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Bifurcation Curves In Minkowski-Curvature Problem With Nonlinearity $u^p + u$*†

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Abstract

We study the exact shapes of bifurcation curve of positive solutions for Minkowski-curvature problem

$$\begin{cases} -\left(u'/\sqrt{1-u'^2}\right)' = \lambda(u^p + u), & \text{in } (-L, L), \\ u(-L) = u(L) = 0, \end{cases}$$

where $\lambda, L > 0$, and $p > 1$. In 2018, Huang gave a conjecture for the shape of bifurcation curve and further proved some part of the conjecture, c.f. [2, 3]. In this paper, we prove the remaining part of the conjecture. In addition, we point out the wrong result appeared in [5, Commun. Contemp. Math., 2019], see Remark 2.

1 Introduction and Main Result

In this paper, we study the exact shapes of bifurcation curves of positive solutions for Minkowski-curvature problem

$$\begin{cases} -\left(u'/\sqrt{1-u'^2}\right)' = \lambda(u^p + u), & \text{in } (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \tag{1}$$

where $\lambda > 0$ is a bifurcation parameter, $L > 0$ is an evolution parameter, and $p > 1$. For $L > 0$, we define the bifurcation curve S_L of (1) on the $(\lambda, \|u\|_\infty)$-plane by

$$S_L \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1)}\}. \tag{2}$$

It is easy to prove that S_L is increasing for all $L > 0$ when $0 < p \leq 1$, c.f. [2]. Thus we consider the case that $p > 1$ in this paper. Before going into further discussions on problem (1), we give some terminologies in this paper for the shapes of bifurcation curves S_L on the $(\lambda, \|u\|_\infty)$-plane.

⊂-like shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_L is *⊂-like shaped* if S_L initially continues to the *left* and eventually continues to the *right*.

⊂-shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_L is *⊂-shaped* if S_L is ⊂-like shaped and has exactly one turning point.

S-like shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_L is *S-like shaped* if S_L initially continues to the *right* (or starts from $(0, 0)$), eventually continues to the *right* and has a turning point which turns to the *left*.

S-shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_L is *S-shaped* if S_L is S-like shaped and has exactly two turning points.

In [2], Huang gave a conjecture for the problem (1), see Conjecture 1.

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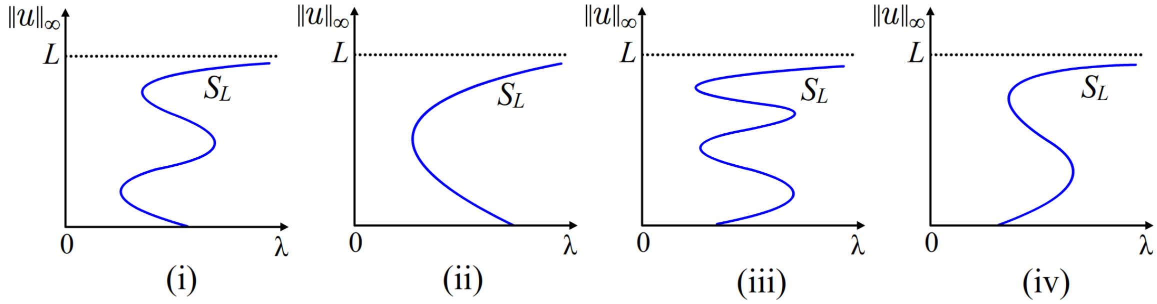


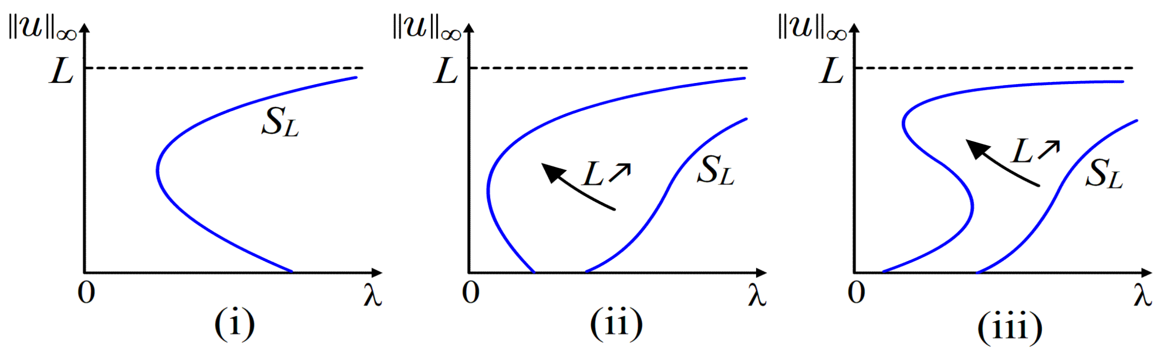
Figure 1: (i) ⊂-like shaped. (ii) ⊂-shaped. (iii) S-like shaped. (iv) S-shaped.

Conjecture 1 ([2, Remark 1], see Figure 1) *Consider (1). Then the following statements (i)–(iii) hold.*

(i) If $1 < p < 3$, the bifurcation curve S_L is ⊂-shaped for all $L > 0$.

(ii) If $p = 3$, the bifurcation curve S_L is from increasing to ⊂-shaped with varying $L > 0$.

(iii) If $p > 3$, the bifurcation curve S_L is from increasing to S-shaped with varying $L > 0$.



Grpahs of bifurcation curve S_L. (i) $1 < p < 3$. (ii) $p = 3$. (iii) $p > 3$.

In 2018, Huang proved the following theorem in order to solve the Conjecture 1.

Theorem 1 ([2, 3]) *Consider (1). Then the following statements (i)–(iv) hold.*

(i) If $1 < p \leq 2$, the bifurcation curve S_L is ⊂-shaped for all $L > 0$.

(ii) If $2 < p < 3$, the bifurcation curve S_L is ⊂-like shaped for all $L > 0$;

(iii) If $p = 3$, the bifurcation curve S_L is increasing for $0 < L \leq \pi/(2\sqrt{2})$ and ⊂-like shaped for $L > \pi/(2\sqrt{2})$; and

(iv) If $p > 3$, there exists $\tilde{L}_p > 0$ such that the bifurcation curve S_L is increasing for $0 < L \leq \tilde{L}_p$ and S-like shaped for $L > \tilde{L}_p$.

By Theorem 1, it is obvious that Conjecture 1 is only resolved when $1 < p \leq 2$. Thus we intend to further study the exact shape of S_L when $p > 2$. The following Theorem 2 is our main result.

Theorem 2 (See Figure 2) *Consider (1) with $p > 2$. Then there exists $\bar{L}_p > 0$ such that*

(i) if $2 < p < 3$, the bifurcation curve S_L is ⊂-shaped for $L > \bar{L}_p$;

(ii) if $p = 3$, the bifurcation curve S_L is $\subset$-shaped for $L > \pi/(2\sqrt{2})$; and

(iii) if $p > 3$, the bifurcation curve S_L is S-shaped for $L > \bar{L}_p$.

Remark 1 *By Theorems 1(iii) and 2(ii), we see that Conjecture 1(ii) has been solved. In addition, we know that it is not easy to prove that the bifurcation curve is S-shaped for Minkowski-curvature problem. Fortunately, we have the result for the S-shaped curve for (1) under $p > 3$.*

Remark 2 *In [5, Theorem 3.6], Zhang and Feng proved that if $p > 2$, then there exists $\lambda^{**} > 0$ such that (1) has no positive solutions for $0 < \lambda < \lambda^{**}$ and exactly one positive solution for $\lambda \geq \lambda^{**}$. Obviously, this result is wrong by Theorems 1 and 2.*

As a comparison, we consider the semilinear problem

$$\begin{cases} -u'' = \lambda(u^p + u), & \text{in } (-1,1)\,, \\ u(-1) = u(1) = 0, \end{cases} \tag{3}$$

and the prescribed mean curvature problem

$$\begin{cases} -\left(u'/\sqrt{1+u'^2}\right)' = \lambda(u^p + u), & \text{in } (-L,L)\,, \\ u(-L) = u(L) = 0. \end{cases} \tag{4}$$

By [4, 6], the corresponding bifurcation curves of (3) and (4) may be decreasing, increasing or reversed $\subset$-shaped, see Figure 3. Obviously, the shapes of bifurcation curve S_L of (1) are more complex than the shapes for (3) and (4).

2 Lemmas

To prove Theorem 2, we first introduce the time-map method used in Corsato [1, p. 127]. We define the time-map formula for (1) by

$$T_\lambda(\alpha) \equiv \int_0^\alpha \frac{\lambda\left[F(\alpha) - F(u)\right] + 1}{\sqrt{\left\{\lambda\left[F(\alpha) - F(u)\right] + 1\right\}^2 - 1}}du \quad \text{for } \alpha > 0 \text{ and } \lambda > 0. \tag{5}$$

where $F(u) \equiv \int_0^u f(t)dt$ and $f(u) \equiv u^p + u$. Observe that positive solutions $u_\lambda \in C^2(-L,L) \cap C[-L,L]$ for (1) correspond to

$$\|u_\lambda\|_\infty = \alpha \quad \text{and} \quad T_\lambda(\alpha) = L.$$

So by definition of S_L in (2), we have that

$$S_L = \{(\lambda,\alpha) : T_\lambda(\alpha) = L \text{ for some } \alpha, \lambda > 0\}\,. \tag{6}$$

Thus, it is important to understand fundamental properties of the time-map $T_\lambda(\alpha)$ on $(0,\infty)$ in order to study the shape of the bifurcation curve S_L of (1) for any fixed $L > 0$. Note that it can be proved that $T_\lambda(\alpha)$ is a twice continuously differentiable function of $\alpha > 0$ and $\lambda > 0$. The proofs are easy but tedious and hence we omit them. Then we compute

$$T'_\lambda(\alpha) = \frac{1}{\alpha}\int_0^\alpha \frac{\lambda^3 B^3 + 3\lambda^2 B^2 + \lambda\,(2B - A)}{\left(\lambda^2 B^2 + 2\lambda B\right)^{3/2}}du \tag{7}$$

and

$$T''_\lambda(\alpha) = \frac{1}{\alpha^2}\int_0^\alpha \frac{\left(3A^2B - B^2C - 2AB^2\right)\lambda^3 + \left(3A^2 - 4AB - 2BC\right)\lambda^2}{\left[\lambda^2 B^2 + 2\lambda B\right]^{5/2}}du \tag{8}$$

where $A = \alpha f(\alpha) - uf(u)$, $B = F(\alpha) - F(u)$ and $C = \alpha^2 f'(\alpha) - u^2 f'(u)$.

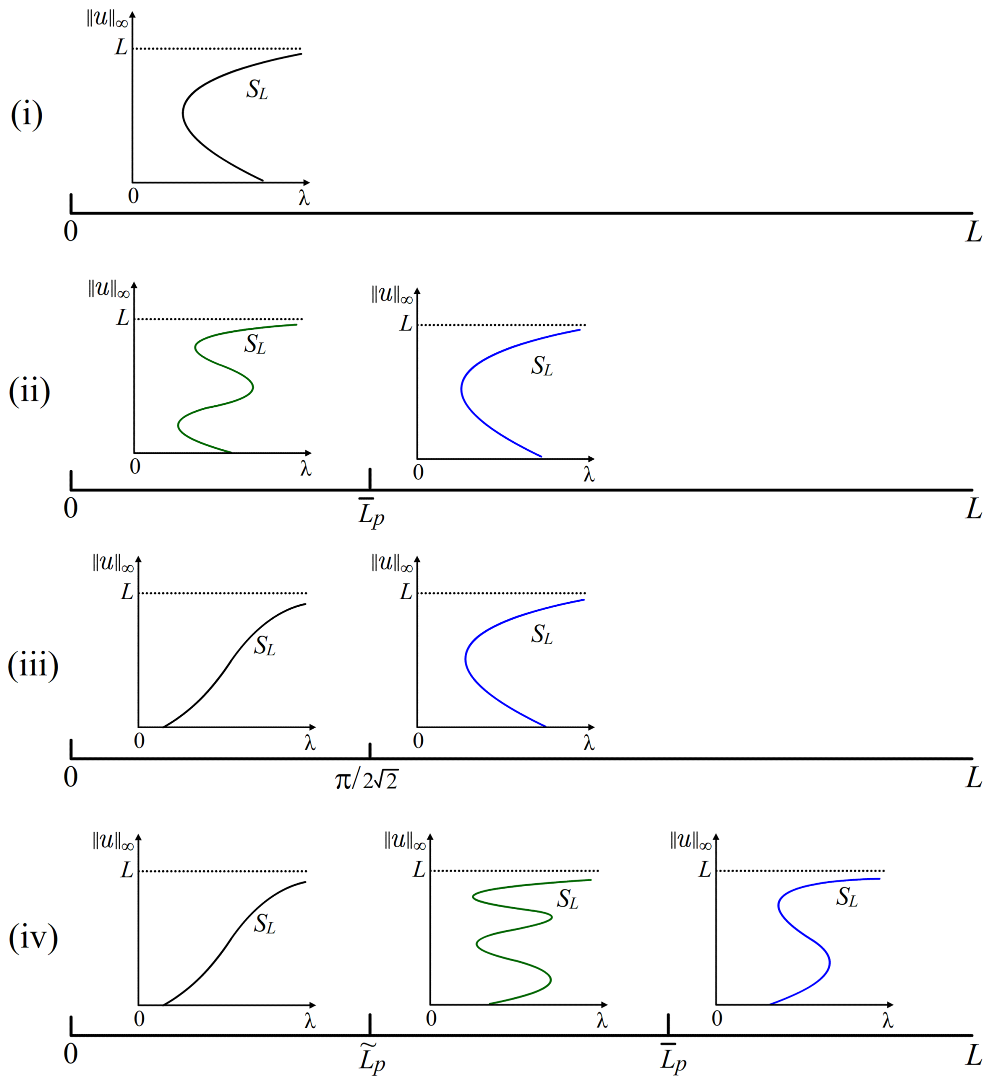


Figure 2: Graphs of bifurcation curve S_L of (1). (i) $1 < p \leq 2$. (ii) $2 < p < 3$. (iii) $p = 3$. (iv) $p > 3$. Notice that the black curves are the known results, the blue curves are obtained in this paper, and the exact shapes of the green curves have yet to be solved.

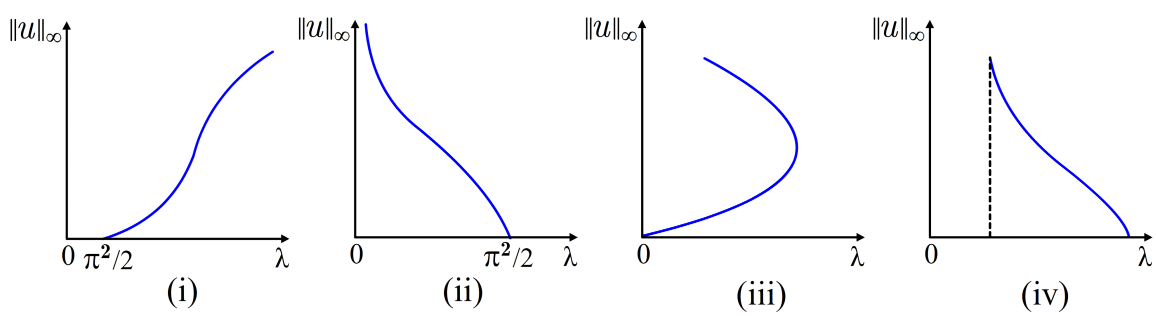


Figure 3: Figures (i)–(ii) are the graphs of bifurcation curve of (3) where (i) $0 < p < 1$. (ii) $p \geq 1$. Figures (iii)–(iv) are the graphs of bifurcation curve of (4) where (iii) $0 < p < 1$. (iv) $p \geq 1$.

Lemma 1 **([3, Lemmas 4.1])** *Consider (1) with $p > 2$. For any $\lambda > 0$, then*

$$\lim_{\alpha \to 0^+} T_\lambda(\alpha) = \frac{\pi}{2\sqrt{\lambda}}, \quad \lim_{\alpha \to \infty} T_\lambda(\alpha) = \infty,$$

$$\lim_{\alpha \to 0^+} T'_\lambda(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to 0^+} T''_\lambda(\alpha) = \begin{cases} -\infty & \text{if } 2 < p < 3, \\ \frac{3\pi}{16\sqrt{\lambda}}(\lambda - 2) & \text{if } p = 3, \\ \frac{3\pi}{16}\sqrt{\lambda} & \text{if } p > 3. \end{cases}$$

Lemma 2 *Consider (1) with $p > 2$. Let $X = \alpha^{p+1} - u^{p+1}$ and $Y = \alpha^2 - u^2$. Then*

$$\alpha^{p-1} < \frac{X}{Y} < \frac{p+1}{2}\alpha^{p-1}$$

and

$$A = X + Y > 0, \;\; B = \frac{X}{p+1} + \frac{Y}{2} > 0 \;\; \text{and} \;\; C = pX + Y > 0 \tag{9}$$

for $0 < u < \alpha$.

Proof. We obtain (9) by definitions of X, Y, A, B and C. Then we compute

$$\frac{\partial}{\partial u}\frac{X}{Y} = \frac{uV(u)}{\left(\alpha^2 - u^2\right)^2}, \tag{10}$$

where $V(u) = (p+1)\left(u^{p+1} - \alpha^2 u^{p-1}\right) + 2\left(\alpha^{p+1} - u^{p+1}\right)$. Since

$$V'(u) = -\left(p^2 - 1\right)\left(\alpha^2 - u^2\right)u^{p-2} < 0 \;\; \text{for } 0 < u < \alpha,$$

we see that $V(u) > V(\alpha) = 0$ for $0 < u < \alpha$. So by (10), we observe that

$$\alpha^{p-1} = \lim_{u \to 0^+}\frac{X}{Y} < \frac{X}{Y} < \lim_{u \to \alpha^-}\frac{X}{Y} = \frac{p+1}{2}\alpha^{p-1} \;\; \text{for } 0 < u < \alpha.$$

The proof is complete. ■

Lemma 3 *Consider (1) with $p > 2$. Then*

$$\frac{\partial}{\partial \lambda}T_\lambda(\alpha) < 0 \;\; \text{and} \;\; \frac{\partial}{\partial \lambda}\sqrt{\lambda}T'_\lambda(\alpha) > 0 \;\; \text{for } \alpha > 0 \text{ and } \lambda > 0.$$

Proof. By (5), (7) and (9), we observe that

$$\frac{\partial}{\partial \lambda}T_\lambda(\alpha) = \frac{1}{\alpha}\int_0^\alpha \frac{-B}{\left(\lambda^2 B^2 + 2\lambda B\right)^{3/2}}du < 0$$

and

$$\frac{\partial}{\partial \lambda}\sqrt{\lambda}T'_\lambda(\alpha) = \frac{1}{\alpha}\int_0^\alpha \frac{B^2\left(B^3\lambda^2 + 5B^2\lambda + 3A + 6B\right)}{2\left(\lambda B^2 + 2B\right)^{5/2}}du > 0$$

for $\alpha > 0$ and $\lambda > 0$. The proof is complete. ■

In order to prove Lemma 4, we need the following Dini's Theorem.

Theorem 3 (Dini's Theorem) *Assume that the monotone sequence of continuous real-valued functions on compact space converges pointwise to a continuous function. Then the convergence is uniform.*

Lemma 4 *Consider (1) with $p > 2$. For any $\rho_2 > \rho_1 > 0$, then there exists $\tilde{\lambda}(p, \rho_1, \rho_2) > 0$ such that*

$$T'_\lambda(\alpha) < 0 \ \text{ for } \rho_1 \le \alpha \le \rho_2 \text{ and } 0 < \lambda < \tilde{\lambda}(p, \rho_1, \rho_2).$$

Proof. Let

$$g_0(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{2B - A}{B^{3/2}} du \ \text{ and } \ g_\lambda(\alpha) = \sqrt{\lambda} T'_\lambda(\alpha) \tag{11}$$

for $\rho_1 \le \alpha \le \rho_2$ and $\lambda > 0$. By (9), we obtain

$$2B - A = -\frac{p-1}{p+1} X < 0 \ \text{ for } 0 < u < \alpha.$$

So by (11), we see that

$$g_0(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{2B - A}{B^{3/2}} du < 0 \ \text{ for } \alpha > 0. \tag{12}$$

By Lemma 3, $\{g_\lambda\}$ is an increasing sequence. Clearly, g_λ and g_0 are continuous on $(0, \infty)$ and

$$\lim_{\lambda \to 0^+} g_\lambda(\alpha) = g_0(\alpha) < 0 \ \text{ for } \alpha > 0. \tag{13}$$

Then by Theorem 3, g_λ uniformly converges to g_0 on $[\rho_1, \rho_2]$. So by (13), there exists $\tilde{\lambda}(p, \rho_1, \rho_2) > 0$ such that $g_\lambda(\alpha) < 0$ for $\rho_1 \le \alpha \le \rho_2$ and $0 < \lambda < \tilde{\lambda}(p, \rho_1, \rho_2)$. It implies that $T'_\lambda(\alpha) < 0$ for $\rho_1 \le \alpha \le \rho_2$ and $0 < \lambda < \tilde{\lambda}(p, \rho_1, \rho_2)$. The proof is complete. ■

Lemma 5 *Consider (1) with $p > 2$. Let $\delta_p \equiv (p-2)^{1/(p-1)}$. Then $T''_\lambda(\alpha) > 0$ for $\alpha \ge \delta_p$ and $\lambda > 0$.*

Proof. By (9), we obtain

$$3A^2 - 4AB - 2BC = \frac{XY(p-1)}{p+1} R_1(\frac{X}{Y}) \tag{14}$$

and

$$3A^2B - B^2C - 2AB^2 = \frac{2XY^2 + (1+p)Y^3}{4(p+1)^2} R_2(\frac{X}{Y}), \tag{15}$$

where $R_1(t) = t - p + 2$ and $R_2(t) = 2(2p+1)t^2 + \left(-p^2 + 9p + 4\right)t + 3(p+1)$. Since

$$R_2(p-2) = 3(p+1)(p-1)^2 > 0 \ \text{ and } \ R'_2(p-2) = (7p+4)(p-1) > 0,$$

we observe that

$$R_1(t) > 0 \ \text{ and } \ R_2(t) > 0 \ \text{ for } t > p - 2. \tag{16}$$

By Lemma 2, then

$$\frac{X}{Y} > \alpha^{p-1} \ge (\delta_p)^{p-1} = p - 2 \ \text{ for } 0 < u < \alpha \text{ and } \alpha \ge \delta_p.$$

So by (16), we see that

$$R_1(\frac{X}{Y}) > 0 \ \text{ and } \ R_2(\frac{X}{Y}) > 0 \ \text{ for } 0 < u < \alpha \text{ and } \alpha \ge \delta_p. \tag{17}$$

By (8), (14), (15) and (17), $T''_\lambda(\alpha) > 0$ for $\alpha \ge \delta_p$ and $\lambda > 0$. The proof is complete. ■

Lemma 6 *Consider (1) with $p>3$. Then there exists $\xi_p>0$ such that $T_\lambda'''(\alpha)<0$ for $0<\alpha\le\xi_p$ and $\lambda>0$.*

Proof. Let $E=\alpha^2 f''(\alpha)-u^2 f''(u)$. Clearly, $E=p\,(p-1)\,X$. Then by (8), we compute

$$T_\lambda'''(\alpha)=\frac{1}{\alpha^3}\int_0^\alpha \frac{\lambda^3\left[M_2\,(B\lambda)^2+M_1 B\lambda+M_0\right]}{\left[\lambda^2B^2+2\lambda B\right]^{7/2}}du, \tag{18}$$

where

$$M_2=9A^2B-3B^2C-B^2E-12A^3+9ABC,$$
$$M_1=27A^2B-12B^2C-4B^2E-24A^3+27ABC$$

and

$$M_0=18A^2B-12B^2C-4B^2E-15A^3+18ABC.$$

We divide this proof into the following three steps.

Step 1. We prove that there exists $\xi_{1,p}>0$ such that $M_2<0$ for $0<u<\alpha\le\xi_{1,p}$. By (9), we see that

$$M_2=\frac{Y^3}{4\,(p+1)^2}P_1(\frac{X}{Y}), \tag{19}$$

where

$$\begin{aligned} P_1(t) \quad = \quad & -4\,(2p+3)\,(2p+1)\,t^3+2\left(7p^3-33p^2-49p-15\right)t^2 \\ & +(p+1)\left(-p^3+15p^2-74p-30\right)t-15\,(p+1)^2\,. \end{aligned}$$

Since $P_1(0)=-15\,(p+1)^2<0$, there exists $\kappa_{1,p}>0$ such that $P_1(t)<0$ for $0<t\le\kappa_{1,p}$. Let

$$\xi_{1,p}=\left(\frac{2\kappa_{1,p}}{p+1}\right)^{\frac{1}{p-1}}.$$

By Lemma 2, then

$$0<\frac{X}{Y}<\frac{p+1}{2}\alpha^{p-1}\le\frac{p+1}{2}\xi_{1,p}^{p-1}=\kappa_{1,p}\quad\text{for } 0<u<\alpha\le\xi_{1,p}.$$

It follows that $P_1(X/Y)<0$ for $0<u<\alpha\le\xi_{1,p}$. So by (19), $M_2<0$ for $0<u<\alpha\le\xi_{1,p}$.

Step 2. We prove that there exists $\xi_{2,p}>0$ such that $M_1^2-4M_2M_0<0$ for $0<u<\alpha\le\xi_{2,p}$. By (9), we see that

$$M_1^2-4M_2M_0=\frac{3XY^3(X+Y)^2(p-1)}{4\,(p+1)^2}P_2(\frac{X}{Y}), \tag{20}$$

where

$$\begin{aligned} P_2(t) \quad = \quad & -20\,(p+3)\,t^3+20\left(5p^2-8p-9\right)t^2+\left(7p^3+99p^2-79p-267\right)t \\ & -20\,(p-2)\,(p-3)\,(p+1)\,. \end{aligned}$$

Since $P_2(0)=-20\,(p-2)\,(p-3)\,(p+1)<0$, there exists $\kappa_{2,p}>0$ such that $P_2(t)<0$ for $0<t\le\kappa_{2,p}$. Let

$$\xi_{2,p}=\left(\frac{2\kappa_{2,p}}{p+1}\right)^{\frac{1}{p-1}}.$$

By Lemma 2, then

$$0<\frac{X}{Y}<\frac{p+1}{2}\alpha^{p-1}\le\frac{p+1}{2}\xi_{2,p}^{p-1}=\kappa_{2,p}\quad\text{for } 0<u<\alpha\le\xi_{2,p}.$$

It follows that $P_2(X/Y) < 0$ for $0 < u < \alpha \le \xi_{2,p}$. So by (20), $M_1^2 - 4M_2M_0 < 0$ for $0 < u < \alpha \le \xi_{2,p}$.

Step 3. We prove Lemma 6. Let $\xi_p = \min\{\xi_{1,p}, \xi_{2,p}\}$. By Steps 1–2, we see that

$$M_2t^2 + M_1t + M_0 < 0 \quad \text{for } t \in \mathbb{R} \text{ and } 0 < u < \alpha \le \xi_p. \tag{21}$$

It follows that $M_2\,(B\lambda)^2 + M_1B\lambda + M_0$ for $0 < u < \alpha \le \xi_p$ and $\lambda > 0$. So by (18), we obtain $T_\lambda'''(\alpha) < 0$ for $0 < \alpha \le \xi_p$ and $\lambda > 0$.

The proof is complete. ■

Lemma 7 *Consider (1) with $p > 2$. Then there exists $\bar\lambda_p > 0$ such that, for $0 < \lambda \le \bar\lambda_p$, the following statements (i)–(ii) hold:*

(i) If $2 < p < 3$, then there exists $\bar\alpha_\lambda > 0$ such that

$$T_\lambda'(\alpha)\begin{cases} < 0 & \text{for } 0 < \alpha < \bar\alpha_\lambda, \\ = 0 & \text{for } \alpha = \bar\alpha_\lambda, \\ > 0 & \text{for } \alpha > \bar\alpha_\lambda. \end{cases} \tag{22}$$

Furthermore, $T_\lambda(\bar\alpha_\lambda)$ is strictly decreasing for $0 < \lambda \le \bar\lambda_p$.

(ii) If $p > 3$, then there exist $\alpha_{2,\lambda} > \alpha_{1,\lambda} > 0$ such that

$$T_\lambda'(\alpha)\begin{cases} < 0 & \text{for } \alpha_{1,\lambda} < \alpha < \alpha_{2,\lambda}, \\ = 0 & \text{for } \alpha = \alpha_{1,\lambda} \text{ and } \alpha = \alpha_{2,\lambda}, \\ > 0 & \text{for } 0 < \alpha < \alpha_{1,\lambda} \text{ and } \alpha > \alpha_{2,\lambda}. \end{cases} \tag{23}$$

Furthermore, $T_\lambda(\alpha_{1,\lambda})$ and $T_\lambda(\alpha_{2,\lambda})$ are strictly decreasing for $0 < \lambda \le \bar\lambda_p$.

Proof. (I) Assume that $2 < p < 3$. By Lemma 1, there exists $\tau_p \in (0, \delta_p)$ such that $T_1'(\alpha) < 0$ for $0 < \alpha \le \tau_p$ where δ_p is defined in Lemma 5. Then by Lemma 3, we see that $\sqrt{\lambda}T_\lambda'(\alpha) \le T_1'(\alpha) < 0$ for $0 < \alpha \le \tau_p$ and $0 < \lambda \le 1$. It implies that

$$T_\lambda'(\alpha) < 0 \quad \text{for } 0 < \alpha \le \tau_p \text{ and } 0 < \lambda \le 1. \tag{24}$$

Let $\bar\lambda_p = \min\left\{1, \tilde\lambda(p, \tau_p, \delta_p)\right\}$ where $\tilde\lambda(p, \tau_p, \delta_p)$ is defined in Lemma 4. It follows that

$$T_\lambda'(\alpha) < 0 \quad \text{for } \tau_p \le \alpha \le \delta_p \text{ and } 0 < \lambda \le \bar\lambda_p. \tag{25}$$

So by (24) and (25), we obtain that $T_\lambda'(\alpha) < 0$ for $0 < \alpha \le \delta_p$ and $0 < \lambda \le \bar\lambda_p$. Let $\lambda \in (0, \bar\lambda_p]$ be given. Since $\lim_{\alpha\to\infty} T_\lambda(\alpha) = \infty$, and by Lemma 5, there exists $\bar\alpha_\lambda \in (\delta_p, \infty)$ such that (22) holds.

Since $\bar\alpha_\lambda \in (\delta_p, \infty)$, and by Lemma 5, we have $T_\lambda''(\bar\alpha_\lambda) > 0$. So by implicit function theorem, $\bar\alpha_\lambda$ is a continuously differentiable function with respective to $\lambda \in (0, \bar\lambda_p]$. Since $T_\lambda'(\bar\alpha_\lambda) = 0$, and by Lemma 3, we observe that

$$\frac{\partial}{\partial\lambda}T_\lambda(\bar\alpha_\lambda) = \left[\frac{\partial}{\partial\lambda}T_\lambda(\alpha)\right]_{\alpha=\bar\alpha_\lambda} + T_\lambda'(\bar\alpha_\lambda)\frac{\partial\bar\alpha_\lambda}{\partial\lambda} = \left[\frac{\partial}{\partial\lambda}T_\lambda(\alpha)\right]_{\alpha=\bar\alpha_\lambda} < 0 \quad \text{for } 0 < \lambda \le \bar\lambda_p.$$

So the statement (i) holds.

(II) Assume that $p > 3$. Let $\bar\lambda_p = \tilde\lambda(p, \xi_p, \delta_p)$ where δ_p and ξ_p are defined in Lemmas 5 and 6, respectively. By Lemma 4, then

$$T_\lambda'(\alpha) < 0 \quad \text{for } \xi_p \le \alpha \le \delta_p \text{ and } 0 < \lambda \le \bar\lambda_p. \tag{26}$$

Let $\lambda \in (0, \bar\lambda_p]$ be given. By Lemma 1, we see that

$$T_\lambda'(\alpha) > 0 \quad \text{for all sufficiently small } \alpha > 0. \tag{27}$$

By Lemma 6, $T'_\lambda(\alpha)$ is concave for $0 < \alpha \le \xi_p$. So by (26) and (27), there exists $\alpha_{1,\lambda} \in \left(0, \xi_p\right)$ such that

$$T'_\lambda(\alpha) \begin{cases} > 0 & \text{for } 0 < \alpha < \alpha_{1,\lambda}, \\ = 0 & \text{for } \alpha = \alpha_{1,\lambda}, \\ < 0 & \text{for } \alpha_{1,\lambda} < \alpha \le \xi_p, \end{cases} \quad \text{and} \quad T''_\lambda(\alpha_{1,\lambda}) < 0. \tag{28}$$

Since $\lim_{\alpha \to \infty} T_\lambda(\alpha) = \infty$, and by (26) and Lemma 5, there exists $\alpha_{2,\lambda} \in (\delta_p, \infty)$ such that

$$T'_\lambda(\alpha) \begin{cases} < 0 & \text{for } \delta_p < \alpha < \alpha_{2,\lambda}, \\ = 0 & \text{for } \alpha = \alpha_{2,\lambda}, \\ > 0 & \text{for } \alpha > \alpha_{2,\lambda}, \end{cases} \quad \text{and} \quad T''_\lambda(\alpha_{2,\lambda}) > 0. \tag{29}$$

By (28), (29) and implicit function theorem, we observe that (23) holds, and $\alpha_{1,\lambda}$ and $\alpha_{2,\lambda}$ are continuously differentiable functions with respective to $\lambda \in (0, \bar{\lambda}_p]$. By Lemma 3, we see that

$$\frac{\partial}{\partial \lambda} T_\lambda(\alpha_{i,\lambda}) = \left[\frac{\partial}{\partial \lambda} T_\lambda(\alpha)\right]_{\alpha = \alpha_{i,\lambda}} + T'_\lambda(\alpha_{i,\lambda}) \frac{\partial \alpha_{i,\lambda}}{\partial \lambda} = \left[\frac{\partial}{\partial \lambda} T_\lambda(\alpha)\right]_{\alpha = \alpha_{i,\lambda}} < 0$$

for $0 < \lambda \le \bar{\lambda}_p$ and $i = 1, 2$. So the statement (ii) holds.

The proof is complete. ■

Lemma 8 *Consider (1) with $p = 3$. Then the following statements (i)–(iii) hold:*

(i) $\alpha T''_\lambda(\alpha) - T'_\lambda(\alpha) > 0$ for $0 < \alpha \le 1$ and $0 < \lambda \le 1$.

(ii) $T'''_\lambda(\alpha) > 0$ for $0 < \alpha \le 1/2$ and $1 \le \lambda \le 2$.

(iii) $\alpha T'''_\lambda(\alpha) + 2T''_\lambda(\alpha) > 0$ for $1/2 < \alpha \le 1$ and $1 \le \lambda \le 2$.

Proof. This proof is easy but tedious. So we put it in Appendix. ■

Lemma 9 *Consider (1) with $p = 3$. Then the following statements (i)–(ii) hold:*

(i) If $\lambda \ge 2$, then $T'_\lambda(\alpha) > 0$ for $\alpha > 0$.

(ii) If $0 < \lambda < 2$, then there exists $\bar{\alpha}_\lambda > 0$ such that

$$T'_\lambda(\alpha) \begin{cases} < 0 & \text{for } 0 < \alpha < \bar{\alpha}_\lambda, \\ = 0 & \text{for } \alpha = \bar{\alpha}_\lambda, \\ > 0 & \text{for } \alpha > \bar{\alpha}_\lambda. \end{cases} \tag{30}$$

Furthermore, $T_\lambda(\bar{\alpha}_\lambda)$ is strictly decreasing for $0 < \lambda < 2$.

Proof. (I) By [3, (5.21)], we have $T'_{\lambda=2}(\alpha) > 0$ for $\alpha > 0$. So by Lemma 3,

$$\sqrt{\lambda} T'_\lambda(\alpha) \ge \sqrt{2} T'_{\lambda=2}(\alpha) > 0 \quad \text{for } \alpha > 0 \text{ and } \lambda \ge 2.$$

It implies that statement (i) holds.

(II) By Lemma 1, we have

$$\lim_{\alpha \to \infty} T_\lambda(\alpha) = \infty, \quad \lim_{\alpha \to 0^+} T'_\lambda(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to 0^+} T''_\lambda(\alpha) = \frac{3\pi}{16\sqrt{\lambda}} (\lambda - 2) < 0 \tag{31}$$

for $0 < \lambda < 2$. By Lemmas 5, we have

$$T''_\lambda(\alpha) > 0 \text{ for } \alpha \geq \delta_{p=3} = 1 \text{ and } \lambda > 0. \tag{32}$$

Then we consider two cases:

Case 1. Assume that $0 < \lambda \leq 1$. If $T'_\lambda(\alpha)$ has a zero β_1 on $(0, \infty)$, by (32) and Lemma 8(i), we see that $T''_\lambda(\beta_1) > 0$. So $T'_\lambda(\alpha)$ has at most one zero on $(0, \infty)$. Then by (31), we observe that $T'_\lambda(\alpha)$ has exactly one zero $\bar{\alpha}_\lambda$ on $(0, \infty)$. Moreover, (30) holds.

Case 2. Assume that $1 < \lambda < 2$. If $T''_\lambda(\alpha)$ has a zero β_2 on $(0, 1]$, by Lemma 8(ii)(iii), we observe that $T'''_\lambda(\beta_2) > 0$. Thus

$$T''_\lambda(\alpha) \text{ has at most one zero on } (0, 1]. \tag{33}$$

By (31)–(33), there exists $\alpha_1 > 0$ such that

$$T''_\lambda(\alpha) \begin{cases} < 0 & \text{for } 0 < \alpha < \alpha_1, \\ = 0 & \text{for } \alpha = \alpha_1, \\ > 0 & \text{for } \alpha > \alpha_1. \end{cases}$$

Then by (31), there exists $\bar{\alpha}_\lambda > 0$ such that (30) holds.

Finally, by similar argument in the proof of Lemma 7, we obtain $T_\lambda(\bar{\alpha}_\lambda)$ is strictly decreasing for $0 < \lambda < 2$. So the statement (ii) holds.

The proof is complete. ■

Lemma 10 ([3, Lemma 4.5]) *Consider (1) with fixed $L > 0$. Then the following assertions (i)–(iii) hold:*

(i) there exists a positive function $\lambda_L(\alpha) \in C^1(0, L)$ such that $T_{\lambda_L(\alpha)}(\alpha) = L$ for $\alpha > 0$. Moreover, the bifurcation curve $S_L = \{(\lambda_L(\alpha), \alpha) : 0 < \alpha < L\}$ is continuous on the $(\lambda, \|u\|_\infty)$-plane.

(ii) $sgn(\lambda'_L(\alpha)) = sgn(T'_{\lambda_L(\alpha)}(\alpha))$ for $\alpha > 0$ where $sgn(u)$ is the signum function.

(iii)

$$\lim_{\alpha \to 0^+} \lambda_L(\alpha) = \frac{\pi^2}{4L^2} \text{ and } \lim_{\alpha \to L^-} \lambda_L(\alpha) = \infty.$$

3 Proof of Theorem 2

(I) Assume that $2 < p < 3$. Let

$$L_p = \frac{\pi}{2\sqrt{\bar{\lambda}_p}},$$

where $\bar{\lambda}_p$ is defined in Lemma 7. Let $L > L_p$ be given. From the proof of [2, Theorem 2.1, p5999-p6000], we find that there exists $\delta > 0$ such that

$$\lambda'_L(\alpha) < 0 \text{ for } 0 < \alpha < \delta. \tag{34}$$

We assert that

$$\lambda_L(\alpha) \text{ has exactly one critical point } \alpha^* \text{ on } (0, L). \tag{35}$$

By Lemma 10(iii), (34) and (35), we obtain

$$\lambda'_L(\alpha) \begin{cases} < 0 & \text{on } (0, \alpha^*), \\ = 0 & \text{for } \alpha = \alpha^*, \\ > 0 & \text{on } (\alpha^*, L). \end{cases}$$

Thus Theorem 2(i) holds.

Next, we prove assertion (35). By Lemma 10(iii) and (34), $\lambda_L(\alpha)$ has at least one critical point on $(0, L)$. Suppose $\lambda_L(\alpha)$ has two critical points α_1 and α_2 on $(0, L)$. Let $\lambda_1 = \lambda_L(\alpha_1)$ and $\lambda_2 = \lambda_L(\alpha_2)$. Since $\lambda'_L(\alpha_1) = \lambda'_L(\alpha_2) = 0$, and by Lemma 10, we obtain

$$T_{\lambda_1}(\alpha_1) = T_{\lambda_2}(\alpha_2) = L \quad \text{and} \quad T'_{\lambda_1}(\alpha_1) = T'_{\lambda_2}(\alpha_2) = 0. \tag{36}$$

Since

$$L > L_p = \frac{\pi}{2\sqrt{\bar{\lambda}_p}},$$

and by Lemmas 1 and 7(i), there exist $\lambda_* \in (0, \bar{\lambda}_p)$ and $\alpha_* \in (\bar{\alpha}_{\lambda_*}, \infty)$ such that

$$L = \frac{\pi}{2\sqrt{\lambda_*}} = \lim_{\alpha \to 0^+} T_{\lambda_*}(\alpha) = T_{\lambda_*}(\alpha_*), \tag{37}$$

see Figure 4(i). By Lemmas 3 and 7(i), we observe that

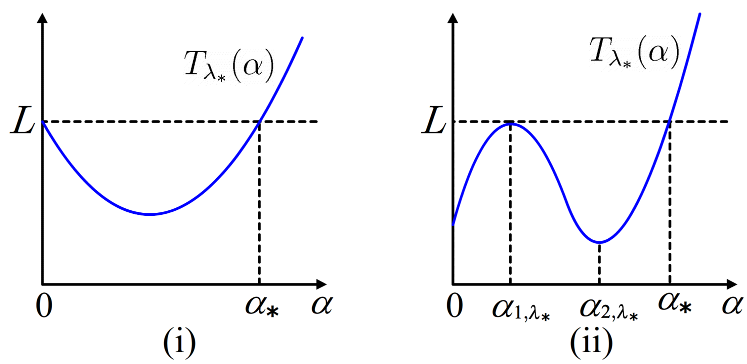


Figure 4: The graph of $T_{\lambda_*}(\alpha)$. (i) $2 < p \leq 3$. (ii) $p > 3$.

$$\sqrt{\lambda}T'_{\lambda}(\alpha) \geq \sqrt{\lambda_*}T'_{\lambda_*}(\alpha) > 0 \quad \text{for } \alpha \geq \alpha_* \text{ and } \lambda \geq \lambda_*. \tag{38}$$

If $\lambda_1 \geq \lambda_*$, by (36) and (38), we observe that $0 < \alpha_1 < \alpha_*$. Then by (36), Lemma 3 and (37), we see that

$$L = T_{\lambda_1}(\alpha_1) \leq T_{\lambda_*}(\alpha_1) < T_{\lambda_*}(\alpha_*) = L,$$

which is a contradiction. Thus $0 < \lambda_1 < \lambda_*$. Similarly, we obtain $0 < \lambda_2 < \lambda_*$. Since $0 < \lambda_1, \lambda_2 < \bar{\lambda}_p$, and by Lemma 7(i) and (36), we observe that

$$\alpha_1 = \bar{\alpha}_{\lambda_1}, \quad \alpha_2 = \bar{\alpha}_{\lambda_2} \quad \text{and} \quad T_{\lambda_1}(\bar{\alpha}_{\lambda_1}) = T_{\lambda_2}(\bar{\alpha}_{\lambda_2}) = L.$$

Since $T_\lambda(\bar{\alpha}_\lambda)$ is strictly decreasing for $0 < \lambda \leq \bar{\lambda}_p$, we see that $\lambda_1 = \lambda_2$ and $\alpha_1 = \alpha_2$. Thus assertion (35) holds.

(II) Assume that $p = 3$. Let $L > \pi/(2\sqrt{2})$ be given. By Lemma 1, we see that

$$L > \frac{\pi}{2\sqrt{2}} = \lim_{\alpha \to 0^+} T_2(\alpha).$$

Suppose $\lambda_L(\alpha)$ has two critical points α_1 and α_2 on $(0, L)$. Let $\lambda_1 = \lambda_L(\alpha_1)$ and $\lambda_2 = \lambda_L(\alpha_2)$. Since $\lambda'_L(\alpha_1) = \lambda'_L(\alpha_2) = 0$, and by Lemma 10, we obtain

$$T_{\lambda_1}(\alpha_1) = T_{\lambda_2}(\alpha_2) = L \quad \text{and} \quad T'_{\lambda_1}(\alpha_1) = T'_{\lambda_2}(\alpha_2) = 0. \tag{39}$$

By Lemma 9(i) and (39), we see that $0<\lambda_1,\lambda_2<2$. By Lemma 9(ii) and (39), we observe that

$$\alpha_1=\bar{\alpha}_{\lambda_1},\ \ \alpha_2=\bar{\alpha}_{\lambda_2}\ \text{ and }\ T_{\lambda_1}(\bar{\alpha}_{\lambda_1})=T_{\lambda_2}(\bar{\alpha}_{\lambda_2})=L.$$

Since $T_\lambda(\bar{\alpha}_\lambda)$ is strictly decreasing for $0<\lambda<2$, we further observe that $\lambda_1=\lambda_2$ and $\alpha_1=\alpha_2$. Thus $\lambda_L(\alpha)$ has at most one critical point on $(0,L)$. By Theorem 1(iii) and Lemma 10(i), $\lambda_L(\alpha)$ has exactly one critical point, a local minimum, on $(0,L)$. So Theorem 2(ii) holds.

(III) Assume that $p>3$. We let

$$L_p=\max\left\{T_{\bar{\lambda}_p}(\alpha_{1,\bar{\lambda}_p}),\tilde{L}_p\right\}+1,$$

where $\bar{\lambda}_p$ and $\alpha_{1,\bar{\lambda}_p}$ are defined in Lemma 7, and $\tilde{L}_p$ is defined in Theorem 1(iv). Let $L>L_p$ be given. We assert that

$$\lambda_L(\alpha)\text{ has exactly two critical points }\alpha_1\text{ and }\alpha_2\text{ on }(0,L). \tag{40}$$

By Theorem 1(iv), Lemma 10 and (40), we obtain

$$\lambda_L'(\alpha)\begin{cases}<0 & \text{on }\ (\alpha_1,\alpha_2)\,,\\ =0 & \text{for }\alpha=\alpha_1\text{ and }\alpha=\alpha_2,\\ >0 & \text{on }\ (0,\alpha_1)\cup(\alpha_2,L)\,,\end{cases} \tag{41}$$

which implies that Theorem 2(iii) holds.

Next, we prove assertion (40). Since $L_p>\tilde{L}_p$, and by Theorem 1(iv) and Lemma 10(i), $\lambda_L(\alpha)$ has two distinct critical points α_1 and α_2 on $(0,L)$. Suppose $\lambda_L(\alpha)$ has a critical point α_3 on $(0,L)$ such that $\alpha_3\neq\alpha_1$ and $\alpha_3\neq\alpha_2$. Let $\lambda_i=\lambda_L(\alpha_i)$ for $i=1,2,3$. Since $\lambda_L'(\alpha_i)=0$ for $i=1,2,3$, and by Lemma 10(ii), we obtain

$$T_{\lambda_1}(\alpha_1)=T_{\lambda_2}(\alpha_2)=T_{\lambda_3}(\alpha_3)=L\ \text{ and }\ T_{\lambda_1}'(\alpha_1)=T_{\lambda_2}'(\alpha_2)=T_{\lambda_3}'(\alpha_3)=0. \tag{42}$$

If $\lambda_1>\bar{\lambda}_p$, by Lemmas 3 and (42), then

$$T_{\bar{\lambda}_p}(\alpha_1)>T_{\lambda_1}(\alpha_1)=L>L_p>T_{\bar{\lambda}_p}(\alpha_{1,\bar{\lambda}_p}).$$

So by Lemma 7(ii), we see that $\alpha_1>\alpha_{2,\bar{\lambda}_p}$. Furthermore, $T_{\bar{\lambda}_p}'(\alpha_1)>0$. So by (42) and Lemmas 3, we see that

$$0=\sqrt{\lambda_1}T_{\lambda_1}'(\alpha_1)>\sqrt{\bar{\lambda}_p}T_{\bar{\lambda}_p}'(\alpha_1)>0,$$

which is a contradiction. Thus $\lambda_1\leq\bar{\lambda}_p$. Similarly, we obtain $\lambda_2\leq\bar{\lambda}_p$ and $\lambda_3\leq\bar{\lambda}_p$. By Lemma 7(ii) and (42), we observe that, for $i=1,2,3$,

$$\text{either }\ (\alpha_i=\alpha_{1,\lambda_i}\text{ and }T_{\lambda_i}(\alpha_{1,\lambda_i})=L)\ \text{ or }\ (\alpha_i=\alpha_{2,\lambda_i}\text{ and }T_{\lambda_i}(\alpha_{2,\lambda_i})=L). \tag{43}$$

Since $T_\lambda(\alpha_{1,\lambda})$ and $T_\lambda(\alpha_{2,\lambda})$ are strictly decreasing for $0<\lambda\leq\bar{\lambda}_p$, and by (43), we observe that one of $\alpha_1=\alpha_2$, $\alpha_1=\alpha_3$ and $\alpha_2=\alpha_3$ holds. It is a contradiction. Thus $\lambda_L(\alpha)$ has exactly two critical points α_1 and α_2. Thus assertion (40) holds.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest The corresponding author states that there is no conflict of interest.

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4 Appendix

In this section, we prove Lemma 8.

Proof of Lemma 8(i). By (7) and (8), we obtain

$$\alpha T_\lambda''(\alpha) - T_\lambda'(\alpha) = \frac{1}{\alpha}\int_0^\alpha \frac{\lambda^2\phi_1(\lambda)}{\left(\lambda^2B^2+2\lambda B\right)^{5/2}}du, \tag{44}$$

where

$$\begin{aligned}\phi_1(\lambda) &= -B^5\lambda^3 - 5B^4\lambda^2 + B\left(-AB - BC + 3A^2 - 8B^2\right)\lambda \\ &\quad -2AB - 2BC + 3A^2 - 4B^2.\end{aligned}$$

By Lemma 2, it is easy to see that

$$\phi_1(0) = \frac{3}{4}X^2 > 0 \ \text{ for } 0 < u < \alpha. \tag{45}$$

We assert that

$$\phi_1(1) > 0 \ \text{ for } 0 < u < \alpha \le 1. \tag{46}$$

Since $\phi_1''(\lambda) = -6B^5\lambda - 10B^4 < 0$ for $0 < u < \alpha$ and $\lambda > 0$, and by (45) and (46), we see that $\phi_1(\lambda) > 0$ for $0 < u < \alpha \le 1$ and $0 < \lambda \le 1$. Then Lemma 8(i) holds by (44).

Next, we prove assertion (46). By Lemma 2, we compute that

$$\phi_1(1) = Y^2\bar{\phi}_1(Y, \frac{X}{Y}) \ , \tag{47}$$

where

$$\bar{\phi}_1(s,t) = -\frac{1}{1024}(t+2)^5 s^3 - \frac{5}{256}(t+2)^4 s^2 + \frac{3}{8}t(t+2)(t+1)s + \frac{3}{4}t^2.$$

It is easy to see that

$$\bar{\phi}_1(0,t) > 0 \ \text{ and } \ \frac{\partial^2}{\partial s^2}\bar{\phi}_1(s,t) < 0 \ \text{ for } s,t > 0. \tag{48}$$

By applying Maple soft, we observe that

$$\bar{\phi}_1(t,t) = \frac{1}{1024}t^2\left(-t^6 - 10t^5 - 60t^4 - 240t^3 - 176t^2 + 480t + 1216\right) > 0 \tag{49}$$

for $0 < t \le 1$, and

$$\bar{\phi}_1(1,t) = -\frac{1}{1024}t^5 - \frac{15}{512}t^4 + \frac{23}{128}t^3 + \frac{85}{64}t^2 + \frac{3}{64}t - \frac{11}{32} > 0 \tag{50}$$

for $1 < t \le 2$. So by (48)–(50), we obtain

$$\bar{\phi}_1(s,t) > 0 \ \text{ for } 0 < s \le t \le 1 \tag{51}$$

and

$$\bar{\phi}_1(s,t) > 0 \ \text{ for } 0 < s \le 1 < t \le 2. \tag{52}$$

In addition, since $p = 3$, and by Lemma 2, we observe that

$$0 < Y < \alpha^2 \le 1 \ \text{ and } \ Y < \alpha^2 + u^2 = \frac{X}{Y} < 2 \ \text{ for } 0 < u < \alpha \le 1.$$

So by (51) and (52), we obtain $\bar{\phi}_1(Y, \frac{X}{Y}) > 0$ for $0 < u < \alpha \le 1$. Then assertion (46) holds by (47). ■

Proof of Lemma 8(ii). We compute

$$T_\lambda'''(\alpha) = \frac{1}{\alpha^3}\int_0^\alpha \frac{\lambda^3}{\left[\lambda^2 B^2 + 2\lambda B\right]^{7/2}}\phi_2(\lambda)du, \tag{53}$$

where $\phi_2(\lambda) = B^2 M_2 \lambda^2 + BM_1\lambda + M_0$, and M_2, M_1 and M_0 are defined in the proof of Lemma 6. Since $p = 3$, and by Lemma 2, we observe that

$$0 < Y < \alpha^2 \le \frac{1}{4} \ \text{ and } \ Y < \alpha^2 + u^2 = \frac{X}{Y} < \frac{1}{2} \ \text{ for } 0 < u < \alpha \le \frac{1}{2}. \tag{54}$$

Then by Lemma 2 and (54),

$$\phi_2''(\lambda) = 2B^2 M_2 = -\frac{3\left(21X^3 + 45X^2Y + 48XY^2 + 20Y^3\right)}{8}B^2 < 0 \tag{55}$$

and

$$\phi_2(0) = M_0 = \frac{3}{4}X^2Y\left(3 - \frac{X}{Y}\right) > 0 \tag{56}$$

for $0 < u < \alpha \le 1/2$ and $\lambda > 0$. We assert that

$$\phi_2(2) > 0 \ \text{ for } 0 < u < \alpha \le \frac{1}{2}. \tag{57}$$

By (55)–(57), we see that $\phi_2(\lambda) > 0$ for $0 < u < \alpha \le 1/2$ and $0 < \lambda \le 2$. Then Lemma 8(ii) holds by (53).

Next, we prove assertion (57). By Lemma 2, we compute that

$$\phi_2(2) = Y^3\bar{\phi}_2(Y, \frac{X}{Y}), \tag{58}$$

where

$$\bar{\phi}_2(s,t) = -\frac{3}{64}\left(48t + 45t^2 + 21t^3 + 20\right)(t+2)^2 s^2 - \frac{3}{8}t(t+2)\left(t^2 - 9t - 6\right)s - \frac{3}{4}t^2(t-3).$$

It is easy to see that

$$\bar{\phi}_2(0,t) > 0 \text{ and } \frac{\partial^2}{\partial s^2}\bar{\phi}_2(s,t) < 0 \text{ for } 0 < s < t \le 2. \tag{59}$$

By applying Maple soft, we observe that

$$\bar{\phi}_2(t,t) = \frac{3}{64}t^2\left(-21t^5 - 129t^4 - 320t^3 - 336t^2 - 96t + 64\right) > 0 \tag{60}$$

for $0 < t \le 1/4$, and

$$\bar{\phi}_2(\frac{1}{4},t) = \frac{3}{1024}\left(-21t^5 - 161t^4 - 344t^3 + 1144t^2 + 112t - 80\right) > 0 \tag{61}$$

for $1/4 < t \le 1/2$. By (59)–(61), we obtain

$$\bar{\phi}_2(s,t) > 0 \text{ for } 0 < s \le t \le \frac{1}{4}, \tag{62}$$

and

$$\bar{\phi}_2(s,t) > 0 \text{ for } 0 < s \le \frac{1}{4} < t \le \frac{1}{2}. \tag{63}$$

So by (54), (62) and (63), we obtain $\bar{\phi}_2(Y, \frac{X}{Y}) > 0$ for $0 < u < \alpha \le 1/2$. Then assertion (57) holds by (58). ■

Proof of Lemma 8(iii). We compute

$$\alpha T_\lambda^{(3)}(\alpha) + 2T_\lambda''(\alpha) = \frac{1}{\alpha^2}\int_0^\alpha \frac{\lambda^3}{\left[\lambda^2 B^2 + 2\lambda B\right]^{7/2}}\phi_3(\lambda)du, \tag{64}$$

where $\phi_3(\lambda) = B^2K_2\lambda^2 + BK_1\lambda + K_0$,

$$K_2 = 15A^2B - 4AB^2 - 5B^2C - B^2E - 12A^3 + 9ABC,$$

$$K_1 = 45A^2B - 16AB^2 - 20B^2C - 4B^2E - 24A^3 + 27ABC,$$

and

$$K_0 = 30A^2B - 16AB^2 - 20B^2C - 4B^2E - 15A^3 + 18ABC.$$

By Lemma 2, we compute and find that

$$\phi_3''(\lambda) = 2B^2K_2 = 2B^2\left(-\frac{49}{16}X^3 - \frac{85}{16}X^2Y - \frac{11}{2}XY^2 - \frac{9}{4}Y^3\right) < 0 \tag{65}$$

for $0 < u < \alpha$ and $\lambda > 0$. We assert that

$$\phi_3(1) > 0 \text{ and } \phi_3(2) > 0 \text{ for } 0 < u < \alpha \text{ and } \frac{1}{2} < \alpha \le 1. \tag{66}$$

By (65) and (66), $\phi_3(\lambda) > 0$ for $0 < u < \alpha$, $1/2 < \alpha \le 1$ and $1 \le \lambda \le 2$. So Lemma 8(iii) holds by (64).

Next, we prove assertion (66). By Lemma 2, we compute

$$\phi_3(1) = Y^3\bar{\phi}_3(Y, \frac{X}{Y}) \text{ and } \phi_3(2) = Y^3\tilde{\phi}_3(Y, \frac{X}{Y}), \tag{67}$$

where

$$\begin{aligned}\bar{\phi}_3(s,t) \quad = \quad & \left(\frac{t}{4}+\frac{1}{2}\right)^2\left(-\frac{49}{16}t^3 - \frac{85}{16}t^2 - \frac{11}{2}t - \frac{9}{4}\right)s^2 \\ & + \left(\frac{t}{4}+\frac{1}{2}\right)\left(\frac{5}{4}t^3 + \frac{53}{4}t^2 + 11t + 3\right)s \\ & + \frac{137}{2}t^3 + \frac{1111}{4}t^2 + \frac{1371}{4}t + \frac{275}{2}\end{aligned}$$

and

$$\begin{aligned}\tilde{\phi}_3(s,t) &= \left(\frac{t}{2}+1\right)^2\left(-\frac{49}{16}t^3-\frac{85}{16}t^2-\frac{11}{2}t-\frac{9}{4}\right)s^2\\ &+\left(\frac{t}{2}+1\right)\left(\frac{5}{4}t^3+\frac{53}{4}t^2+11t+3\right)s\\ &+\frac{137}{2}t^3+\frac{1111}{4}t^2+\frac{1371}{4}t+\frac{275}{2}.\end{aligned}$$

It is easy to see that

$$\bar{\phi}_3(0,t)>0,\ \ \tilde{\phi}_3(0,t)>0,\ \ \frac{\partial^2}{\partial s^2}\bar{\phi}_3(s,t)<0 \ \text{ and } \ \frac{\partial^2}{\partial s^2}\tilde{\phi}_3(s,t)<0 \ \text{ for } t>0. \tag{68}$$

By applying Maple soft, we find that

$$\bar{\phi}_3(1,t)=-\frac{49}{256}t^5-\frac{201}{256}t^4+70t^3+\frac{9097}{32}t^2+\frac{5553}{16}t+\frac{2215}{16}>0 \tag{69}$$

and

$$\tilde{\phi}_3(1,t)=-\frac{49}{64}t^5-\frac{241}{64}t^4+\frac{533}{8}t^3+\frac{2281}{8}t^2+\frac{695}{2}t+\frac{553}{4}>0, \tag{70}$$

for $\frac{1}{4}<t<2$. So by (68)–(70), we observe that

$$\bar{\phi}_3(s,t)>0 \ \text{ and } \ \tilde{\phi}_3(s,t)>0 \ \text{ for } 0<s<1 \text{ and } \frac{1}{4}<t<2. \tag{71}$$

In addition, since $p=3$, and by Lemma 2, we observe that

$$0<Y<\alpha^2\leq 1 \ \text{ and } \ \frac{1}{4}<\frac{X}{Y}<2 \ \text{ for } 0<u<\alpha \text{ and } \frac{1}{2}<\alpha\leq 1. \tag{72}$$

Then assertion (66) holds by (67), (71) and (72). ∎