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Calderón's Type Reproducing Formula Related To The q-Dunkl Two Wavelet Theory*

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Abstract

In this paper, using some elements of the q-harmonic analysis associated to the q-Dunkl operator introduced by N. Bettaibi et al. in [1], for fixed $0 < q < 1$, the notion of a q-Dunkl two-wavelet is introduced. The resolution of the identity formula for the q-Dunkl continuous wavelet transform is then formulated and proved. Calderón's type reproducing formula in the context of the q-Dunkl two wavelet theory is proved.

1 Introduction

In recent years, the q-theory, called also in some literature "quantum calculus", began to arise. Interest in this theory is grown at an explosive rate by both physicists and mathematicians due to a large number of its application domains. For instance, a lot of work has been carried out while developing some q-analogues of Fourier analysis using elements of quantum calculus (see [1, 4, 5, 6, 26, 22] and references therein). In the recent mathematical literature, we find many articles that deal with the theory of q-Fourier analysis associated with the q-Dunkl transform. In [1], Bettaibi et al. introduced a new q-analogue of the classical Dunkl operator and studied its related Fourier transform, which is a q-analogue of the classical Bessel-Dunkl one and called the q-Dunkl transform. The q-analogue of the Dunkl operator is defined in terms of Rubin's q-differential operator ∂_q, introduced in [23, 22].

Calderón formula [3] involving convolution related to the Fourier transform is useful in obtaining reconstruction formula for wavelet transform besides many other applications in decomposition of certain function spaces. It is expressed as follows:

$$\Phi(\xi) = \frac{1}{c_{\varphi,\phi}} \int_0^{+\infty} \Phi * \varphi_t * \phi_t(\xi) \frac{dt}{t}, \quad \xi \in \mathbb{R}, \tag{1}$$

where

$$\varphi_t(x) := \frac{1}{t}\varphi\left(\frac{x}{t}\right), \quad \phi_t(x) := \frac{1}{t}\phi\left(\frac{x}{t}\right), \quad \forall x \in \mathbb{R},$$

$c_{\varphi,\phi}$ is a constant depending on functions φ, ϕ and $*$ denotes a convolution operation.

Formula (1) first appeared in the pioneering paper [3] by Calderón. It went on to play an important role as a tool in harmonic analysis, see e.g. [11] or, more recently, [9]. It also occurs in wavelet theory [7, 12, 14, 18].

It is believed that Calderón's type reproducing formula as discussed here will be of great utility in Inversion Problems (see [19, 24, 25]), and in wavelet theory on Bessel-Kingman hypergroups (see [25]). Our investigation in the present work consists to study similar questions when in (1), the classical convolution $*$ is replaced by a generalized q-Dunkl convolution a on the real line generated by the q-Dunkl differential operator $\Lambda_{q,\alpha}$, $\alpha \geq -1/2$, introduced in [1].

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In this paper, using some new elements of q-harmonic analysis related to the q-Dunkl transform $\mathcal{F}_D^{q,\alpha}$ introduced by N. Bettaibi et al. in [1], we define and study the q-Dunkl two wavelet and the continuous q-wavelet transform associated with this q-harmonic analysis. In addition to several properties, we establish a Plancherel formula and an inversion theorem for this transform. As applications, we prove a Calderón's type reproducing formula in the context of the q-Dunkl two wavelet theory.

The outline of this paper is arranged as follows. In Section 2, we state some basic notions and results from the q-harmonic analysis related to the q-Dunkl transform that will be needed throughout this paper. In Section 3, we define and study the q-Dunkl two wavelet and the continuous q-wavelet transform associated with the q-Dunkl operator $\Lambda_{q,\alpha}$. Thus, some results (Plancherel's formula, inversion formula, etc.) are established. Section 4 is devoted to giving the main results of this paper, Calderón's type reproducing formula in the context of the q-Dunkl two wavelet theory is proved.

2 Harmonic Analysis Associated with the q-Dunkl Operator

We recall some usual notions and notations used in the q-theory (see [16] and [20]). We refer to the book by G. Gasper and M. Rahman [16] for the definitions, notations and properties of the q-shifted factorials. The references [1, 2, 4, 22] are devoted to the q-Dunkl Fourier analysis. Throughout this paper, we assume $0 < q < 1$, $\alpha \geq -1/2$ and we denote

$$\mathbb{R}_q = \{\pm q^n,\, n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n,\, n \in \mathbb{Z}\} \quad \text{and} \quad \widehat{\mathbb{R}_q} = \mathbb{R}_q \cup \{0\}.$$

For complex number a, the q-shifted factorials are defined by

$$(a;q)_0 = 1,\;\; (a;q)_n = \prod_{l=0}^{n-1}(1-aq^l),\; n = 1,2,...,\;\; (a;q)_\infty = \prod_{l=0}^{+\infty}(1-aq^l).$$

We also denote for all $x \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$[x]_q = \frac{1-q^x}{1-q}, \quad [n]_q! = [1]_q \times [2]_q \times ... \times [n]_q = \frac{(q;q)_n}{(1-q)^n}.$$

The q-Gamma function is given by (see [13])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, ...$$

It satisfies the following relations

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x) = [x]_q!,\;\; \Gamma_q(1) = 1 \;\; \text{and} \;\; \lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x),\;\; \mathrm{Re}(x) > 0.$$

The q-Jackson integrals from 0 to a, from 0 to $+\infty$ and from $-\infty$ to $+\infty$ are defined by (see [13])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{+\infty} q^n f(aq^n),$$

$$\int_0^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{+\infty} q^n f(q^n),$$

$$\int_{-\infty}^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{+\infty} q^n$$

provided the sums converge absolutely. In particular, for $a \in \mathbb{R}_q^+$,

$$\int_a^{+\infty} f(x) d_q x = (1-q)a \sum_{n=-\infty}^{-1} q^n f(aq^n).$$

The q-Jackson integral in a generic interval $[a,b]$ is given by (see [13])

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

The q-derivative $\mathcal{D}_q f$ of a function f is given by

$$\mathcal{D}_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if } x \neq 0,$$

where $\mathcal{D}_q f(0) = f^{'}(0)$ provided $f^{'}(0)$ exists. The Rubin's q-differential operator is defined in [22, 23] by

$$\partial_q f(x) = \begin{cases} \dfrac{f(q^{-1}x) + f(-q^{-1}x) - f(qx) + f(-qx) - 2f(-x)}{2(1-q)x} & \text{if } x \neq 0, \\ \lim\limits_{x \to 0} \partial_q f(x), \qquad (\text{in } \mathbb{R}_q) & \text{if } x = 0. \end{cases}$$

We remark that if f is differentiable at x, then $\partial_q f(x)$ tends to $f^{'}(x)$ as q tends to 1 and by using the definition of ∂_q, we can see that

$$\partial_q f(x) = q^{-1} \mathcal{D}_{q^{-1}} f_e(x) + \mathcal{D}_q f_o(x),$$

with f_e and f_o are respectively, the even and the odd parts of f defined by

$$f_e(x) = \frac{f(x) + f(-x)}{2} \text{ and } f_o(x) = \frac{f(x) - f(-x)}{2}.$$

A repeated application of the Rubin's q-differential operator n times is denoted by

$$\partial_q^0 f = f, \quad \partial_q^{n+1} f = \partial_q(\partial_q^n f).$$

The q-analogue of the integration theorem by a change of variable can be stated as follows

$$\int_a^b f\left(\frac{\lambda}{\chi}\right) |\lambda|^{2\alpha+1} d_q\lambda = \chi^{2\alpha+2} \int_{\frac{a}{\chi}}^{\frac{b}{\chi}} f(x)|x|^{2\alpha+1} d_q x, \quad \forall \chi \in \mathbb{R}_q.$$

The third q-Bessel function is defined as follows (see [1, 15])

$$J_\alpha(x;q^2) = \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1) q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)} \left(\frac{x}{1+q}\right)^{2n}.$$

$J_\alpha(\cdot;q^2)$ has the normalized form

$$j_\alpha(x;q^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1) q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)} \left(\frac{x}{1+q}\right)^{2n}.$$

For $\alpha \geq -1/2$, the q-Dunkl operator $\Lambda_{q,\alpha} f$ is defined by

$$\Lambda_{q,\alpha} f(x) = \partial_q [\mathcal{H}_{q,\alpha}(f)](x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{2x},$$

where

$$\mathcal{H}_{q,\alpha} : f = f_e + f_o \mapsto f_e + q^{2\alpha+1} f_o.$$

It satisfies the following relations:

- For $\alpha = -1/2$, $\Lambda_{q,\alpha} = \partial_q$.
- $\Lambda_{q,\alpha}$ leaves $\mathcal{S}_q(\mathbb{R}_q)$ invariant.
- For all $a \in \mathbb{C}$, $\Lambda_{q,\alpha}\left[f(ax)\right] = a\Lambda_{q,\alpha}(f)(ax)$.
- If f is odd, then
$$\Lambda_{q,\alpha}(f)(x) = q^{2\alpha+1}\partial_q f(x) + [2\alpha+1]_q \frac{f(x)}{x}$$
and if f is even, then
$$\Lambda_{q,\alpha} f(x) = \partial_q f(x).$$

It was shown in [1] that for all $\lambda \in \mathbb{C}$, the function

$$x \mapsto \Psi_{q,\alpha}(\lambda x) = j_\alpha(\lambda x, q^2) + \frac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x, q^2), \quad \forall x \in \mathbb{R}_q \tag{2}$$

is the unique analytic solution of the q-differential-difference equation:

$$\Lambda_{q,\alpha} f(x) = i\lambda f(x), \quad f(0) = 1.$$

Some other properties of the q-Dunkl kernel are given in the following results.

Theorem 1 *The following properties are checked:*

i) For all $\lambda, x \in \mathbb{R}$, $\quad \overline{\Psi_{q,\alpha}(\lambda x)} = \Psi_{q,\alpha}(-\lambda x)$.

ii) For all $\lambda, x \in \mathbb{R}_q$, $\quad \Lambda_{q,\alpha}\Psi_{q,\alpha}(\lambda x) = i\lambda\Psi_{q,\alpha}(\lambda x)$.

iii) For all $\lambda \in \mathbb{R}_q$, $\Psi_{q,\alpha}(\lambda .)$ *is bounded on* $\widehat{\mathbb{R}_q}$ *and we have*

$$|\Psi_{q,\alpha}(\lambda x)| \leq \frac{4}{(q,q)_\infty}, \quad \forall x \in \widehat{\mathbb{R}_q}. \tag{3}$$

Proof. See [1, Proposition 6]. ■

In what follows, let us fix some notations:

- $\mathcal{C}_q^p(\mathbb{R}_q)$, the space of functions f, p times q-differentiable on $\widehat{\mathbb{R}_q}$ such that for all $0 \leq n \leq p$, $\Lambda_{q,\alpha}^n f$ is continuous on $\widehat{\mathbb{R}_q}$.
- $\mathcal{S}_q(\mathbb{R}_q)$, the space of functions f defined on $\mathbb{R}_q$ satisfying
$$\forall n, m \in \mathbb{N}, \quad P_{n,m,q}(f) = \sup_{x \in \mathbb{R}_q} |x^m \partial_q^n f(x)| < \infty$$
and
$$\lim_{x \to 0} \partial_q^n(f)(x) \quad (\text{in } \mathbb{R}_q) \quad \text{exists.}$$
- $\mathcal{D}_q(\mathbb{R}_q)$, the subspace of $\mathcal{S}_q(\mathbb{R}_q)$ constituted of functions with compact supports.

We denote by $L^p_{q,\alpha}(\mathbb{R}_q)$, $p \in [1, +\infty]$, the set of all real functions on $\mathbb{R}_q$ for which

$$\|f\|_{q,p,\alpha} = \begin{cases} \left(\int_{-\infty}^{+\infty} |f(x)|^p |x|^{2\alpha+1} d_q x\right)^{1/p} < +\infty & \text{if } 1 \leq p < +\infty, \\ \sup\limits_{xx \in \mathbb{R}_q} |f((x)| < +\infty & \text{if } p = +\infty. \end{cases}$$

For $p = 2$, we provide this space with the scalar product

$$\langle f, g\rangle_{q,\alpha} = c_{q,\alpha} \int_{-\infty}^{+\infty} f(x)\overline{g(x)}|x|^{2\alpha+1} d_q x. \tag{4}$$

By the q-integration by parts one can verify the correlation (see [1])

$$\langle \Lambda_{q,\alpha} f, g\rangle_{q,\alpha} = -\langle f, \Lambda_{q,\alpha} g\rangle_{q,\alpha}, \tag{5}$$

for any functions $f, g \in \mathcal{D}_q(\mathbb{R}_q)$.

Definition 1 ([1]) *The q-Dunkl transform $\mathcal{F}_D^{q,\alpha}$ associated with the q-Dunkl operator $\Lambda_{q,\alpha}$ is defined for every function in $L^1_{q,\alpha}(\mathbb{R}_q)$ by*

$$\mathcal{F}_D^{q,\alpha}(f)(\lambda) = c_{q,\alpha} \int_{-\infty}^{+\infty} f(x)\Psi_{q,\alpha}(-\lambda x)|x|^{2\alpha+1} d_q x, \quad \forall \lambda \in \widehat{\mathbb{R}_q}, \tag{6}$$

where

$$c_{q,\alpha} = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}.$$

The q-Dunkl transform $\mathcal{F}_D^{q,\alpha}$ satisfies the following properties:

- $L^1 - L^\infty$-boundedness: For all $f \in L^1_{q,\alpha}(\mathbb{R}_q)$, we have $\mathcal{F}_D^{q,\alpha}(f) \in \mathcal{C}_{q,0}(\mathbb{R}_q)$ and we get

$$\|\mathcal{F}_D^{q,\alpha}(f)\|_{q,\infty} \leq \frac{4c_{q,\alpha}}{(q,q)_\infty}\|f\|_{q,1,\alpha}. \tag{7}$$

- *Riemann-Lebesque Lemma:* If $f \in L^1_{q,\alpha}(\mathbb{R}_q)$, then

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\mathbb{R}_q}} \mathcal{F}_D^{q,\alpha}(f)(\lambda) = 0.$$

- *q-Plancherel formula:* The q-Dunkl transform $\mathcal{F}_D^{q,\alpha}$ is an isomorphism from $\mathcal{S}_q(\mathbb{R}_q)$ onto itself and extends uniquely to an isometric isomorphism on $L^2_{q,\alpha}(\mathbb{R}_q)$ with:

$$\|\mathcal{F}_D^{q,\alpha}(f)\|_{q,2,\alpha} = \|f\|_{q,2,\alpha}. \tag{8}$$

- q-Parseval's formula: For all f, g in $L^2_{q,\alpha}(\mathbb{R}_q)$, we have

$$\int_{-\infty}^{+\infty} f(x)\overline{g(x)}|x|^{2\alpha+1} d_q x = \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\xi)\overline{\mathcal{F}_D^{q,\alpha}(g)(\xi)}|\xi|^{2\alpha+1} d_q \xi. \tag{9}$$

- *q-Inversion formula:* If $f \in L^1_{q,\alpha}(\mathbb{R}_q)$ such that $\mathcal{F}_D^{q,\alpha}(f) \in L^1_{q,\alpha}(\mathbb{R}_q)$, then the q-inversion formula holds and we have

$$\begin{aligned} f(x) &= c_{q,\alpha} \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda)\Psi_{q,\alpha}(\lambda x)|\lambda|^{2\alpha+1} d_q \lambda \\ &= \overline{\mathcal{F}_D^{q,\alpha}(\overline{\mathcal{F}_D^{q,\alpha}(f)})}(x). \end{aligned} \tag{10}$$

Note that a consequence of the q-Plancherel theorem says that for every function $f \in L^2_{q,\alpha}(\mathbb{R}_q)$, its inverse transform $(\mathcal{F}_D^{q,\alpha})^{-1}$ is given by

$$\begin{aligned} (\mathcal{F}_D^{q,\alpha})^{-1}(f)(x) &= c_{q,\alpha} \int_{-\infty}^{+\infty} f(\lambda)\Psi_{q,\alpha}(\lambda x)|\lambda|^{2\alpha+1} d_q \lambda \\ &= \mathcal{F}_D^{q,\alpha}(f)(-x). \end{aligned} \tag{11}$$

Proposition 1 (q-hausdorff inequality) *Let $f \in L^p_{q,\alpha}(\mathbb{R}_q)$, with $p \geq 1$. Then $\mathcal{F}_D^{q,\alpha}(f) \in L^{p'}_{q,\alpha}(\mathbb{R}_q)$. Moreover, if $1 \leq p \leq 2$, hence we have*

$$\|\mathcal{F}_D^{q,\alpha}(f)\|_{q,p',\alpha} \leq \left(\frac{4c_{q,\alpha}}{(q,q)_\infty}\right)^{\frac{2}{p}-1} \|f\|_{q,p,\alpha}, \tag{12}$$

where the numbers p and p' above are conjugate exponents

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Proof. This is an immediate consequence of (7), q-Plancherel formula (8), q-inversion formula (10) and the Riesz-Thorin Theorem. ■

Proposition 2 *If f in $\mathcal{D}_q(\mathbb{R}_q)$, then for all $n = 1, 2, ...,$ we have*

$$\mathcal{F}_D^{q,\alpha}(\Lambda^n_{q,\alpha} f)(\lambda) = (i\lambda)^n \mathcal{F}_D^{q,\alpha}(f)(\lambda).$$

Proof. *The proof is immediate by using the formulas (5), (6) and the proof by induction for n.* ■

Proposition 3 *For all f in $\mathcal{D}_q(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$), we have the following relations*

$$\mathcal{F}_D^{q,\alpha}(\overline{f})(\lambda) = \overline{\mathcal{F}_D^{q,\alpha}(\check{f})(\lambda)}, \quad \forall \lambda \in \mathbb{R}_q, \tag{13}$$

$$\mathcal{F}_D^{q,\alpha}(f)(\lambda) = \mathcal{F}_D^{q,\alpha}(\check{f})(-\lambda), \quad \forall \lambda \in \mathbb{R}_q, \tag{14}$$

where $\check{f}$ is the function defined by

$$\check{f}(x) = f(-x), \quad \forall x \in \mathbb{R}_q.$$

Proof. For all f in $\mathcal{D}_q(\mathbb{R}_q)$, it immediately follows from i) of the Theorem 1 and (6) that

$$\begin{aligned} \mathcal{F}_D^{q,\alpha}(\overline{f})(\lambda) &= c_{q,\alpha} \int_{-\infty}^{+\infty} \overline{f(x)} \Psi_{q,\alpha}(-\lambda x)|x|^{2\alpha+1} d_q x \\ &= \overline{c_{q,\alpha} \int_{-\infty}^{+\infty} f(x) \Psi_{q,\alpha}(\lambda x)|x|^{2\alpha+1} d_q x} \\ &= \overline{c_{q,\alpha} \int_{-\infty}^{+\infty} \check{f}(x) \Psi_{q,\alpha}(-\lambda x)|x|^{2\alpha+1} d_q x} \\ &= \overline{\mathcal{F}_D^{q,\alpha}(\check{f})(\lambda)}. \end{aligned}$$

Then the formula (13) is proved. In the same way, we prove (14). ■

Definition 2 *The generalized q-Dunkl translation operator is defined for $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ and $x, y \in \mathbb{R}_q$ by*

$$\mathcal{T}_x^{q,\alpha}(f)(y) = c_{q,\alpha} \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda) \Psi_{q,\alpha}(\lambda x) \Psi_{q,\alpha}(\lambda y)|\lambda|^{2\alpha+1} d_q \lambda$$

and

$$\mathcal{T}_0^{q,\alpha}(f) = f.$$

In particular the product formula

$$\mathcal{T}_x^{q,\alpha}(\Psi_{q,\alpha}(\lambda \cdot))(y) = \Psi_{q,\alpha}(\lambda x) \Psi_{q,\alpha}(\lambda y), \quad \forall x, \lambda, y \in \mathbb{R}_q$$

holds.

Proposition 4 *If $f \in L^2_{q,\alpha}(\mathbb{R}_q)$, then $\mathcal{T}^{q,\alpha}_x(f) \in L^2_{q,\alpha}(\mathbb{R}_q)$ and we have*

$$\|\mathcal{T}^{q,\alpha}_x(f)\|_{q,2,\alpha} \leq \frac{4}{(q,q)_\infty}\|f\|_{q,2,\alpha}, \quad \forall x \in \mathbb{R}_q. \tag{15}$$

Proof. See [2, Proposition 2]. ■

Proposition 5 *The q-Dunkl translation operator checks the following statements:*

i) For $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ and $x, y \in \mathbb{R}_q$, we have

$$\mathcal{F}^{q,\alpha}_D(T^{q,\alpha}_x f)(\lambda) = \Psi_{q,\alpha}(\lambda x)\mathcal{F}^{q,\alpha}_D(f)(\lambda). \tag{16}$$

ii) For all $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ and $x, y \in \mathbb{R}_q$, we have

$$T^{q,\alpha}_x(f)(y) = T^{q,\alpha}_y(f)(x) \tag{17}$$

and

$$\overline{\mathcal{T}^{q,\alpha}_x(f)(y)} = \mathcal{T}^{q,\alpha}_x(\overline{f})(y). \tag{18}$$

Proof. See [2, Proposition 2]. ■

The generalized q-Dunkl translation operators allow us to define a q-Dunkl convolution product $*_{q,\alpha}$:

Definition 3 *For all $f, g \in \mathcal{S}_q(\mathbb{R}_q)$, we have*

$$f *_{q,\alpha} g(x) = c_{q,\alpha}\int_{-\infty}^{\infty} T^{q,\alpha}_x(f)(-y)g(y)|y|^{2\alpha+1}d_qy, \quad \forall x, y \in \mathbb{R}_q, \tag{19}$$

provided the q-integral exists.

Theorem 2 *For $f, g \in \mathcal{S}_q(\mathbb{R}_q)$, we have*

i) $\mathcal{F}^{q,\alpha}_D(f *_{q,\alpha} g) = \mathcal{F}^{q,\alpha}_D(f).\mathcal{F}^{q,\alpha}_D(g)$.

ii) $f *_{q,\alpha} g = g *_{q,\alpha} f$.

iii) $(f *_{q,\alpha} g) *_{q,\alpha} h = f *_{q,\alpha} (g *_{q,\alpha} h)$.

iv) $f *_{q,\alpha} g \in \mathcal{S}_q(\mathbb{R}_q)$. Moreover,

$$\int_{-\infty}^{\infty} |f *_{q,\alpha} g(x)|^2|x|^{2\alpha+1}d_qx = \int_{-\infty}^{\infty} |\mathcal{F}^{q,\alpha}_D(f)(x)|^2|\mathcal{F}^{q,\alpha}_D(g)(x)|^2|x|^{2\alpha+1}d_qx.$$

Proof. See [2, Proposition 3 and Proposition 4]. ■

Proposition 6 *Let $p, r, s \geq 1$ such that*

$$\frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{s}.$$

Given two functions $f \in L^p_{q,\alpha}(\mathbb{R}_q)$ and $g \in L^r_{q,\alpha}(\mathbb{R}_q)$. Then $f *_{q,\alpha} g$ exists and we have

$$f *_{q,\alpha} g \in L^s_{q,\alpha}(\mathbb{R}_q), \quad \mathcal{F}^{q,\alpha}_D(f *_{q,\alpha} g) = \mathcal{F}^{q,\alpha}_D(f)\mathcal{F}^{q,\alpha}_D(g). \tag{20}$$

Moreover, if $s \geq 2$, then

$$\|f *_{q,\alpha} g\|_{q,s,\alpha} \leq \frac{4c_{q,\alpha}}{(q,q)_\infty}\|f\|_{q,p,\alpha}\|f\|_{q,r,\alpha}. \tag{21}$$

Proof. Using similar ideas as in the proof for Theorem 5 in [8]. ■

In the sequel, we need some technical Propositions.

Proposition 7 *For all $f, g \in L^2_{q,\alpha}(\mathbb{R}_q)$ and all $\varphi \in \mathcal{S}_q(\mathbb{R}_q)$, we have the identity*

$$\int_{-\infty}^{+\infty} f *_{q,\alpha} g(x)\overline{(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(x)}|x|^{2\alpha+1}d_qx = \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda)\mathcal{F}_D^{q,\alpha}(g)(\lambda)\overline{\varphi(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda.$$

Proof. Fix $g \in L^2_{q,\alpha}(\mathbb{R}_q)$ and $\varphi \in \mathcal{S}_q(\mathbb{R}_q)$. For $f \in L^2_{q,\alpha}(\mathbb{R}_q)$, we put

$$\mathcal{J}_1 = \int_{-\infty}^{+\infty} f *_{q,\alpha} g(x)\overline{(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(x)}|x|^{2\alpha+1}d_qx$$

and

$$\mathcal{J}_2 = \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda)\mathcal{F}_D^{q,\alpha}(g)(\lambda)\overline{\varphi(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda.$$

By using the formula (21), Hölder's inequality and q-Plancherel formula (8), we have

$$\begin{aligned}
|\mathcal{J}_1| &\leq \|f *_{q,\alpha} g\|_{q,\infty}\|(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)\|_{q,1,\alpha} \\
&\leq \frac{4c_{q,\alpha}}{(q,q)_\infty}\|f\|_{q,2,\alpha}\|g\|_{q,2,\alpha}\|\mathcal{F}_D^{q,\alpha}(\varphi)\|_{q,1,\alpha} \\
&\leq \left(\frac{4c_{q,\alpha}}{(q,q)_\infty}\right)\left(\frac{4c_{q,\alpha}}{(q,q)_\infty}\right)^{\frac{2}{\infty}-1}\|f\|_{q,2,\alpha}\|g\|_{q,2,\alpha}\|\varphi\|_{q,\infty} \\
&= \|f\|_{q,2,\alpha}\|g\|_{q,2,\alpha}\|\varphi\|_{q,\infty} < \infty
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{J}_2| &\leq \|\mathcal{F}_D^{q,\alpha}(f)\mathcal{F}_D^{q,\alpha}(g)\|_{q,1,\alpha}\|\varphi\|_{q,\infty} \\
&\leq \|\mathcal{F}_D^{q,\alpha}(f)\|_{q,2,\alpha}\|\mathcal{F}_D^{q,\alpha}(g)\|_{q,2,\alpha}\|\varphi\|_{q,\infty} \\
&= \|f\|_{q,2,\alpha}\|g\|_{q,2,\alpha}\|\varphi\|_{q,\infty} < \infty,
\end{aligned}$$

which shows that these two integrals are well defined. Furthermore, by formulas (20) and q-Parseval'S formula (9), we get

$$\begin{aligned}
\int_{-\infty}^{+\infty} f *_{q,\alpha} g(x)\overline{(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(x)}|x|^{2\alpha+1}d_qx &= \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f *_{q,\alpha} g)(\lambda)\overline{\varphi(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda \\
&= \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda)\mathcal{F}_D^{q,\alpha}(g)(\lambda)\overline{\varphi(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda.
\end{aligned}$$

Then, Proposition 7 is proved. ■

Proposition 8 *Let $f, g \in L^2_{q,\alpha}(\mathbb{R}_q)$. Then $f *_{q,\alpha} g$ belongs to $L^2_{q,\alpha}(\mathbb{R}_q)$ if and only if $\mathcal{F}_D^{q,\alpha}(f)\mathcal{F}_D^{q,\alpha}(g) \in L^2_{q,\alpha}(\mathbb{R}_q)$ and we have*

$$\mathcal{F}_D^{q,\alpha}(f *_{q,\alpha} g) = \mathcal{F}_D^{q,\alpha}(f) \times \mathcal{F}_D^{q,\alpha}(g),$$

in the $L^2_{q,\alpha}(\mathbb{R}_q)$-case.

Proof. Suppose $f *_{q,\alpha} g \in L^2_{q,\alpha}(\mathbb{R}_q)$. By Proposition 7 and q-Parseval'S formula (9), we have for any $\varphi \in \mathcal{S}_q(\mathbb{R}_q)$

$$\begin{aligned}
\int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda)\mathcal{F}_D^{q,\alpha}(g)(\lambda)\overline{\varphi(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda &= \int_{-\infty}^{+\infty} f *_{q,\alpha} g(x)\overline{(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(x)}|x|^{2\alpha+1}d_qx \\
&= \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f *_{q,\alpha} g)(\lambda)\overline{\varphi(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda,
\end{aligned}$$

which shows that $\mathcal{F}_D^{q,\alpha}(f *_{q,\alpha} g) = \mathcal{F}_D^{q,\alpha}(f)\mathcal{F}_D^{q,\alpha}(g)$. Conversely, if $\mathcal{F}_D^{q,\alpha}(f)\mathcal{F}_D^{q,\alpha}(g) \in L^2_{q,\alpha}(\mathbb{R}_q)$ then by Proposition 7 again and q-Parseval'S formula (9), we have for any $\varphi \in \mathcal{S}_q(\mathbb{R}_q)$:

$$\begin{aligned}&\int_{-\infty}^{+\infty} f *_{q,\alpha} g(x)\overline{(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(x)}|x|^{2\alpha+1}d_qx\\ &= \int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda)\mathcal{F}_D^{q,\alpha}(g)(\lambda)\overline{\varphi(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda\\ &= \int_{-\infty}^{+\infty} (\mathcal{F}_D^{q,\alpha})^{-1}(\mathcal{F}_D^{q,\alpha}(f)\mathcal{F}_D^{q,\alpha}(g))(x)\overline{(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(x)}|x|^{2\alpha+1}d_qx,\end{aligned}$$

which shows, that $f *_{q,\alpha} g = (\mathcal{F}_D^{q,\alpha})^{-1}(\mathcal{F}_D^{q,\alpha}(f)\mathcal{F}_D^{q,\alpha}(g))$. This achieves the proof of Proposition 8. ■

3 q-Dunkl Two-Wavelet Theory

The concept of "wavelets" or "ondelettes" started to appear in the literature only in the early 1980's. This new concept can be viewed as a synthesis of various ideas which orginated from different disciplines including mathematics, physics and engineering. In 1982, Jean Morlet a French geophysicist engineer, first, introduced the idea of wavelets transform as a new mathematics tool for seismic signal analysis. The mathematical foundations were given by Grossmann and Morlet [17]. The harmonic analyst Meyer and many other mathematicians became aware of this theory and they recognized many classical results inside it (see [18, 21]).

Next, the theory of wavelets and continuous wavelet transform has been extended to the harmonic analysis associated with a class of singular differential operators (see [10]). Recently Trimèche [24], with the aid of the harmonic analysis associated to the Dunkl theory, has defined and studied the Dunkl wavelet transform. In the same paper [24], Trimèche has proved for this transform the Plancherel and inversion formulas.

In this Section, we define and study the q-Dunkl two wavelet and the continuous q-wavelet transform associated with the q-Dunkl operator and we establish a Plancherel formula and an inversion theorem for this transform.

Notations: We denote by

(i) $\mathbb{R}^2_{q,+} = \mathbb{R}^+_q \times \mathbb{R}_q$.

(ii) $L^p_{q,\alpha}(\mathbb{R}^2_{q,+})$, $p \in [1, +\infty]$, the space of measurable functions on $\mathbb{R}^2_{q,+}$ for which

$$\|f\|_{\mu_{q,\alpha},p} = \begin{cases} \left(\int_{\mathbb{R}^2_{q,+}} |f(a,x)|^p d\mu_{q,\alpha}(a,x)\right)^{1/p} < +\infty & \text{if } 1 \le p < +\infty,\\ \sup\limits_{(a,x)\in\mathbb{R}^2_{q,+}} |f(a,x)| < +\infty & \text{if } p = +\infty,\end{cases}$$

where the measure $\mu_{q,\alpha}$ is defined by

$$d\mu_{q,\alpha}(a,x) = \frac{|x|^{2\alpha+1}d_qxd_qa}{a^{2\alpha+3}}, \quad \forall (a,x) \in \mathbb{R}^2_{q,+}.$$

Definition 4 ([27]) *A q-wavelet associated with the q-Dunkl operator $\Lambda_{q,\alpha}$, is a square q-integrable function h on $\mathbb{R}_q$ satisfying the following admissibility condition*

$$0 < \mathcal{C}_{\alpha,h} = \int_0^{+\infty} |\mathcal{F}_D^{q,\alpha}(h)(a)|^2 \frac{d_qa}{a} = \int_0^{+\infty} |\mathcal{F}_D^{q,\alpha}(h)(-a)|^2 \frac{d_qa}{a} < \infty.$$

Remark 1 *For all $\lambda \in \mathbb{R}_q$, we have*

$$\mathcal{C}_{\alpha,h} = \int_0^{+\infty} |\mathcal{F}_D^{q,\alpha}(h)(a\lambda)|^2 \frac{d_qa}{a}.$$

Definition 5 *Let u and v be in $L^2_{q,\alpha}(\mathbb{R}_q)$. We say that the pair (u,v) is a q-Dunkl two-wavelet on $\mathbb{R}_q$ if the following integral, noted by $\mathcal{C}_{\alpha,u,v}$,*

$$\int_0^{+\infty} \mathcal{F}_D^{q,\alpha}(v)(a\lambda)\overline{\mathcal{F}_D^{q,\alpha}(u)(a\lambda)}\frac{d_q a}{a} \tag{22}$$

is constant for almost all $\lambda \in \mathbb{R}_q$. We call the number $\mathcal{C}_{\alpha,u,v}$, the q-Dunkl two-wavelet constant associated to the functions u and v.

Remark 2 *It is obvious that if u is a q-Dunkl wavelet then the pair (u,u) is a q-Dunkl two-wavelet, and $\mathcal{C}_{\alpha,u,u}$ coincides with $\mathcal{C}_{\alpha,u}$.*

Let $a > 0$, $a \in \mathbb{R}_q^+$ and $h \in L^2_{q,\alpha}(\mathbb{R}_q)$. We consider the function h_a defined by

$$h_a(x) = \frac{1}{a^{2\alpha+2}} h\left(\frac{x}{a}\right), \quad \forall x \in \mathbb{R}_q.$$

Proposition 9 *(i) For all function h belongs to $L^2_{q,\alpha}(\mathbb{R}_q)$. We have*

$$\|h_a\|_{q,2,\alpha} = \frac{1}{a^{\alpha+1}}\|h\|_{q,2,\alpha}. \tag{23}$$

(ii) Let h be in $L^1_{q,\alpha}(\mathbb{R}_q)\bigcup L^2_{q,\alpha}(\mathbb{R}_q)$. Then, we get

$$\mathcal{F}_D^{q,\alpha}(h_a)(\lambda) = \mathcal{F}_D^{q,\alpha}(h)(a\lambda), \quad \forall a \in \mathbb{R}_q^+, : \forall \lambda \in \mathbb{R}_q. \tag{24}$$

Proof. The change of variables $u = \dfrac{x}{a}$ gives the result. ■

Remark 3 *Let $h \in L^p_{q,\alpha}(\mathbb{R}_q), p \in [1,+\infty]$. The function h_a belongs to $L^p_{q,\alpha}(\mathbb{R}_q)$ and we have*

$$\|h_a\|_{q,p,\alpha} = a^{(2\alpha+2)(\frac{1}{p}-1)}\|h\|_{q,p,\alpha}.$$

Proposition 10 *Let h be a q-Dunkl wavelet in $L^2_{q,\alpha}(\mathbb{R}_q)$. Then, for all $x \in \widehat{\mathbb{R}_q}$ and $a \in \mathbb{R}_q^+$, the function $h_{a,x}$, defined by*

$$h_{a,x}(\lambda) = a^{\alpha+1} T_x^{q,\alpha}(h_a)(\lambda) \tag{25}$$

is a q-Dunkl wavelet in $L^2_{q,\alpha}(\mathbb{R}_q)$ and we have

$$\mathcal{C}_{\alpha,h_{a,x}} = a^{2\alpha+2}\int_0^{+\infty}\left|\Psi_{q,\alpha}\left(\frac{\lambda x}{a}\right)\right|^2 |\mathcal{F}_D^{q,\alpha}(h)(\lambda)|^2\frac{d_q\lambda}{\lambda},$$

with $T_x^{q,\alpha}$ is the generalized q-Dunkl translation operator defined by Definition 2.
Proof. Let $h \in L^2_{q,\alpha}(\mathbb{R}_q)$. We first ascertain that $h_{a,x}$ is in $L^2_{q,\alpha}(\mathbb{R}_q)$. Indeed, in view of formulas (15), (23), we have

$$\begin{aligned}
\|h_{a,x}\|^2_{q,2,\alpha} &= a^{2\alpha+2}\int_{-\infty}^{+\infty} |T_x^{q,\alpha}(h_a)(\lambda)|^2|\lambda|^{2\alpha+1}d_q\lambda \\
&\leq \frac{16a^{2\alpha+2}}{(q;q)^2_\infty}\|h_a\|^2_{q,2,\alpha} \\
&= \frac{16}{(q;q)^2_\infty}\|h\|^2_{q,2,\alpha}.
\end{aligned}$$

Then

$$\|h_{a,x}\|_{q,2,\alpha} \leq \frac{4}{(q;q)_\infty}\|h\|_{q,2,\alpha} < \infty. \tag{26}$$

On other hand, it follows from (16) and (24) that

$$\begin{aligned}\mathcal{C}_{\alpha,h_{a,x}} &= \int_0^{+\infty} |\mathcal{F}_D^{q,\alpha}(h_{a,x})(\lambda)|^2 \frac{d_q\lambda}{\lambda} \\ &= a^{2\alpha+2}\int_0^{+\infty} |\Psi_{q,\alpha}(\lambda x)|^2 |\mathcal{F}_D^{q,\alpha}(h)(a\lambda)|^2 \frac{d_q\lambda}{\lambda} \\ &= a^{2\alpha+2}\int_0^{+\infty} \left|\Psi_{q,\alpha}\left(\frac{\lambda x}{a}\right)\right|^2 |\mathcal{F}_D^{q,\alpha}(h)(\lambda)|^2 \frac{d_q\lambda}{\lambda}.\end{aligned}$$

Moreover, we have

$$0 < \mathcal{C}_{\alpha,h_{a,x}} \leq \frac{16a^{2\alpha+2}}{(q;q)_\infty^2}\int_0^{+\infty} |\mathcal{F}_D^{q,\alpha}(h)(\lambda)|^2 \frac{d_q\lambda}{\lambda} = \frac{16a^{2\alpha+2}}{(q;q)_\infty^2}\mathcal{C}_{\alpha,h} < \infty.$$

So, the essential is proved. ■

Definition 6 *Let h be a q-Dunkl wavelet on $\mathbb{R}_q$ in $L^2_{q,\alpha}(\mathbb{R}_q)$. We define the continuous q-wavelet transform associated with the q-Dunkl operator for all $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ by*

$$\begin{aligned}\psi_{q,h}^{\alpha,D}(f)(a,x) &= c_{q,\alpha}\int_{-\infty}^{+\infty} f(\lambda)\overline{h_{a,x}(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda \\ &= \langle f, h_{a,x}\rangle_{q,\alpha}.\end{aligned} \tag{27}$$

Remark that (27) is equivalent to

$$\begin{aligned}\psi_{q,h}^{\alpha,D}(f)(a,x) &= c_{q,\alpha}\int_{-\infty}^{+\infty} f(\lambda)\overline{h_{a,x}(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda \\ &= c_{q,\alpha}a^{\alpha+1}\int_{-\infty}^{+\infty} f(\lambda)\overline{T_x^{q,\alpha}(h_a)(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda \\ &= c_{q,\alpha}a^{\alpha+1}\int_{-\infty}^{+\infty} \check{f}(\lambda)T_x^{q,\alpha}(\overline{h_a})(-\lambda)|\lambda|^{2\alpha+1}d_q\lambda \\ &= a^{\alpha+1}\check{f} *_{q,\alpha} \overline{h_a}(x),\end{aligned} \tag{28}$$

and also equivalent to

$$\begin{aligned}\psi_{q,h}^{\alpha,D}(f)(a,x) &= a^{\alpha+1}\mathcal{F}_D^{q,\alpha}\left[\mathcal{F}_D^{q,\alpha}(\check{f} *_{q,\alpha} \overline{h_a})\right](-x) \\ &= a^{\alpha+1}\mathcal{F}_D^{q,\alpha}\left[\mathcal{F}_D^{q,\alpha}(\check{f}).\mathcal{F}_D^{q,\alpha}(\overline{h_a})\right](-x) \\ &= c_{q,\alpha}a^{\alpha+1}\int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(\check{f})(\lambda)\mathcal{F}_D^{q,\alpha}(\overline{h_a})(\lambda)\Psi_{q,\alpha}(\lambda x)|\lambda|^{2\alpha+1}d_q\lambda \\ &= c_{q,\alpha}a^{\alpha+1}\int_{-\infty}^{+\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda)\overline{\mathcal{F}_D^{q,\alpha}(h)(a\lambda)}\Psi_{q,\alpha}(-\lambda x)|\lambda|^{2\alpha+1}d_q\lambda.\end{aligned}$$

Proposition 11 *Let h be a q-Dunkl wavelet in $L^2_{q,\alpha}(\mathbb{R}_q)$. Then for all f in $L^2_{q,\alpha}(\mathbb{R}_q)$, we have*

$$\|\psi_{q,h}^{\alpha,D}(f)\|_{\mu_{q,\alpha},\infty} \leq \frac{4c_{q,\alpha}}{(q;q)_\infty}\|f\|_{q,2,\alpha}\|h\|_{q,2,\alpha}.$$

Proof. Suppose that $f \in L^2_{q,\alpha}(\mathbb{R}_q)$, $x \in \widehat{\mathbb{R}_q}$ and $a \in \mathbb{R}_q^+$, it follows from (26), (27) and the Cauchy-Schwartz inequality that

$$\begin{aligned}|\psi_{q,h}^{\alpha,D}(f)(a,x)| &\leq c_{q,\alpha}\int_{-\infty}^{+\infty} |f(\lambda)||\overline{h_{a,x}(\lambda)}||\lambda|^{2\alpha+1}d_q\lambda \\ &\leq c_{q,\alpha}\|f\|_{q,2,\alpha}\|h_{a,x}\|_{q,2,\alpha} \\ &= \frac{4c_{q,\alpha}}{(q;q)_\infty}\|f\|_{q,2,\alpha}\|h\|_{q,2,\alpha}.\end{aligned}$$

Then, the necessity is proved. ■

Theorem 3 (Parseval formula) *Let (u, v) be a q-Dunkl two-wavelet. Then for all f and g in $L^2_{q,\alpha}(\mathbb{R}_q)$, there holds*

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\psi_{q,u}^{\alpha,D}(f)(a,x)\overline{\psi_{q,v}^{\alpha,D}(g)(a,x)}d\mu_{q,\alpha}(a,x)=\mathcal{C}_{\alpha,u,v}\int_{-\infty}^{+\infty}f(x)\overline{g(x)}|x|^{2\alpha+1}d_qx, \tag{29}$$

where

$$\mathcal{C}_{\alpha,u,v}=\int_0^{+\infty}\mathcal{F}_D^{q,\alpha}(v)(a\lambda)\overline{\mathcal{F}_D^{q,\alpha}(u)(a\lambda)}\frac{d_qa}{a}.$$

Proof. Using Fubini's Theorem, formula (28) and Parseval's formula (9), we get

$$\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\psi_{q,u}^{\alpha,D}(f)(a,x)\overline{\psi_{q,v}^{\alpha,D}(g)(a,x)}d\mu_{q,\alpha}(a,x)\\
=&\int_0^{+\infty}a^{2\alpha+2}\int_{-\infty}^{+\infty}\check{f}*_{q,\alpha}\overline{u_a}(x)\overline{\check{g}*_{q,\alpha}\overline{v_a}(x)}d\mu_{q,\alpha}(a,x)\\
=&\int_0^{+\infty}\int_{-\infty}^{+\infty}\check{f}*_{q,\alpha}\overline{u_a}(x)\overline{\check{g}*_{q,\alpha}\overline{v_a}(x)}|x|^{2\alpha+1}\frac{d_qxd_qa}{a}\\
=&\int_0^{+\infty}\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(\check{f})(\lambda)\overline{\mathcal{F}_D^{q,\alpha}(\check{g})(\lambda)}\mathcal{F}_D^{q,\alpha}(\overline{u_a})(\lambda)\overline{\mathcal{F}_D^{q,\alpha}(\overline{v_a})(\lambda)}\|\lambda|^{2\alpha+1}\frac{d_q\lambda d_qa}{a}\\
=&\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(\check{f})(\lambda)\overline{\mathcal{F}_D^{q,\alpha}(\check{g})(\lambda)}\left(\int_0^{+\infty}\mathcal{F}_D^{q,\alpha}(\overline{u})(a\lambda)\mathcal{F}_D^{q,\alpha}(v)(-a\lambda)\frac{d_qa}{a}\right)|\lambda|^{2\alpha+1}d_q\lambda.
\end{aligned}$$

On the other hand using the formulas (13) and (14), we deduce that

$$\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\psi_{q,u}^{\alpha,D}(f)(a,x)\overline{\psi_{q,v}^{\alpha,D}(g)(a,x)}d\mu_{q,\alpha}(a,x)\\
=&\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f)(\lambda)\overline{\mathcal{F}_D^{q,\alpha}(g)(\lambda)}\left(\int_0^{+\infty}\overline{\mathcal{F}_D^{q,\alpha}(u)(a\lambda)}\mathcal{F}_D^{q,\alpha}(v)(a\lambda)\frac{d_qa}{a}\right)|\lambda|^{2\alpha+1}d_q\lambda\\
=&\ \mathcal{C}_{\alpha,u,v}\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f)(\lambda)\overline{\mathcal{F}_D^{q,\alpha}(g)(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda\\
=&\ \mathcal{C}_{\alpha,u,v}\int_{-\infty}^{+\infty}f(x)\overline{g(x)}|x|^{2\alpha+1}d_qx,
\end{aligned}$$

which completes the proof. ■

Remark 4 *The previous theorem generalizes the Parseval formula for the continuous Dunkl wavelet transform proved by Trimèche [24].*

Theorem 4 (Plancherel formula) *Let u be a q-Dunkl wavelet. Then for all f in $L^2_{q,\alpha}(\mathbb{R}_q)$, we have*

$$\frac{1}{\mathcal{C}_{\alpha,u}}\int_0^{+\infty}\int_{-\infty}^{+\infty}\left|\psi_{q,u}^{\alpha,D}(f)(a,x)\right|^2d\mu_{q,\alpha}(a,x)=\int_{-\infty}^{+\infty}|f(x)|^2|x|^{2\alpha+1}d_qx,$$

where

$$\mathcal{C}_{\alpha,u}=\int_0^{+\infty}|\mathcal{F}_D^{q,\alpha}(u)(\lambda a)|^2\frac{d_qa}{a}. \tag{30}$$

Proof. Taking on account the result of Theorem 3. If $u = v$ is a q-Dunkl wavelet and $f = g$, we obtain

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left|\psi_{q,u}^{\alpha,D}(f)(a,x)\right|^2 d\mu_{q,\alpha}(a,x) = \mathcal{C}_{\alpha,u}\int_{-\infty}^{+\infty}|f(x)|^2|x|^{2\alpha+1}d_qx,$$

where

$$\mathcal{C}_{\alpha,u} = \mathcal{C}_{\alpha,u,u} = \int_0^{+\infty}|\mathcal{F}_D^{q,\alpha}(u)(\lambda a)|^2\frac{d_qa}{a}.$$

■

Corollary 1 *Let (u, v) be a q-Dunkl two-wavelet. We have the following: if $\mathcal{C}_{\alpha,u,v} = 0$, then $\psi_{q,u}^{\alpha,D}(L^2_{q,\alpha}(\mathbb{R}_q))$ and $\psi_{q,v}^{\alpha,D}(L^2_{q,\alpha}(\mathbb{R}_q))$ are orthogonal.*

Proof. This immediately follows from (29). ■

Theorem 5 (Inversion formula) *Let (u, v) be a q-Dunkl two-wavelet. For all f in $L^1_{q,\alpha}(\mathbb{R}_q)$ (resp. $L^2_{q,\alpha}(\mathbb{R}_q)$) such that $\mathcal{F}_D^{q,\alpha}(f)$ belongs to $L^1_{q,\alpha}(\mathbb{R}_q)$ (resp. $L^1_{q,\alpha}(\mathbb{R}_q)\bigcap L^\infty_{q,\alpha}(\mathbb{R}_q)$), we have*

$$f(\lambda) = \frac{c_{q,\alpha}}{\mathcal{C}_{\alpha,u,v}}\int_0^{+\infty}\int_{-\infty}^{+\infty}\psi_{q,u}^{\alpha,D}(f)(a,x)v_{a,x}(\lambda)d\mu_{q,\alpha}(a,x), \quad a.e., \tag{31}$$

where for each $\lambda \in \mathbb{R}_q$, both the inner integral and the outer integral are absolutely convergent, but eventually not the double integral.

Proof. In view of formulas (16), (17), (18), (25), (28) and the q-Parseval formula (9) for the q-Dunkl transform $\mathcal{F}_D^{q,\alpha}$, we deduce that

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\psi_{q,u}^{\alpha,D}(f)(a,x)v_{a,x}(\lambda)|x|^{2\alpha+1}d_qx\\
= \;& a^{2\alpha+2}\int_{-\infty}^{+\infty}\check{f}*_{q,\alpha}\overline{u_a}(x)T_x^{q,\alpha}(v_a)(\lambda)|x|^{2\alpha+1}d_qx\\
= \;& a^{2\alpha+2}\int_{-\infty}^{+\infty}\check{f}*_{q,\alpha}\overline{u_a}(x)T_\lambda^{q,\alpha}(v_a)(x)|x|^{2\alpha+1}d_qx\\
= \;& a^{2\alpha+2}\int_{-\infty}^{+\infty}\check{f}*_{q,\alpha}\overline{u_a}(x)\overline{T_\lambda^{q,\alpha}(\overline{v_a})(x)}|x|^{2\alpha+1}d_qx\\
= \;& a^{2\alpha+2}\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(\check{f})(\xi)\mathcal{F}_D^{q,\alpha}(\overline{u_a})(\xi)\overline{\Psi_{q,\alpha}(\lambda\xi)\mathcal{F}_D^{q,\alpha}(\overline{v_a})(\xi)}|\xi|^{2\alpha+1}d_q\xi\\
= \;& a^{2\alpha+2}\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f)(-\xi)\overline{\mathcal{F}_D^{q,\alpha}(u_a)(-\xi)}\Psi_{q,\alpha}(-\lambda\xi)\mathcal{F}_D^{q,\alpha}(v_a)(-\xi)|\xi|^{2\alpha+1}d_q\xi\\
= \;& a^{2\alpha+2}\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f)(\xi)\overline{\mathcal{F}_D^{q,\alpha}(u_a)(\xi)}\Psi_{q,\alpha}(\lambda\xi)\mathcal{F}_D^{q,\alpha}(v_a)(\xi)|\xi|^{2\alpha+1}d_q\xi.
\end{aligned}$$

Therefore, by using the formula (24), we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\psi_{q,u}^{\alpha,D}(f)(a,x)v_{a,x}(\lambda)|x|^{2\alpha+1}d_qx\\
= \;& a^{2\alpha+2}\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f)(\xi)\Psi_{q,\alpha}(\lambda\xi)\overline{\mathcal{F}_D^{q,\alpha}(u)(a\xi)}\mathcal{F}_D^{q,\alpha}(v)(a\xi)|\xi|^{2\alpha+1}d_q\xi.
\end{aligned}$$

Taking into account the q-inversion formula (10) and integrating both sides of this equality over $\mathbb{R}_q^+$ with respect to the measure $d_qa/a^{2\alpha+3}$, we get

$$\begin{aligned} & \frac{c_{q,\alpha}}{\mathcal{C}_{\alpha,u,v}}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty}\psi_{q,u}^{\alpha,D}(f)(a,x)v_{a,x}(\lambda)|x|^{2\alpha+1}d_qx\right)\frac{d_qa}{a^{2\alpha+3}} \\ = & \frac{c_{q,\alpha}}{\mathcal{C}_{\alpha,u,v}}\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f)(\xi)\Psi_{q,\alpha}(\lambda\xi)\left(\int_0^{+\infty}\overline{\mathcal{F}_D^{q,\alpha}(u)(a\xi)}\mathcal{F}_D^{q,\alpha}(v)(a\xi)\frac{d_qa}{a}\right)|\xi|^{2\alpha+1}d_q\xi. \\ = & c_{q,\alpha}\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f)(\xi)\Psi_{q,\alpha}(\lambda\xi)|\xi|^{2\alpha+1}d_q\xi=f(\lambda). \end{aligned}$$

Then, the proof of this theorem is finished. ■

4 Calderón's Type Reproducing Formula in the Context of the q-Dunkl Two-Wavelet

In this Section, we will prove a Calderón-reproducing formula for the continuous q-wavelet transform in the context of the q-Dunkl two-wavelet. More precisely, we prove the following theorem:

Theorem 6 (Calderón's type formula) *Let u and v be two q-Dunkl wavelets in $L^2_{q,\alpha}(\mathbb{R}_q)$ such that (u,v) is a q-Dunkl two-wavelet, $\mathcal{C}_{\alpha,u,v}\neq 0$, and $\mathcal{F}_D^{q,\alpha}(u)$ and $\mathcal{F}_D^{q,\alpha}(v)$ both belong to $L^\infty_{q,\alpha}(\mathbb{R}_q)$. Then, for all f in $L^2_{q,\alpha}(\mathbb{R}_q)$ and $0<\varepsilon<\delta<\infty$, the function*

$$f^{\varepsilon,\delta}(\lambda)=\frac{c_{q,\alpha}}{\mathcal{C}_{\alpha,u,v}}\int_\varepsilon^\delta\int_{-\infty}^{+\infty}\psi_{q,u}^{\alpha,D}(f)(a,x)v_{a,x}(\lambda)|x|^{2\alpha+1}d_qx\frac{d_qa}{a^{2\alpha+3}} \tag{32}$$

belongs to $L^2_{q,\alpha}(\mathbb{R}_q)$, and satisfies

$$\lim_{\varepsilon\to0,\,\delta\to\infty}\|f^{\varepsilon,\delta}-f\|_{q,2,\alpha}=0. \tag{33}$$

To prove this theorem we need the following lemmas:

Lemma 1 *Let u and v be two q-Dunkl wavelets satisfying the conditions of Theorem 6 and f in $L^2_{q,\alpha}(\mathbb{R}_q)$. Then,*

(i) The functions $(\check{f}*_{q,\alpha}\overline{u_a})\check{}$ and $(\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a$ are in $L^2_{q,\alpha}(\mathbb{R}_q)$, and we have

$$\mathcal{F}_D^{q,\alpha}((\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a)(\lambda)=\mathcal{F}_D^{q,\alpha}(f)(\lambda)\overline{\mathcal{F}_D^{q,\alpha}(u_a)(\lambda)}\mathcal{F}_D^{q,\alpha}(v_a)(\lambda),\ \forall\lambda\in\mathbb{R}_q. \tag{34}$$

(ii) The following inequality is checked

$$\|(\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a\|_{q,2,\alpha}\leq\|\mathcal{F}_D^{q,\alpha}(u)\|_{q,\infty}\|\mathcal{F}_D^{q,\alpha}(v)\|_{q,\infty}\|f\|_{q,2,\alpha}. \tag{35}$$

Proof. Taking on account the relations (13) and (14) of Proposition 3, we have

$$\begin{aligned}\mathcal{F}_D^{q,\alpha}((\check{f}*_{q,\alpha}\overline{u_a})\check{})(\lambda) &= \mathcal{F}_D^{q,\alpha}(\check{f}*_{q,\alpha}\overline{u_a})(-\lambda)\\ &= \mathcal{F}_D^{q,\alpha}(\check{f})(-\lambda)\mathcal{F}_D^{q,\alpha}(\overline{u_a})(-\lambda)\\ &= \mathcal{F}_D^{q,\alpha}(f)(\lambda)\overline{\mathcal{F}_D^{q,\alpha}(\check{u}_a)(-\lambda)}.\end{aligned}$$

Therefore,

$$\mathcal{F}_D^{q,\alpha}((\check{f}*_{q,\alpha}\overline{u_a})\check{})(\lambda)=\mathcal{F}_D^{q,\alpha}(f)(\lambda)\overline{\mathcal{F}_D^{q,\alpha}(u_a)(\lambda)}. \tag{36}$$

Furthermore, we put

$$w(x) = (\check{f} *_{q,\alpha} \overline{u_a})\check{}(x), \quad x \in \mathbb{R}_q.$$

Thus,

$$\mathcal{F}_D^{q,\alpha}((\check{f} *_{q,\alpha} \overline{u_a})\check{} *_{q,\alpha} v_a)(\lambda) = \mathcal{F}_D^{q,\alpha}(w *_{q,\alpha} v_a)(\lambda).$$

By using Proposition 8, we deduce that the function w belongs to $L^2_{q,\alpha}(\mathbb{R}_q)$, and we have

$$\mathcal{F}_D^{q,\alpha}(w *_{q,\alpha} v_a)(\lambda) = \mathcal{F}_D^{q,\alpha}(w)(\lambda)\mathcal{F}_D^{q,\alpha}(v_a)(\lambda). \tag{37}$$

We deduce (34) from (36) and (37).

From (i) we have

$$\begin{aligned} &\int_{-\infty}^{+\infty} |\mathcal{F}_D^{q,\alpha}((\check{f} *_{q,\alpha} \overline{u_a})\check{} *_{q,\alpha} v_a)(\lambda)|^2 |\lambda|^{2\alpha+1} d_q\lambda \\ = &\int_{-\infty}^{+\infty} |\mathcal{F}_D^{q,\alpha}(f)(\lambda)|^2 |\mathcal{F}_D^{q,\alpha}(u_a)(\lambda)|^2 |\mathcal{F}_D^{q,\alpha}(v_a)(\lambda)|^2 |\lambda|^{2\alpha+1} d_q\lambda. \end{aligned}$$

Then, from the Plancherel formula (8) and the fact that $\mathcal{F}_D^{q,\alpha}(u_a)$ and $\mathcal{F}_D^{q,\alpha}(v_a)$ both belong to $L^\infty_{q,\alpha}(\mathbb{R}_q)$, we obtain

$$\|(\check{f} *_{q,\alpha} \overline{u_a})\check{} *_{q,\alpha} v_a\|_{q,2,\alpha} \leq \|\mathcal{F}_D^{q,\alpha}(u_a)\|_{q,\infty} \|\mathcal{F}_D^{q,\alpha}(v_a)\|_{q,\infty} \|f\|_{q,2,\alpha}. \tag{38}$$

We deduce the result from the relation (24). ■

Lemma 2 *Let u and v be two q-Dunkl wavelets satisfying the conditions of Theorem 6. Then, the function $K_{\varepsilon,\delta}$ defined by*

$$K_{\varepsilon,\delta}(\lambda) = \frac{1}{\mathcal{C}_{\alpha,u,v}} \int_\varepsilon^\delta \overline{\mathcal{F}_D^{q,\alpha}(u_a)(\lambda)} \mathcal{F}_D^{q,\alpha}(v_a)(\lambda) \frac{d_q a}{a}, \tag{39}$$

satisfies, for almost all $\lambda \in \mathbb{R}_q$:

$$0 < |K_{\varepsilon,\delta}(\lambda)| < \frac{\sqrt{\mathcal{C}_{\alpha,u}\mathcal{C}_{\alpha,v}}}{|\mathcal{C}_{\alpha,u,v}|} \tag{40}$$

and

$$\lim_{\varepsilon\to 0,\, \delta\to\infty} K_{\varepsilon,\delta}(\lambda) = 1.$$

Proof. From the Cauchy-Schwarz inequality and the relation (30), for almost all $\lambda \in \mathbb{R}_q$, we have

$$\begin{aligned} |K_{\varepsilon,\delta}(\lambda)| &\leq \frac{1}{|\mathcal{C}_{\alpha,u,v}|} \left(\int_\varepsilon^\delta |\mathcal{F}_D^{q,\alpha}(u_a)(\lambda)|^2 \frac{d_q a}{a}\right)^{1/2} \left(\int_\varepsilon^\delta |\mathcal{F}_D^{q,\alpha}(v_a)(\lambda)|^2 \frac{d_q a}{a}\right)^{1/2} \\ &\leq \frac{1}{|\mathcal{C}_{\alpha,u,v}|} \left(\int_0^{+\infty} |\mathcal{F}_D^{q,\alpha}(u)(a\lambda)|^2 \frac{d_q a}{a}\right)^{1/2} \left(\int_0^{+\infty} |\mathcal{F}_D^{q,\alpha}(v)(a\lambda)|^2 \frac{d_q a}{a}\right)^{1/2} \\ &\leq \frac{\sqrt{\mathcal{C}_{\alpha,u}\mathcal{C}_{\alpha,v}}}{|\mathcal{C}_{\alpha,u,v}|}. \end{aligned}$$

On the other hand, for almost all $\lambda \in \mathbb{R}_q$, we have

$$\lim_{\varepsilon\to 0,:\delta\to\infty} K_{\varepsilon,\delta}(\lambda) = 1.$$

This completes the proof. ■

Lemma 3 *Let (u,v) be a q-Dunkl two-wavelet satisfying the admissibility condition (22). Then for all $f\in L^1_{q,\alpha}(\mathbb{R}_q)\bigcap L^2_{q,\alpha}(\mathbb{R}_q)$, we have*

$$f(\lambda)=\frac{1}{\mathcal{C}_{\alpha,u,v}}\int_0^{+\infty}((\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a)(\lambda)\frac{d_qa}{a}.$$

Proof. From Theorem 5, the formulas (25) and (28), we have

$$\begin{aligned}f(\lambda) &= \frac{c_{q,\alpha}}{\mathcal{C}_{\alpha,u,v}}\int_0^{+\infty}\int_{-\infty}^{+\infty}\psi^{\alpha,D}_{q,u}(f)(a,x)v_{a,x}(\lambda)d\mu_{q,\alpha}(a,x)\\ &= \frac{c_{q,\alpha}}{\mathcal{C}_{\alpha,u,v}}\int_0^{+\infty}\int_{-\infty}^{+\infty}\check{f}*_{q,\alpha}\overline{u_a}(x)T^{q,\alpha}_x(v_a)(\lambda)|x|^{2\alpha+1}d_qx\frac{d_qa}{a}\\ &= \frac{c_{q,\alpha}}{\mathcal{C}_{\alpha,u,v}}\int_0^{+\infty}\int_{-\infty}^{+\infty}(\check{f}*_{q,\alpha}\overline{u_a})\check{}(x)T^{q,\alpha}_\lambda(v_a)(-x)|x|^{2\alpha+1}d_qx\frac{d_qa}{a}.\end{aligned}$$

By Proposition 6, we see that $(\check{f}*_{q,\alpha}\overline{u_a})$ belong to $L^2_{q,\alpha}(\mathbb{R}_q)$ for each $f\in L^1_{q,\alpha}(\mathbb{R}_q)$. Hence also $(\check{f}*_{q,\alpha}\overline{u_a})\check{}$ is in $L^2_{q,\alpha}(\mathbb{R}_q)$. Then $(\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a$ exists and we have

$$(\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a(\lambda)=c_{q,\alpha}\int_{-\infty}^{+\infty}(\check{f}*_{q,\alpha}\overline{u_a})\check{}(x)T^{q,\alpha}_\lambda(v_a)(-x)|x|^{2\alpha+1}d_qx.$$

Then

$$f(\lambda)=\frac{1}{\mathcal{C}_{\alpha,u,v}}\int_0^{+\infty}((\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a)(\lambda)\frac{d_qa}{a}.$$

■

Lemma 4 *We consider the functions u, v and f satisfying the conditions of Theorem 6. Then the function $f^{\varepsilon,\delta}$ defined by the relation (32) belongs to $L^2_{q,\alpha}(\mathbb{R}_q)$ and satisfies*

$$\mathcal{F}^{q,\alpha}_D(f^{\varepsilon,\delta})(\xi)=K_{\varepsilon,\delta}(\xi)\mathcal{F}^{q,\alpha}_D(f)(\xi),:\ \forall\xi\in\mathbb{R}_q, \tag{41}$$

where $K_{\varepsilon,\delta}$ is the function given by the relation (39).

Proof. We first prove that the function $f^{\varepsilon,\delta}$ belongs to $L^2_{q,\alpha}(\mathbb{R}_q)$. From Lemma 3, we have

$$f^{\varepsilon,\delta}(\lambda)=\frac{1}{\mathcal{C}_{\alpha,u,v}}\int_\varepsilon^{\delta}((\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a)(\lambda)\frac{d_qa}{a}.$$

By using Hölder's inequality for the measure $\frac{d_qa}{a}$, we get

$$|f^{\varepsilon,\delta}(\lambda)|^2\le\frac{1}{|\mathcal{C}_{\alpha,u,v}|^2}\left(\int_\varepsilon^\delta\frac{d_qa}{a}\right)\int_\varepsilon^\delta|(\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a(\lambda)|^2\frac{d_qa}{a}.$$

So, by applying Fubini-Tonelli's theorem, we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty}|f^{\varepsilon,\delta}(\lambda)|^2|\lambda|^{2\alpha+1}d_q\lambda &\le \frac{1}{|\mathcal{C}_{\alpha,u,v}|^2}\left(\int_\varepsilon^\delta\frac{d_qa}{a}\right)\\ &\quad\times\int_\varepsilon^\delta\left(\int_{-\infty}^{+\infty}|(\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a(\lambda)|^2|\lambda|^{2\alpha+1}d_q\lambda\right)\frac{d_qa}{a}.\end{aligned}$$

From the Parseval formula (9) and the relation (34), we deduce that

$$\int_{-\infty}^{+\infty} |f^{\varepsilon,\delta}(\lambda)|^2|\lambda|^{2\alpha+1}d_q\lambda \leq \frac{1}{|\mathcal{C}_{\alpha,u,v}|^2}\left(\int_\varepsilon^\delta \frac{d_qa}{a}\right)\int_{-\infty}^{+\infty}|\mathcal{F}_D^{q,\alpha}(f)(\xi)|^2 \times\left(\int_\varepsilon^\delta |\mathcal{F}_D^{q,\alpha}(u_a)(\xi)|^2|\mathcal{F}_D^{q,\alpha}(v_a)(\xi)|^2\frac{d_qa}{a}\right)|\xi|^{2\alpha+1}d_q\xi.$$

On the other hand, from the relations (24) and (30), we have

$$\int_\varepsilon^\delta |\mathcal{F}_D^{q,\alpha}(u_a)(\xi)|^2|\mathcal{F}_D^{q,\alpha}(v_a)(\xi)|^2\frac{d_qa}{a} \leq \mathcal{C}_{\alpha,v}\|\mathcal{F}_D^{q,\alpha}(u)\|^2_{q,\infty}.$$

Thus,

$$\int_{-\infty}^{+\infty} |f^{\varepsilon,\delta}(\lambda)|^2|\lambda|^{2\alpha+1}d_q\lambda \leq \frac{\mathcal{C}_{\alpha,v}}{|\mathcal{C}_{\alpha,u,v}|^2}\left(\int_\varepsilon^\delta \frac{d_qa}{a}\right)\|\mathcal{F}_D^{q,\alpha}(u)\|^2_{q,\infty}\|\mathcal{F}_D^{q,\alpha}(f)\|^2_{q,2,\alpha},$$

and the Plancherel formula (8) implies

$$\int_{-\infty}^{+\infty} |f^{\varepsilon,\delta}(\lambda)|^2|\lambda|^{2\alpha+1}d_q\lambda \leq \frac{\mathcal{C}_{\alpha,v}}{|\mathcal{C}_{\alpha,u,v}|^2}\left(\int_\varepsilon^\delta \frac{d_qa}{a}\right)\|\mathcal{F}_D^{q,\alpha}(u)\|^2_{q,\infty}\|f\|^2_{q,2,\alpha} < \infty.$$

Then, $f^{\varepsilon,\delta}$ belongs to $L^2_{q,\alpha}(\mathbb{R}_q)$.

We now prove the formula (41). Let φ in $\mathcal{S}_q(\mathbb{R}_q)$. We have the function $(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)$ is in $\mathcal{S}_q(\mathbb{R}_q)$. From the Lemma 3, we have

$$\begin{aligned}&\int_{-\infty}^{+\infty} f^{\varepsilon,\delta}(\lambda)\overline{(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda \\ =& \int_{-\infty}^{+\infty}\left(\frac{1}{\mathcal{C}_{\alpha,u,v}}\int_\varepsilon^\delta (\check{f} *_{q,\alpha} \overline{u_a})\check{}*_{q,\alpha} v_a(\lambda)\frac{d_qa}{a}\right)\overline{(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda.\end{aligned} \tag{42}$$

We consider

$$\begin{aligned}&\left|\frac{1}{\mathcal{C}_{\alpha,u,v}}\right|\int_{-\infty}^{+\infty}\int_\varepsilon^\delta |(\check{f} *_{q,\alpha} \overline{u_a})\check{}*_{q,\alpha} v_a(\lambda)\overline{(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(\lambda)}|\frac{d_qa}{a}|\lambda|^{2\alpha+1}d_q\lambda \\ =& \left|\frac{1}{\mathcal{C}_{\alpha,u,v}}\right|\int_\varepsilon^\delta\left[\int_{-\infty}^{+\infty}|(\check{f} *_{q,\alpha} \overline{u_a})\check{}*_{q,\alpha} v_a(\lambda)||(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(\lambda)||\lambda|^{2\alpha+1}d_q\lambda\right]\frac{d_qa}{a}.\end{aligned}$$

By applying Hölder's inequality to the second member, we get

$$\begin{aligned}&\left|\frac{1}{\mathcal{C}_{\alpha,u,v}}\right|\int_\varepsilon^\delta\left[\int_{-\infty}^{+\infty}|(\check{f} *_{q,\alpha} \overline{u_a})\check{}*_{q,\alpha} v_a(\lambda)||(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(\lambda)||\lambda|^{2\alpha+1}d_q\lambda\right]\frac{d_qa}{a} \\ \leq& \left|\frac{1}{\mathcal{C}_{\alpha,u,v}}\right|\int_\varepsilon^\delta \|(\check{f} *_{q,\alpha} \overline{u_a})\check{}*_{q,\alpha} v_a\|_{q,2,\alpha}\|(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)\|_{q,2,\alpha}\frac{d_qa}{a}.\end{aligned}$$

In view of formula (35) and the q-Plancherel formula (8), we obtain

$$\begin{aligned}&\left|\frac{1}{\mathcal{C}_{\alpha,u,v}}\right|\int_\varepsilon^\delta\left[\int_{-\infty}^{+\infty}|(\check{f} *_{q,\alpha} \overline{u_a})\check{}*_{q,\alpha} v_a(\lambda)||(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(\lambda)||\lambda|^{2\alpha+1}d_q\lambda\right]\frac{d_qa}{a} \\ \leq& \left|\frac{1}{\mathcal{C}_{\alpha,u,v}}\right|\left(\int_\varepsilon^\delta\frac{d_qa}{a}\right)\|\mathcal{F}_D^{q,\alpha}(u)\|_{q,\infty}\|\mathcal{F}_D^{q,\alpha}(v)\|_{q,\infty}\|f\|_{q,2,\alpha}\|\varphi\|_{q,2,\alpha} < \infty.\end{aligned}$$

Then, from Fubini's theorem, the second member of the formula (42) can also be written in the form

$$\frac{1}{\mathcal{C}_{\alpha,u,v}}\int_{\varepsilon}^{\delta}\left(\int_{-\infty}^{+\infty}(\check{f}*_{q,\alpha}\overline{u_a})\check{}*_{q,\alpha}v_a(\lambda)\overline{(\mathcal{F}_D^{q,\alpha})^{-1}(\varphi)(\lambda)}|\lambda|^{2\alpha+1}d_q\lambda\right)\frac{d_qa}{a}. \tag{43}$$

Now, by using the q-Parseval formula (9) and the formula (34), the formula (43) becomes

$$\frac{1}{\mathcal{C}_{\alpha,u,v}}\int_{\varepsilon}^{\delta}\left(\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f)(\xi)\overline{\mathcal{F}_D^{q,\alpha}(u_a)(\xi)}\mathcal{F}_D^{q,\alpha}(v_a)(\xi)\overline{\varphi(\xi)}|\xi|^{2\alpha+1}d_q\xi\right)\frac{d_qa}{a}.$$

By applying Fubini's theorem to this integral, it takes the form

$$\begin{aligned}&\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f)(\xi)\left(\frac{1}{\mathcal{C}_{\alpha,u,v}}\int_{\varepsilon}^{\delta}\overline{\mathcal{F}_D^{q,\alpha}(u_a)(\xi)}\mathcal{F}_D^{q,\alpha}(v_a)(\xi)\frac{d_qa}{a}\right)\overline{\varphi(\xi)}|\xi|^{2\alpha+1}d_q\xi\\ =\;&\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f)(\xi)K_{\varepsilon,\delta}(\xi)\overline{\varphi(\xi)}|\xi|^{2\alpha+1}d_q\xi.\end{aligned} \tag{44}$$

On the other hand, by applying the q-Parseval formula (9) to the first member of the formula (42), we get

$$\int_{-\infty}^{+\infty}\mathcal{F}_D^{q,\alpha}(f^{\varepsilon,\delta})(\xi)\overline{\varphi(\xi)}|\xi|^{2\alpha+1}d_q\xi. \tag{45}$$

From the formulas (44) and (45), we obtain for all φ in $\mathcal{S}_q(\mathbb{R}_q)$:

$$\int_{-\infty}^{+\infty}(\mathcal{F}_D^{q,\alpha}(f^{\varepsilon,\delta})(\xi)-\mathcal{F}_D^{q,\alpha}(f)(\xi)K_{\varepsilon,\delta}(\xi))\overline{\varphi(\xi)}|\xi|^{2\alpha+1}d_q\xi=0.$$

Hence,

$$\mathcal{F}_D^{q,\alpha}(f^{\varepsilon,\delta})(\xi)=K_{\varepsilon,\delta}(\xi)\mathcal{F}_D^{q,\alpha}(f)(\xi),:\ \forall\xi\in\mathbb{R}_q.$$

This puts the end of the desired proof. ■

Proof of Theorem 6. From Lemma 4, the function $f^{\varepsilon,\delta}$ belongs to $L^2_{q,\alpha}(\mathbb{R}_q)$. By using the q-Plancherel formula (8) and Lemma 4, we obtain

$$\begin{aligned}\|f^{\varepsilon,\delta}-f\|_{q,2,\alpha} &= \int_{-\infty}^{+\infty}|\mathcal{F}_D^{q,\alpha}(f^{\varepsilon,\delta}-f)(\xi)|^2|\xi|^{2\alpha+1}d_q\xi\\ &= \int_{-\infty}^{+\infty}|\mathcal{F}_D^{q,\alpha}(\xi)(K_{\varepsilon,\delta}(\xi)-1)|^2|\xi|^{2\alpha+1}d_q\xi\\ &= \int_{-\infty}^{+\infty}|\mathcal{F}_D^{q,\alpha}(\xi)|^2|1-K_{\varepsilon,\delta}(\xi)|^2|\xi|^{2\alpha+1}d_q\xi.\end{aligned}$$

Furthermore, from Lemma 4 again and formula (40), for almost all $\xi\in\mathbb{R}_q$, we have

$$\lim_{\varepsilon\to0,\,\delta\to\infty}|\mathcal{F}_D^{q,\alpha}(\xi)|^2|1-K_{\varepsilon,\delta}(\xi)|^2=0,$$

and

$$\begin{aligned}|\mathcal{F}_D^{q,\alpha}(\xi)|^2|1-K_{\varepsilon,\delta}(\xi)|^2 &\leq |\mathcal{F}_D^{q,\alpha}(\xi)|^2(1+|K_{\varepsilon,\delta}(\xi)|)^2\\ &\leq C|\mathcal{F}_D^{q,\alpha}(\xi)|^2,\end{aligned}$$

where

$$C=\left(1+\frac{\sqrt{\mathcal{C}_{\alpha,u}\mathcal{C}_{\alpha,v}}}{|\mathcal{C}_{\alpha,u,v}|}\right)^2$$

is a positive constant.

Moreover, since the function $\xi \mapsto |\mathcal{F}_D^{q,\alpha}(\xi)|^2$ is in $L^2_{q,\alpha}(\mathbb{R}_q)$. The dominated convergence theorem yields (33). ■

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