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The Multiplicity of Positive Solutions For A Certain Logistic Problem*

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Abstract

We study the exact multiplicity and bifurcation curves of positive solutions for the logistic problem

$$\begin{cases} -u'' = \lambda \left(\frac{u}{1+u}\right)^p, & \text{in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases}$$

where $\lambda, p > 0$. We prove that the bifurcation curve is monotone increasing for $0 < p \leq 1$ and $\subset$-shaped for $p > 1$.

1 Introduction and Main Result

In this paper, we study the exact multiplicity and bifurcation curves of positive solutions for the logistic problem

$$\begin{cases} -u'' = \lambda \left(\frac{u}{1+u}\right)^p, & \text{in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \tag{1}$$

where $\lambda, p > 0$. For any $p > 0$, on the $(\lambda, \|u\|_\infty)$-plane, we define the bifurcation curve S by

$$S_p \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1)}\}. \tag{2}$$

It is well-known that studying the exact shape of bifurcation curve S_p of (1) is equivalent to studying of the exact multiplicity of positive solutions of (1). Therefore, this issue has been extensively researched, cf. [2, 3, 4, 6] and references therein.

Let

$$f(u) \equiv \left(\frac{u}{1+u}\right)^p. \tag{3}$$

When $p = 1$, the nonlinearity f becomes $f(u) = u/(1+u)$. It is a standard logistic function, which is widely used as a logarithmic linear model for classification and regression scenarios. The other significance of studying nonlinearity (3) can be found in [9]. Zhang et al. [9] studied the following prescribed mean curvature problem

$$\begin{cases} -\left(u'/\sqrt{1+u'^2}\right)' = \lambda \left(\frac{u}{1+u}\right)^p, & \text{in } (-L,L), \\ u(-L) = u(L) = 0, \end{cases} \tag{4}$$

where $\lambda, L, p > 0$, and have the following result:

Theorem 1 ([9, Theorems 1–3]) *Consider (4). Then the following statement (i)–(iii) hold.*

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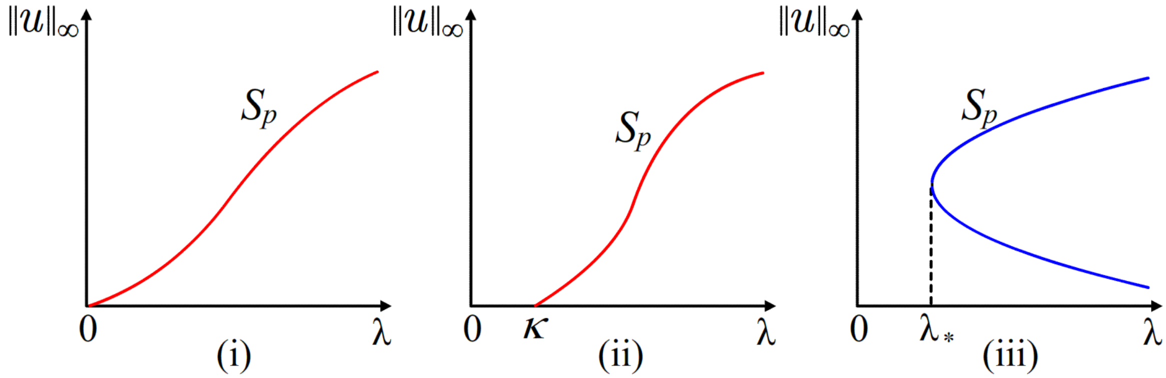

Figure 1: Grpahs of bifurcation curve S_p. (i) $0 < p < 1$. (i) $p = 1$ and $\kappa = \frac{\pi}{\sqrt{p}}$. (iii) $p > 1$.

(i) *If* $0 < p < \sqrt{2}/4$, *then the corresponding bifurcation curve is* $\supset$*-shaped for* $L > 0$.

(ii) *If* $p = 1$, *then the corresponding bifurcation curve is* $\supset$*-shaped for* $0 < L < \pi/2$ *or* $L > \pi^2/4$.

(iii) *If* $p > 1$, *then there exist* $L^* > L_* > 0$ *such that the corresponding bifurcation curve is monotone decreasing for* $0 < L < L_*$, *and reversed S-like shaped for* $L > L^*$.

Although problem (4) seems to be more complex than problem (1), we find that problem (1) has not yet been completely solved. Therefore, we intend to address the issue of the exact multiplicity of positive solutions of (1) in this paper. In addition, f is a convex-concave function if $p > 1$. There are many researchers to study the shape of bifurcation curve of positive solutions for the semilinear problem

$$\begin{cases} -u'' = \lambda g(u), & \text{in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases}$$

where $\lambda > 0$ and g is a convex-concave function on $(0, \infty)$, cf. [5, 8, 1] and references therein. However, no references can solve our problem (1). To this end, we study this problem (1) and have the following main result:

Theorem 2 (See Figure 1) *Consider (1). Then the following statements (i)–(iii) hold.*

(i) *If* $0 < p < 1$, *then the bifurcation curve* S_p *is monotone increasing, starts from* $(0, 0)$ *and goes to* (∞, ∞). *Furthermore, (1) has exactly one positive solution for* $\lambda > 0$.

(ii) *If* $p = 1$, *then the bifurcation curve* S_p *is monotone increasing, starts from* $(\frac{\pi}{\sqrt{p}}, 0)$ *and goes to* (∞, ∞) *Furthermore, (1) has no positive solutions for* $0 < \lambda \leq \frac{\pi}{\sqrt{p}}$, *and exactly one positive solution for* $\lambda > \frac{\pi}{\sqrt{p}}$.

(iii) *If* $p > 1$, *then the bifurcation curve* S_p *is* $\subset$*-shaped, starts from* $(\infty, 0)$ *and goes to* (∞, ∞). *Furthermore, there exists* $\lambda_* > 0$ *such that*

(a) *(1) has no positive solutions for* $0 < \lambda < \lambda_*$;

(b) *(1) has exactly one positive solution for* $\lambda = \lambda_*$; *and*

(c) *(1) has exactly two positive solutions for* $\lambda > \lambda_*$.

2 The Proof of Theorem 2

To prove Theorem 2, we develop some new time-map techniques. The time-map formula which we apply to study (1) takes the form as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha [F(\alpha) - F(u)]^{-1/2} \, du \equiv T(\alpha) \ \text{ for } \alpha > 0, \tag{5}$$

where $F(u) \equiv \int_0^u f(t)dt$, see Laetsch [7]. Observe that positive solutions u_λ for (1) correspond to

$$\|u_\lambda\|_\infty = \alpha \ \text{ and } \ T(\alpha) = \sqrt{\lambda}. \tag{6}$$

Thus, studying of the exact number of positive solutions of (1) is equivalent to studying the shape of the time map $T(\alpha)$ on $(0, \infty)$. Since $f \in C^2(0, \infty)$, it can be proved that $T_\varepsilon(\alpha)$ is a twice differentiable function of $\alpha > 0$. The proof is easy but tedious and hence we omit them.

By (**??**), we compute

$$T'(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{2B(\alpha,u) - A(\alpha,u)}{B^{3/2}(\alpha,u)} du \tag{7}$$

and

$$T''(\alpha) = \frac{1}{4\sqrt{2}\alpha^2} \int_0^\alpha \frac{3A^2(\alpha,u) - 4A(\alpha,u)B(\alpha,u) - 2B(\alpha,u)C(\alpha,u)}{B^{5/2}(\alpha,u)} du \tag{8}$$

where $A(\alpha,u) \equiv \alpha f(\alpha) - uf(u)$, $B(\alpha,u) \equiv F(\alpha) - F(u)$ and $C(\alpha,u) \equiv \alpha^2 f'(\alpha) - u^2 f'(u)$. Clearly,

$$B(\alpha,u) = \int_0^\alpha f(t) > 0 \ \text{ for } 0 < u < \alpha. \tag{9}$$

We further compute

$$\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to 0^+} \frac{u^{p-1}}{(1+u)^p} \begin{cases} \infty & \text{if } 0 < p < 1, \\ 1 & \text{if } p = 1, \\ 0 & \text{if } p > 1, \end{cases} \tag{10}$$

$$\lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} \frac{u^{p-1}}{(1+u)^p} = 0, \tag{11}$$

$$f'(u) = \frac{pu^{p-1}}{(1+u)^{p+1}} \ \text{ and } \ f''(u) = \frac{2pu^{p-2}}{(1+u)^{p+2}} \left(\frac{p-1}{2} - u \right). \tag{12}$$

Let $\theta(u) \equiv 2F(u) - uf(u)$. Then

$$\theta'(u) = f(u) - uf'(u) \ \text{ and } \ \theta''(u) = -uf''(u). \tag{13}$$

Next, we consider three cases.

Case 1. Assume that $0 < p \le 1$. By (10), (11) and [7, Theorems 2.6, 2.9 and 2.10], we have

$$\lim_{\alpha\to 0^+} T(\alpha) = \begin{cases} 0 & \text{if } 0 < p < 1, \\ \frac{\pi^2}{p} & \text{if } p = 1, \end{cases} \quad \text{and} \quad \lim_{\alpha\to\infty} T(\alpha) = \infty. \tag{14}$$

By (12) and (13), then $\theta''(u) > 0$ for $u > 0$. It follows that

$$\theta'(u) > \lim_{u\to 0^+} \theta'(u) = \lim_{u\to 0^+} \frac{u^p\,(u-p+1)}{(u+1)^{p+1}} = 0 \ \text{ for } u > 0.$$

Then $\theta(\alpha) - \theta(u) > 0$ for $0 < u < \alpha$. Since $2B(\alpha,u) - A(\alpha,u) = \theta(\alpha) - \theta(u)$, and by (7), we see that

$$T'(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{B(\alpha,u)} du > 0 \ \text{ for } \alpha > 0. \tag{15}$$

So by (14) and (15), the bifurcation curve S is monotone increasing, starts from $(\kappa, 0)$ and goes to (∞, ∞) where

$$\kappa = \begin{cases} 0 & \text{if } 0 < p < 1, \\ \frac{\pi}{\sqrt{p}} & \text{if } p = 1. \end{cases}$$

Thus the statements (i) and (ii) hold.

Case 2. Assume that $p > 1$. By (10), (11) and [7, Theorems 2.6 and 2.9], we have

$$\lim_{\alpha \to 0^+} T(\alpha) = \infty \quad \text{and} \quad \lim_{\alpha \to \infty} T(\alpha) = \infty. \tag{16}$$

By (13), we find that

$$\theta'(u) = \frac{u^p\,(u - p + 1)}{(u+1)^{p+1}} \begin{cases} < 0 & \text{for } 0 < u < p-1, \\ = 0 & \text{for } u = p-1, \\ > 0 & \text{for } u > p-1. \end{cases} \tag{17}$$

Since $\lim_{u \to \infty} \theta'(u) = 1$, there exists $N > 0$ such that $\theta'(u) > 1/2$ for $u > N$. Since $\theta(0) = 0$, we observe that

$$\lim_{u \to \infty} \theta(u) = \int_0^\infty \theta'(t)dt = \int_0^N \theta'(t)dt + \int_N^\infty \theta'(t)dt > \int_0^N \theta'(t)dt + \int_N^\infty \frac{1}{2}dt = \infty.$$

So by (17), there exists $\tau > p - 1$ such that

$$\theta(u) \begin{cases} < 0 & \text{for } 0 < u < \tau, \\ = 0 & \text{for } u = \tau, \\ > 0 & \text{for } u > \tau. \end{cases} \tag{18}$$

By (15)–(18), we obtain that

$$T'(\alpha) < 0 \text{ for } 0 < \alpha < p - 1, \quad \text{and} \quad T'(\alpha) > 0 \text{ for } \alpha > \tau. \tag{19}$$

On the other hand, by (7) and (8), we see that

$$T''(\alpha) + \frac{2}{\alpha}T'(\alpha) = \frac{1}{4\sqrt{2}\alpha^2}\int_0^\alpha \frac{H(\alpha,u)}{B^{5/2}(\alpha,u)}du > 0, \tag{20}$$

where

$$H(\alpha, u) \equiv 3\,(A - 2B)^2 + 2B\,(2A - 2B - C)\,.$$

Let $\varphi(u) \equiv u\theta'(u) - \theta(u)$. By (12), (17) and (18), we see that

$$\varphi(p-1) = (p-1)\,\theta'(p-1) - \theta(p-1) = -\theta(p-1) > 0 \tag{21}$$

and

$$\varphi'(u) = -u^2 f''(u) \begin{cases} < 0 & \text{for } 0 < u < \frac{p-1}{2}, \\ = 0 & \text{for } u = \frac{p-1}{2}, \\ > 0 & \text{for } u > \frac{p-1}{2}. \end{cases} \tag{22}$$

Since $\varphi(0) = 0$ and by (21) and (22), we obtain that $\varphi(\alpha) - \varphi(u) > 0$ for $0 \le u < \alpha$ and $\alpha \ge p - 1$. So by (9), then

$$\begin{aligned} H(\alpha,u) &\ge 2B(\alpha,u)\,[2A(\alpha,u) - 2B(\alpha,u) - C(\alpha,u)] \\ &= 2B(\alpha,u)\,[\varphi(\alpha) - \varphi(u)] > 0 \end{aligned} \tag{23}$$

or $0 < u < \alpha$ and $\alpha \ge p - 1$. By (20) and (23), we obtain that

$$T''(\alpha) + \frac{2}{\alpha}T'(\alpha) > 0 \quad \text{for } \alpha \ge p - 1. \tag{24}$$

By (19), $T(\alpha)$ has a critical point $\bar{\alpha}$ in $(p-1,\tau)$. Then by (24), we see that

$$T''(\bar{\alpha}) = T''(\bar{\alpha}) + \frac{2}{\bar{\alpha}}T'(\bar{\alpha}) > 0.$$

So $T(\alpha)$ has exactly one local minimum on $(p-1,\tau)$. It implies that

$$T'(\alpha)\begin{cases} <0 & \text{for } 0<\alpha<\bar{\alpha}, \\ =0 & \text{for } \alpha=\bar{\alpha}, \\ >0 & \text{for } \alpha>\bar{\alpha}. \end{cases} \tag{25}$$

So by (16) and (25), the bifurcation curve S is $\subset$-shaped, starts from $(\infty,0)$ and goes to (∞,∞). Thus the statement (iii) holds.

The proof is complete.

References

[1] V. Anuradha, R. Shivaji and J. Zhu, On S-shaped bifurcation curves for multiparameter positone problems. Differential equations and computational simulations, I (Mississippi State, MS, 1993), Appl. Math. Comput., 65(1994), 171–182.

[2] J.-Y. Ciou and T.-S. Yeh, Complete classification of bifurcation curves for a multiparameter diffusive logistic problem with generalized Holling type-IV functional response, Electron. J. Differential Equations, (2021), 12 pp.

[3] S.-Y. Huang and P.-H. Hsieh, Exact multiplicity and bifurcation curves of positive solutions of generalized logistic problems, Czechoslovak Math. J., 73(2023), 1081–1098.

[4] S.-Y. Huang, Bifurcation diagrams of positive solutions for one-dimensional Minkowski-curvature problem and its applications, Discrete Contin. Dyn. Syst., 39(2019), 3443–3462.

[5] K.-C. Hung and S.-H. Wang, A theorem on S-shaped bifurcation curve for a positone problem with convex-concave nonlinearity and its applications to the perturbed Gelfand problem, J. Differential Equations, 251(2011), 223–237.

[6] K.-C. Hung, Y.-N. Suen and S.-H. Wang, Structures and evolution of bifurcation diagrams for a one-dimensional diffusive generalized logistic problem with constant yield harvesting, J. Differential Equations, 269(2020), 3456–3488.

[7] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, Indiana Univ. Math. J. 20(1970), 1–13.

[8] B. L. Xu and Z. L. Wang, On S-shaped bifurcation curves for a class of perturbed semilinear equations, Chinese Ann. Math. Ser. B, 29(2008), 641–662.

[9] J. Zhang, Y. Qiao, L. Duan and J. Miao, On the evolutionary bifurcation curves for the one-dimensional prescribed mean curvature equation with logistic type, Open Mathematics 2021; 19: 927–939.