:

```
%%% For solution of (28) %%%
% a=\alpha, b=beta, c=u_0, d= backslash
function [U]=Rmlf(t,a,b,c,d)
K=power(t,a);
K1=gamma(b+1);
K2=-d/a;
U1=(mlf1(1,1,(K.*(K2).*(K1)),10);
U=c.*(U1);
end
```

2. One more well known physical problem, the Newton's law of cooling is generalized with the help of the α-LLGT M-fractional derivative which is represented by

$${}_i\mathfrak{D}^{\alpha,\beta}_{M,0}\ T = -K\ (T - T_m), \tag{29}$$

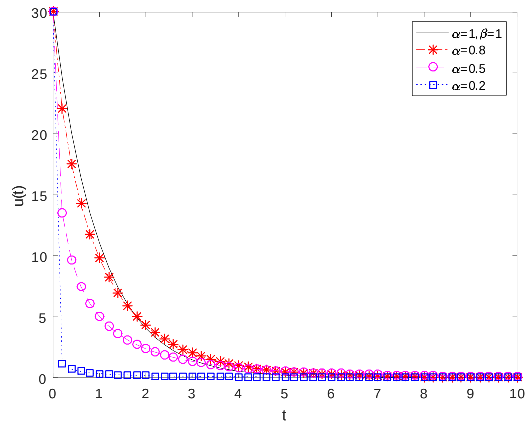

Figure 1: Solutions of (28) for $\beta = 0.6$, $\mu = 1$ and $u_0 = 30$.

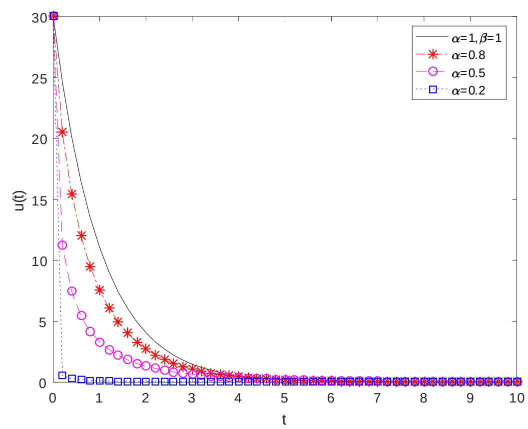

Figure 2: Solutions of (28) for $\beta = 1.2$, $\mu = 1$ and $u_0 = 30$.

where T_m is the temperature of the medium with $T(0) = T_0$, which is considered to be constant and K is the positive constant that depends on the area and nature of the body under consideration. One can easily compute the solution of (29) which is obtained as

$$T(t) = T_m + E_1\left(-\frac{K}{\alpha}\Gamma(\beta+1)t^{\alpha}\right)(T_0 - T_m).$$

Again, by our earlier choices of the parameters, i.e. $\alpha = 1, a = 0$ and $\beta = 1$ in (29) we obtain the classical differential equation of the Newton's law of cooling and its solution as $\frac{dT}{dt} = -K\ (T - T_m),\ T(0) = T_0$ and $T(t) = T_m + (T_0 - T_m)\ e^{-Kt}$ respectively.

The following graphs show the comparison of this problem in terms of our newly defined α-LLGT M-fractional derivative with the ordinary derivative. Here, we can observe from the graph that if we choose the value of the parameter β of the truncated Mittag-Leffler function of our newly defined operator wisely, then we can easily approach towards the analytical solution as shown in the Figures 4, 5 and 6.

MATLAB Code:

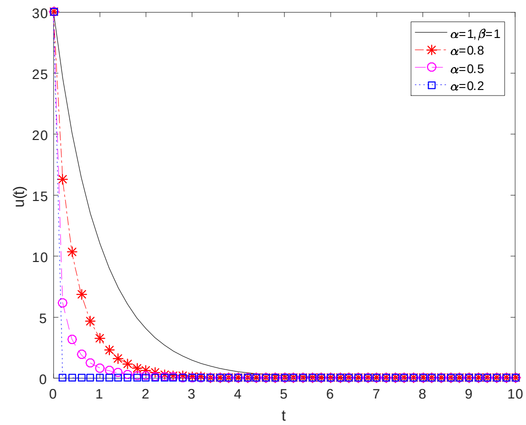

Figure 3: Solutions of (28) for $\beta = -0.5$, $\mu = 1$ and $u_0 = 30$.

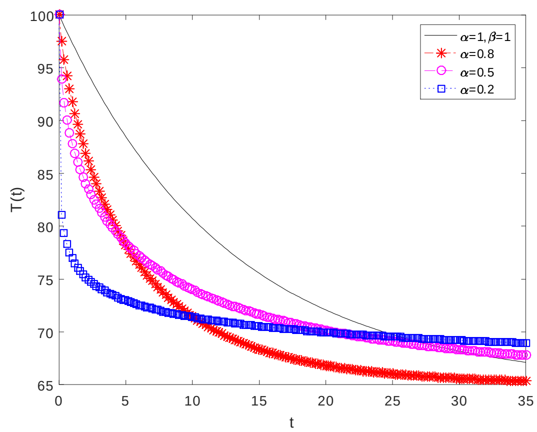

Figure 4: Solutions of (29) for $\beta = 2.3$, $K = 0.08$, $T_m = 65°C$ and $T_0 = 100°C$.

```
%%% For solution of (29) %%%
% a=\alpha, b=beta, c=T_0, d=k, e=T_m.
function [T]=nlcmlf(t,a,b,c,d)
e=input('Enter the value of e: ')
K=power(t,a);
K1=gamma(b+1);
K2=-d/a;
T1=(mlf1(1,1,(K.*(K2).*(K1)),10);
T=e+(c-e).*(T1);
end
```

3. At last, as an another application, we generalize the Kirchoff's voltage law in terms of the α-LLGT M-fractional derivative which is represented by

$$ {}_i\mathfrak{D}^{\alpha,\beta}_{M,0}\ I + \frac{R}{L}\ I = \frac{E}{L}, \tag{30} $$

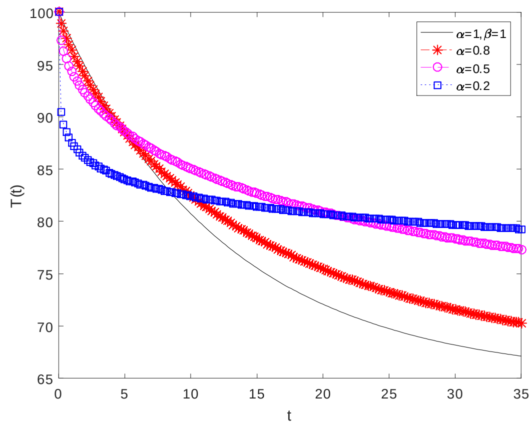

Figure 5: Solutions of (29) for $\beta = 1.2$, $K = 0.08$, $T_m = 65°C$ and $T_0 = 100°C$.

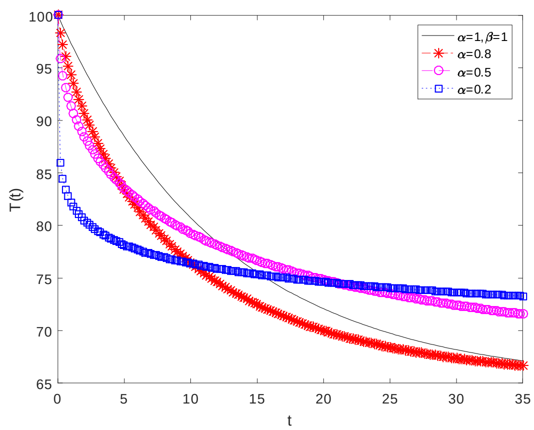

Figure 6: Solutions of (29) for $\beta = -0.5$, $K = 0.08$, $T_m = 65°C$ and $T_0 = 100°C$.

where I is the current with $I(0) = I_0$, R is the resistance, L is the inductance and E is the emf of the circuit whose solution for $E = 0$ is obtained as

$$I(t) = I_0 \ \mathbb{E}_1 \left(-\frac{R}{L\alpha} \Gamma(\beta + 1) t^{\alpha} \right).$$

By restricting the parameters $\alpha = 1$, $a = 0$ and $\beta = 1$ of the α-LLGT M-fractional derivative and then applying the Theorem 2, for $E = 0$, (30) reduces to the classical Kirchoff's voltage law

$$\frac{dI}{dt} + \frac{R}{L} I = 0, \ I(0) = I_0,$$

whose solution is given by $I(t) = I_0 e^{-\frac{R}{L} t}$.

Again, the comparison of the solutions in terms of the α-LLGT M-fractional derivative with the classical order derivative is shown by taking different parametric values in the following graphs, from which we can conclude that we can obtain the solution analogous to the analytical solution if we are allowed to choose the β parameter of the truncated Mittag-Leffler function involved in the definition of the α-LLGT M-fractional derivative as shown in the Figures 7, 8 and 9.

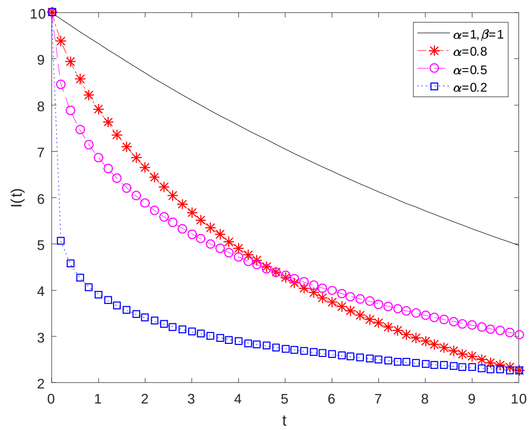

Figure 7: Solutions of (30) for $E = 0, \beta = 2.3$, $R = 3.5\Omega$, $L = 50mH$ and $I_0 = 10$.

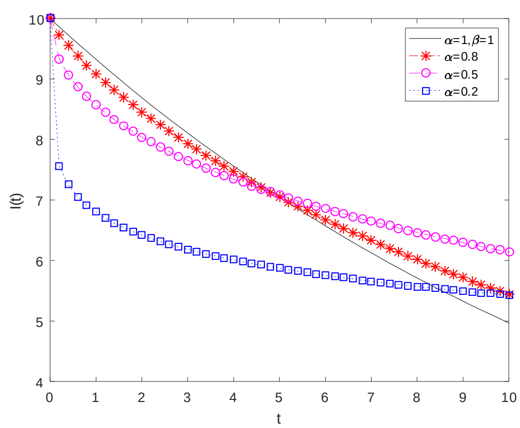

Figure 8: Solutions of (30) for $E = 0, \beta = 1.2$, $R = 3.5\Omega$, $L = 50mH$ and $I_0 = 10$.

```
MATLAB Code:
 %%% For solution of (\QTSN{ref}{eqn:KVL}) %%%
% a=\alpha, b=beta, c=I_0, d=R/L.
function [I]=kvlmlf(t,a,b,c,d)
K=power(t,a);
K1=gamma(b+1);
K2=-d/a;
I1=(mlf1(1,1,(K.*(K2).*(K1)),10);
I=c.*(I1);
end
```

4.2 Role of α-LLGT M-Fractional Derivative in the Image Processing

Texture enhancements is one of the important aspects in image processing, interpretation of image data, signal-processing, robotics, pattern recognition and remote sensing.

The fractional derivative mask maintains high frequency marginal characteristics nonlinearly in places where the fluctuations in grey level are negligible, while low frequency contour features are preserved in

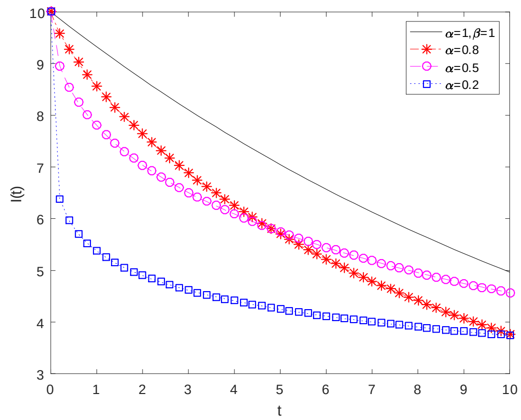

Figure 9: Solutions of (30) for $E = 0$, $\beta = -0.5$, $R = 3.5\Omega$, $L = 50mH$ and $I_0 = 10$.

smooth areas by the fractional derivative mask. The Grünwald-Letnikov (G-L) derivative is used for image enhancement in [20, 21]. Also, in [9], He et al. have used the G-L fractional differential operator to improve denoising operator mask.

In this subsection, we introduce α-LLGT M-fractional derivative mask functioning like G-L fractional derivative mask which enhances the evaluation index of the images. The Grünwald-Letnikov (G-L) derivative is an extension of the ordinary derivative in fractional calculus which allows to take the non-integer number of times the derivative and it is defined as follows:

Definition 10 *The G-L derivative of fractional order $q, q > 0$ is defined as [8]*

$$ {}_a\mathbf{D}_t^q = \lim_{h\to 0} \frac{1}{h^q} \sum_{m=0}^{\left[\frac{t-a}{h}\right]} (-1)^m \binom{q}{m} f(t - mh), \tag{31}$$

where $\binom{q}{m}$ is the binomial coefficient.

Now, from the Definition 6 for $a = 0$, we have

$$ {}_i\mathfrak{D}_{M,0}^{\alpha,\beta} f(t) = \lim_{\xi\to 0} \frac{f\left(t \ {}_iE_\beta(\xi t)^{-\alpha}\right) - f(t)}{\xi}.$$

Therefore,

$$\begin{aligned} \left({}_i\mathfrak{D}_{M,0}^{\alpha,\beta} f(t)\right)^2 &= \lim_{\xi_1\to 0} \frac{{}_i\mathfrak{D}_{M,0}^{\alpha,\beta} f\left(t \ {}_iE_\beta(\xi_1 t)^{-\alpha}\right) - \ {}_i\mathfrak{D}_{M,0}^{\alpha,\beta} f(t)}{\xi_1} \\ &= \lim_{\xi_1\to 0} \frac{\lim\limits_{\xi_2\to 0} \dfrac{f\left(t \ {}_iE_\beta(\xi_1 t)^{-\alpha} \ {}_iE_\beta(\xi_2 t)^{-\alpha}\right) - f\left(t \ {}_iE_\beta(\xi_1 t)^{-\alpha}\right)}{\xi_2}}{\xi_1} \\ &\quad - \lim_{\xi_1\to 0} \frac{\lim\limits_{\xi_2\to 0} \dfrac{f\left(t \ {}_iE_\beta(\xi_2 t)^{-\alpha}\right) - f(t)}{\xi_2}}{\xi_1}. \end{aligned}$$

Assuming that ξ's converge synchronously, we get

$$\left({}_i\mathfrak{D}_{M,0}^{\alpha,\beta} f(t)\right)^2 = \lim_{\xi\to 0} \frac{f\left(t \ \left({}_iE_\beta(\xi t)^{-\alpha}\right)^2\right) - 2f\left(t \ {}_iE_\beta(\xi t)^{-\alpha}\right) + f(t)}{\xi^2}.$$

Similarly, we get

$$\left({}_i\mathfrak{D}^{\alpha,\beta}_{M,0}f(t)\right)^3 = \lim_{\xi\to 0}\frac{f\left(t\ \left({}_iE_\beta(\xi t)^{-\alpha}\right)^3\right) - 3f\left(t\ \left({}_iE_\beta(\xi t)^{-\alpha}\right)^2\right) + 3f\left(t\ {}_iE_\beta(\xi t)^{-\alpha}\right) - f(t)}{\xi^3}.$$

Hence, in general for $n \in \mathbb{N}$, we have

$$\left({}_i\mathfrak{D}^{\alpha,\beta}_{M,0}f(t)\right)^n = \lim_{\xi\to 0}\frac{1}{\xi^n}\sum_{m=0}^{n}(-1)^m\binom{n}{m}f\left(t\ \left({}_iE_\beta(\xi t)^{-\alpha}\right)^{n-m}\right).$$

Now, removing the restriction that n be a positive integer, we have

$$\begin{aligned}\left({}_i\mathfrak{D}^{\alpha,\beta}_{M,0}f(t)\right)^q &= \lim_{\xi\to 0}\frac{1}{\xi^q}\sum_{m=0}^{\infty}(-1)^m\binom{q}{m}f\left(t\ \left({}_iE_\beta(\xi t)^{-\alpha}\right)^{q-m}\right)\\ &= \lim_{\xi\to 0}\frac{1}{\xi^q}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(q+1)}{\Gamma(m+1)\Gamma(q-m+1)}f\left(t\ \left({}_iE_\beta(\xi t)^{-\alpha}\right)^{q-m}\right).\end{aligned} \tag{32}$$

The numerical approximation of (32) is as follows:

$$\begin{aligned}\left({}_i\mathfrak{D}^{\alpha,\beta}_{M,0}f(t)\right)^q &\approx f(t) + (-q)f\left((t-1)\left({}_iE_\beta(\xi(t-1))^{-\alpha}\right)\right)\\ &+\frac{(-q)(-q+1)}{2}f\left((t-2)\left({}_iE_\beta(\xi(t-2))^{-\alpha}\right)\right)\\ &+\cdots+\frac{\Gamma(-q+1)}{\Gamma(m+1)\Gamma(-q-m+1)}f\left((t-m)\ \left({}_iE_\beta(\xi(t-m))^{-\alpha}\right)\right).\end{aligned} \tag{33}$$

Now from (33), to generate the α-LLGT M-fractional derivative mask, the coefficients are obtained as follows:

$$1, -q, \frac{(-q)(-q+1)}{2}, \ldots, \frac{\Gamma(-q+1)}{\Gamma(m+1)\Gamma(-q-m+1)}.$$

In the similar aspect adopted from the numerical approximation of G-L fractional derivative, we get the following 3×3 and 5×5 masks for α-LLGT M-fractional derivative:

$-q$	$-q$	$-q$
$-q$	$8*1$	$-q$
$-q$	$-q$	$-q$

$\frac{(-q)(-q+1)}{2*\xi^q}$	0	$\frac{(-q)(-q+1)}{2*\xi^q}$	0	$\frac{(-q)(-q+1)}{2*\xi^q}$
0	$-q$	$-q$	$-q$	0
$\frac{(-q)(-q+1)}{2*\xi^q}$	$-q$	$8*1$	$-q$	$\frac{(-q)(-q+1)}{2*\xi^q}$
0	$-q$	$-q$	$-q$	0
$\frac{(-q)(-q+1)}{2*\xi^q}$	0	$\frac{(-q)(-q+1)}{2*\xi^q}$	0	$\frac{(-q)(-q+1)}{2*\xi^q}$

We choose an original image as the 4D ultrasound of a fetal and we select various fractional order $q = 0.5, 0.8, 1.0, 1.2, 1.5$ to test the effect of image enhancement as shown in the Figure 10.

When the order of α-LLGT M-fractional derivative is fractional, the brightness of the image improved significantly. But, when the level of gray level images decreases, the local texture details also disappear. When the order of the operator is 1, that is an integer order derivative, the brightness of the image and texture are very weak, in fact here image processing giving void. Here, again from the Figure 10, observe that when the order is 1.5, the brightness of the image is improved and it can preserve image texture information well. In addition, the image gray has not been destroyed. The results are shown in the Figure 10 which indicate that the α-LLGT M-fractional derivative mask can not only enhance the image quality, but also can preserve weak texture and smooth area containing both global and local information in the image.

To interpretate the image data we need (i) Entropy (ii) PSNR.

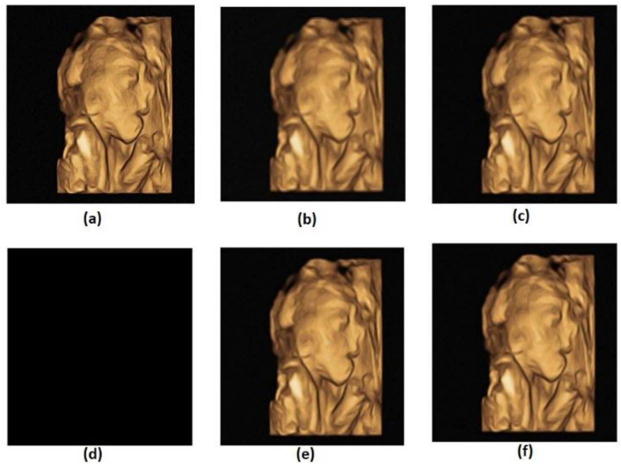

Figure 10: Comparison of 4D ultrasound of fetal images: (a) Original image (b) $q = 0.5$-order (c) $q = 0.8$-order (d) $q = 1.0$-order (e) $q = 1.2$-order (f) $q = 1.5$-order.

(i) **Entropy:** Entropy is an important measure of the uncertainty with regard to randomness. It is a statistical measure of randomness that can be used to characterize the texture of an image. The formula of an entropy is given as [14]

$$E_0 = \sum_{i=1}^{N} \rho_i \log_2(\rho_i),$$

where ρ_i is the probability gray value and N is the maximum gray value. Entropy represents the average amount of information in an image. The small entropy indicates that the image has less details, and a high value indicates that the image has more details.

(ii) **PSNR:** Peak Signal to Noise Ratio (PSNR) is the ratio between the maximum possible power of a signal and the power of corresponding noise. The PSNR is defined as [14]

$$\text{PSNR} = 10 \log_{10} \left(\frac{Max^2}{MSE} \right),$$

where Max is the maximum possible pixel value of the image and MSE is the Mean Square Error. The PSNR block computes the peak signal to noise ratio between two images. The higher the PSNR, the better the quality of the compressed or reconstructed image.

The following table shows the evaluation index of the images represented in Figure 10. It can be observed from the fourth column of the above table that the original image has Entropy 6.6770 which is lesser as compare to the entropy of the images obtained after applying α-LLGT M-fractional derivative mask of different orders. Also, from the third column of the table, it can be conclude that the figures (b), (c), (e) and (f) have better quality compare to figure (d). In the second column of the Table 1, the mean values are computed which will be useful for noise reduction.

5 Conclusion

We have established and studied a new conformable fractional derivative and its integral analogue which we have called as the α-LLGT M-fractional derivative. We have proved that this newly defined derivative responds well with respect to classical results of integer order calculus. Additionally, we could find the

Figures		Mean value	PSNR	Entropy
(a)	Original image	61.1308	–	6.6770
	α	**Mean value**	**PSNR**	**Entropy**
(b)	0.5	62.4296	19.6096	6.9510
(c)	0.8	61.1509	23.8937	6.8685
(d)	1.0	0	8.9611	0
(e)	1.2	61.0685	26.2530	6.7262
(f)	1.5	61.0737	26.0578	6.7202

Table 1: Evaluation indices of the images

associations between the α-LLGT M-fractional derivative and left M-integral. The well known results of the calculus like the Rolle's theorem, the MVT, the fundamental theorem of calculus and the theorem containing integration by parts are also generalized for our newly defined fractional derivative.

Also, we have shown the other fractional derivatives available in the literature as the particular cases to our new generalizations. Using the proved result in the previous sections, we have obtained and solved the generalized versions of some of the well known physical problems Like Bernoulli type fractional differential equation, Newton's Law of cooling and Kirchoff's voltage law by our newly defined α-LLGT M-fractional derivative and with the use of MATLAB software, we have compared their solutions with the ordinary versions of the same. From Figures 1 to 9, it can be concluded that the physical problem described by the α-LLGT M-fractional derivative, then by assigning appropriate parametric value of the parameter β from the truncated Mittag-Leffler function, one can easily approach to the existing ordinary solution. Even a faster convergence rate can be obtained.

At last, we have generalized the α-LLGT M-fractional derivative mask through which the images of the 4D ultrasound of a fetal are enhanced by different fractional value α as shown in the Figures 10 [(a), (b), (c), (d), (f)] whereas the Figure 10 [(e)] gives a void as it is concerned about $\alpha = 1$. The image quality, texture and smooth areas are enhanced through this newly defined mask. Comparison of the evaluation indices for this image enhancement can be observed from Table 1.

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