Applied Mathematics E-Notes, 23(2023), 383-392 © ISSN 1607-2510
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Decay Rate Estimates For A Von Karman System With Infinite Memory And Distributed Delay Terms*

Mohamed Ferhat†, Tayeb Blouhi‡

Received 18 November 2022

Abstract

We consider the following nonlinear von Karman system in a bounded domain with infinite memory and distributed delay

$$u_{tt}(x,t)+\Delta^2 u(x,t)-\int_0^\infty g(\gamma)\Delta^2 u(x,t-\gamma)d\gamma+\mu_1 u_t(x,t)+\int_{\tau_1}^{\tau_2}\mu_2(s)u_t(x,t-s)ds=[u,F(u)],$$

under suitable condition on relaxation function, we obtain the decay rate of the system in using an appropriate Lyapunov functional. Our result generalizes the previous one in [1].

1 Introduction

We omit the space variable x of $u(x,t)$, $u_t(x,t)$ and for simplicity reason denote $u(x,t)=u$. We denote

$$W_0=\left\{u\in H^3(\Omega)\mid u=\Delta u=0 \ \text{ on } \partial\Omega\right\}$$

and

$$W=\left\{u\in H^4(\Omega)\mid u=\Delta u=0 \ \text{ on } \ \partial\Omega\right\}.$$

In this paper we investigate the decay properties of solutions for a von Karman equation of the form

$$\begin{cases} u_{tt}(x,t)+\Delta^2 u(x,t)-\int_0^\infty g(\gamma)\Delta^2 u(x,t-\gamma)d\gamma & \\ +\mu_1 u_t(x,t)+\int_{\tau_1}^{\tau_2}\mu_2(s)u_t(x,t-s)ds=[u,F(u)], & \text{in } \Omega\times]0,+\infty[, \\ \Delta^2 F(u)+[u,u]=0, & \text{on } \Omega\times]0,+\infty[, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & \text{in } \Omega, \\ \partial_\nu u=\partial_\nu \upsilon=0, & \text{on } \Gamma\times]0,+\infty[, \\ u=\nu=0, F(u)=\frac{\partial F(u)}{\partial\nu}=0, & \text{in } \Gamma\times]0,+\infty[\\ u_t(x,-t)=f_0(x,t), & \text{in } \Omega\times]0,\tau_2[, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $\nu=(\nu_1,\nu_2)$ is the outward unit normal vector to $\partial\Omega$, τ_1 and τ_2 are nonnegative constants with $\tau_1<\tau_2$ and $\mu_2:[\tau_1,\tau_2]\to\mathbb{R}$ is a bounded function $x=(x_1,x_2)\in\Omega$, g is a kernel function which will be specified later and Von Karman bracket is given by

$$[u,\nu]=\frac{\partial^2 u}{\partial x^2}\frac{\partial^2\nu}{\partial y^2}-2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2\nu}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\frac{\partial^2\nu}{\partial x^2},$$

*Mathematics Subject Classifications: 35B40, 35L70, 74D99.

†Departement of Mathematics, University of Science and Technology of Oran Mohamed-Boudiaf USTOMB El Mnaouar, BP 1505, Bir El Djir 31000, Oran, Algeria

‡Departement of Mathematics, University of Science and Technology of Oran Mohamed-Boudiaf USTOMB El Mnaouar, BP 1505, Bir El Djir 31000, Oran, Algeria

belonging to a suitable space.

To motivate our work, let us recall some results regarding von Karman system.

Theodore von Karman (1910) [16] started the nonlinear system of partial differential equations for great deflections and for the airy stress function of a thin elastic plate. For several years this system was studied in different situations. Using frictional dissipation at boundary, I. Lasiecka et al. [9, 10, 11] proved the uniform decay of the solution. G. P. Menzela and E. Zuazua [3] by semigroup properties gave the exponential decay when thermal damping was considered. For viscoelastic plates with memory, Rivera et al. [4, 8] proved that the energy decays uniformly, exponentially or algebraically with the same rate of decay of the relaxation function. C. A. Raposo and M. L. Santos [2] gave a general decay of solution for the finite memory case. Though the presence of rotational inertia $-\Delta u_{tt}$ is quite legitimate from the physical point of view, it gives the amount of regularity necessary to compute via a suitable Lyapunov functional. Recently, Cavalcanti et al. [12] considered problem (1) under the condition $g'(t) \leq -H(g(t))$, where $H(s)$ is a given continuous, positive, increasing and convex function such that $H(0) = 0$. The feature of the work [12] is to provide wellposedness of both weak and regular solutions, and sharp and general decay rate estimates without accounting for regulazing effects of rotational inertia by pursuing the strategy introduced in [7, 5, 6].

On the other hand, Fabrizio and Polidoro [13] obtained exponential decay rates of solutions to a linear viscoelastic wave equation under the condition $g'(t) \leq 0$ and $e^{\alpha t}g(t) \in L^1(0,\infty)$ for some $\alpha > 0$. Our method of proof uses some ideas developed in [17] for the wave equation with delay and some estimates of the viscoelastic wave equation, enabling us to obtain suitable Lyapunov functionals, from which are derived the desired results. We recall that for $\mu_1 = \mu_2$, Nicaise and Pignotti showed in [17] that some instabilities may occur due to the presence of the viscoelastic damping.

Motivated by the previous works, it is interesting to show more general decay result to that in [15, 1], we analyze the influence of the viscoelastic and distributed delay terms on the solutions to (1). Under suitable assumptions on functions $g(.)$, the initial data and the parameters in the equations. The content of this paper is organized as follows. In Section 2, we provide assumptions that will be used later, In Section 3, The decay result is given in the last section by exploiting the perturbed Lyapunov functionals.

2 Preliminary Results

Let us introduce the following new variable

$$z(x,\rho,s,t) = u_t(x,t-\rho s), \quad (x,\rho,s,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty). \tag{2}$$

Then, system (1) is equivalent to

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^\infty g(\gamma)\Delta^2 u(t-\gamma)d\gamma & \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)ds = [u, F(u)], & \text{in } \Omega\times]0,+\infty[, \\ \Delta^2 F(u) + [u,u] = 0, & \text{on } \Omega\times]0,+\infty[, \\ s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2)\times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu \upsilon = 0, & \text{on } \Gamma\times]0,+\infty[, \\ u = \nu = 0, F(u) = \frac{\partial F(u)}{\partial \nu} = 0, & \text{in } \Gamma\times]0,+\infty[, \\ z(x,\rho,s,0) = f_0(x,\rho s), & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2). \end{cases} \tag{3}$$

In this section, we present some material for the proof of our result. We denote $(u,\upsilon) = \int_\Omega u(x)\upsilon(x)dx$. For a Hilbert space E, we denote $(u,\upsilon)_E$ and $\|.\|_E$ the inner product and norm of E, respectively. For simplicity, we denote $\|.\|_{L^2(\Omega)}$ by $\|.\|$. Let λ_0 and λ be the smallest positive constants such that

$$\lambda_0\|u\|^2 \leq \|\nabla u\|^2 \quad \text{and} \quad \lambda\|u\|^2 \leq \|\Delta u\|^2 \quad \text{for } u \in H_0^2(\Omega). \tag{4}$$

Now we introduce some results that can be found in [6, 13, 14, 15].

Lemma 1 *If $u \in H^2(\Omega)$, then $\|F(u)\|_{W^{2,\infty}(\Omega)} \le c\|u\|^2_{H^2(\Omega)}$.*

Lemma 2 *If u, ϕ and ψ belong in $H^2(\Omega)$ and at least one of them belongs to $H^2_0(\Omega)$, then $([u,\phi],\psi) = ([u,\psi],\phi)$.*

Lemma 3 *If $u \in H^2(\Omega)$ and $\phi \in W^{2,\infty}(\Omega)$, then $\|[u,\phi]\| \le c\|u\|_{H^2(\Omega)}\|\phi\|_{W^{2,\infty}(\Omega)}$.*

Now, we have some assumptions. For the relaxation function g, we assume

(A$_0$) $g : R_+ \to R_+$ is a C^1 function satisfying

$$g(0) > 0,\ 1 - \int_0^t g(\gamma)d\gamma = \ell \ \text{ for } t > 0, \tag{5}$$

and there exists a nonincreasing differentiable function $\sigma : R_+ \to R_+$ with

$$\sigma(t) > 0,\ g'(t) \le -\sigma(t)g(t) \ \text{ for } t > 0. \tag{6}$$

(A$_1$) There exists $m_0 > 0$ such that

$$\|\Delta u_0(.,s)\|_2 \le m_0 \quad \forall s \ge 0. \tag{7}$$

By combining the arguments of [6, 13, 14], we recall the existence result (see [6]).

3 Energy Decay Result

In this section, we study the asymptotic behavior of the solutions to the system (3). We define the energy associated with (3) by

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{4}\|\Delta F(u(t))\|_2^2, \tag{8}$$

and a modified energy by

$$\begin{aligned}\varepsilon(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^\infty g(\gamma)d\gamma\right)\|\Delta u(t)\|_2^2 + \frac{1}{4}\|\Delta F(u(t))\|_2^2 + \frac{1}{2}(g \diamond \Delta u)(t)\\ &+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(|\mu_2(s)| + \xi)z^2(x,\rho,s,t)dsd\rho dx,\end{aligned} \tag{9}$$

where ξ is a positive constant satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds + \frac{\xi(\tau_2 - \tau_1)}{2} < \mu_1, \tag{10}$$

and

$$(g \diamond \Delta u))(t) = \int_0^\infty g(\gamma)d\gamma\|\Delta u(t) - \Delta u(t-\gamma)\|_2^2 d\gamma.$$

Theorem 1 *Assume that (A$_0$)–(A$_1$) and (10) hold. Then, for every $(u_0, u_1, f_0) \in W \times (H^2(\Omega) \cap H^2_0(\Omega)) \times W^{1,2}(\Omega \times (0,1) \times (\tau_1,\tau_2))$, Problem (3) admits a unique solution u in the class*

$$u \in L^\infty(0,T;W_0), \quad u_t \in L^\infty(0,T;H^2(\Omega) \cap H^1_0(\Omega)), \quad u \in L^\infty(0,T;L^2(\Omega)).$$

Lemma 4 *Suppose that (A$_0$)–(A$_1$) hold. Let (u,z) be the solution of the system (3). Then the modified energy functional satisfies*

$$\begin{aligned}\frac{d\varepsilon(t)}{dt} &= -\left[\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds - \frac{\xi(\tau_2-\tau_1)}{2}\right]\int_\Omega u_t^2 dx + \frac{1}{2}(g' \diamond \Delta u)(t)\\ &\quad -m\int_\Omega\int_{\tau_1}^{\tau_2} z^2(x,1,s,t)dsdx \le 0.\end{aligned} \tag{11}$$

Proof. Multiplying the first equation in (3) by u_t, and the second by $(\mu_2(s)+\xi)z$ and integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$ using integration by parts, hypotheses (A_0) and (A_1), we obtain

$$\begin{aligned}\frac{1}{2}\left[\|u_t(t)\|_2^2+\|\Delta u(t)\|_2^2\right] &= \int_\Omega\int_0^\infty g(\gamma)(\Delta u(t-\gamma)-\Delta u(t),\Delta u_t(t))d\gamma+([u(t),F(u(t))],u_t(t))\\ &\quad -\int_\Omega u_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)dsdx-\mu_1\|u_t\|_2^2+(1-\ell)\int_\Omega \Delta u(t)\Delta u_t(t)ds\\ &\quad -\int_\Omega u_t\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_0^1 z(x,\rho,s,t)z_\rho(x,\rho,s,t)d\rho dsdx\\ &= \int_\Omega\int_0^\infty g(\gamma)(\Delta u(t-\gamma)-\Delta u(t),\Delta u_t(t))d\gamma+([u(t),F(u(t))],u_t(t))\\ &\quad -\int_\Omega u_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)dsdx-\mu_1\|u_t\|_2^2+(1-\ell)\int_\Omega \Delta u(t)\Delta u_t(t)ds\\ &\quad +\frac{1}{2}\left[\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds+\xi(\tau_2-\tau_1)\right]\int_\Omega u_t^2dx\\ &\quad -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)z^2(x,1,s,t)dsdx, \end{aligned}\tag{12}$$

from Lemma 2, we have

$$\begin{aligned}([u(t),F(u(t))],u_t(t)) &= ([u(t),u_t(t)],)F(u(t))=-\left(\frac{1}{2}\frac{d}{dt}[u(t),u(t)],F(u(t))\right)\\ &= \frac{1}{4}\frac{d}{dt}\|\Delta F(u(t)\|_2^2,\end{aligned}\tag{13}$$

for the first term on the left side of (12), we get

$$\begin{aligned}\int_\Omega \Delta u_t(t)\int_0^{+\infty}g(\gamma)(\Delta u(t-\gamma)-\Delta u(t)d\gamma dx &= -\frac{1}{2}\int_\Omega\int_0^{+\infty}g(\gamma)\frac{\partial}{\partial\gamma}|\Delta u(t-\gamma)-\Delta u(t)|^2d\gamma dx\\ &\quad -\frac{1}{2}\int_\Omega\int_0^{+\infty}g(\gamma)\frac{\partial}{\partial t}|\Delta u(t-\gamma)-\Delta u(t)|^2d\gamma dx.\end{aligned}$$

Using integration by parts, we get

$$\int_\Omega \Delta u_t(t)\int_0^{+\infty}g(\gamma)(\Delta u(t-\gamma)-\Delta u(t))d\gamma dx=\frac{1}{2}(g'\diamond\Delta u)(t)-\frac{1}{2}\frac{d}{dt}(g\diamond\Delta u)(t).\tag{14}$$

Inserting (13)–(14) into (12), we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\left(1-\int_0^\infty g(\gamma)d\gamma\right)\|\Delta u(t)\|_2^2+\frac{1}{4}\|\Delta F(u(t))\|_2^2+\frac{1}{2}(g\diamond\Delta u))(t)\right]\\ &+\frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s(|\mu_2(s)|+\xi)z^2(x,\rho,s,t)dsd\rho dx\\ =\;& -\int_\Omega u_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)dsdx+\frac{1}{2}\left[\int_{\tau_1}^{\tau_2}|\mu_2|ds+\xi(\tau_2-\tau_1)\right]\int_\Omega u_t^2dx\\ &-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)z^2(x,1,s,t)dsdx+\frac{1}{2}(g'\diamond\Delta u)(t)-\mu_1\|u_t\|_2^2.\end{aligned}\tag{15}$$

Using Young's inequality, we obtain

$$\begin{aligned}\varepsilon'(t) &\leq -\left[\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds-\frac{\xi(\tau_2-\tau_1)}{2}\right]\int_\Omega u_t^2dx-\frac{\xi}{2}\int_\Omega\int_{\tau_1}^{\tau_2}z^2(x,1,s,t)dsdx\\ &\quad+\frac{1}{2}(g'\diamond\Delta u)(t).\end{aligned}$$

Hence, (11) is established. ■

Lemma 5 *Under the assumptions (A_0)–(A_1), the functional*

$$\psi(t) := \int_\Omega u_t u dx \tag{16}$$

satisfies, along the solution of system (3), the estimate

$$\begin{aligned}\psi'(t) \quad &\leq \quad \|u_t(t)\|_2^2 - \frac{1}{2}\left(1-(1+\mu_1)(1-\ell)^2\right)\|\Delta u(t)\|_2^2 \\ &+\frac{1}{2}\left(1+\frac{1}{\mu_1}\right)(1-\ell)(g\diamond\Delta u)(t) - \|\Delta F(u(t))\|_2^2.\end{aligned}$$

Proof. Direct differentiation of (16) yields

$$\psi'(t) = \|u_t(t)\|_2^2 - \|\Delta u(t)\|_2^2 + \int_\Omega \Delta u.\left(\int_0^{+\infty} g(\gamma).\Delta u(t-\gamma)d\gamma\right)dx - \|\Delta F(u(t))\|_2^2. \tag{17}$$

Using Cauchy-Schwarz and Young's inequalities, we obtain that, for all μ_1,

$$\begin{aligned}&\int_\Omega \Delta u.\left(\int_0^{+\infty} g(\gamma).\Delta u(t-\gamma)d\gamma\right)dx \\ \leq \quad &\frac{1}{2}\left\{\|\Delta u(t)\|_2^2 + \left(1+\frac{1}{\mu_1}\right)(1-\ell)(g\diamond\Delta u)(t) + (1+\mu_1)(1-\ell)^2\|\Delta u(t)\|_2^2\right\}.\end{aligned} \tag{18}$$

Inserting (18) into (17), we arrive at

$$\begin{aligned}\psi'(t) \quad &\leq \quad \|u_t(t)\|_2^2 - \frac{1}{2}\left(1-(1+\mu_1)(1-\ell)^2\right)\|\Delta u(t)\|_2^2 + \frac{1}{2}\left(1+\frac{1}{\mu_1}\right)(1-\ell)(g\diamond\Delta u)(t) \\ &-\|\Delta F(u(t))\|_2^2.\end{aligned}$$

■

Lemma 6 *Assume that (A_0)–(A_1) hold. Then the functional*

$$\chi(t) := -\int_\Omega u_t \int_0^{+\infty} g(\gamma)(u(t)-u(t-\gamma))d\gamma dx, \tag{19}$$

satisfies, along the solution of system (3) and for all $\alpha > 0$, the estimate

$$\begin{aligned}\chi'(t) \quad &\leq \quad (\delta_2 + 2\delta_2(1-l)^2 + \delta_2 c(E(0))^2)\|\Delta u(t)\|_2^2 - ((1-\ell)-\delta_2)\|u_t\|_2^2 \\ &-\frac{g(0)}{4\delta}C_*^2(g'\diamond\Delta u)(t) + \left\{(2\delta+\frac{1}{4\delta})(1-\ell) + \frac{(1-\ell)}{4\delta} + \frac{\ell}{4\lambda\delta_2}\right\}(g\diamond\Delta u)(t).\end{aligned}$$

Proof. Differentiate (19) and use the first equation in system (3) to get

$$\begin{aligned}\chi'(t) \quad = \quad &\int_\Omega \Delta u(t).\left(\int_0^{+\infty} g(\gamma)(\Delta u(t)-\Delta u(t-\gamma))d\gamma\right)dx \\ &-\int_\Omega \left(\int_0^{+\infty} g(\gamma)\Delta u(t-\gamma)d\gamma\right)\left(\int_0^{+\infty} g(\gamma)(\Delta u(t)-\Delta u(t-\gamma))d\gamma\right) \\ &-\int_\Omega u_t \int_0^{+\infty} g'(\gamma)(u(t)-u(t-\gamma))d\gamma dx - (1-\ell)\|u_t\|_2^2 \\ &-\int_\Omega \int_0^{+\infty} g(\gamma)([u(t),F(u(t))],u(t)-u(t-\gamma))d\gamma dx \\ &+\int_\Omega \left(\int {}_{\tau_1}^{\tau_2}\mu_2(s)u_t(x,t-s)ds\right)\left(\int_0^{+\infty} g(\gamma),u(t)-u(t-\gamma)d\gamma\right)dx\end{aligned} \tag{20}$$

we estimate the right-hand side terms of (20) as follows. By using Young's and Cauchy-Schwarz inequalities, we obtain $\forall \delta > 0$

$$\int_\Omega \Delta u(t). \left(\int_0^{+\infty} g(\gamma)(\Delta u(t) - \Delta u(t-\gamma))d\gamma\right) dx \leq \delta \|\Delta u\|_2^2 + \frac{1-\ell}{4\delta}(g \diamond \Delta u)(t), \tag{21}$$

and for the second term can be estimated as follows

$$\begin{aligned} &\int_\Omega \left(\int_0^{+\infty} g(\gamma)\Delta u(t-\gamma)d\gamma\right)\left(\int_0^{+\infty} g(\gamma)(\Delta u(t) - \Delta u(t-\gamma))d\gamma\right) \\ \leq\ & \left(2\delta + \frac{1}{4\delta}\right)(1-\ell)(g \diamond \Delta u)(t) + 2\delta(1-\ell)^2\|\Delta u\|_2^2. \end{aligned} \tag{22}$$

By exploiting Young's, Poincaré's and Cauchy-Schwarz' inequalities to get

$$\int_\Omega u_t \int_0^{+\infty} g'(\gamma)(u(t) - u(t-\gamma))d\gamma dx \leq \delta\|u_t\|_2^2 - \frac{g(0)}{4\delta} C_\star (g' \diamond \Delta u)(t), \tag{23}$$

and for any $\delta_1 > 0$, we get

$$\begin{aligned} &\int_\Omega \int_0^{+\infty} g(t-\gamma)([u(t), F(u(t))], u(t) - u(\gamma))d\gamma dx \\ \leq\ & \delta_1 \|[u(t), F(u(t))]\|_2^2 + \frac{\ell}{4\lambda\delta_1}(g \diamond \Delta u)(t), \end{aligned} \tag{24}$$

respectively. From Lemmas 2 and 3, and the fact $\varepsilon(t) \leq \varepsilon(0) = E(0)$, we observe that

$$\begin{aligned} \|[u(t), F(u(t))]\|_2^2 &\leq c\|u(t)\|_{H^2(\Omega)}^2 \|u(t)\|_{H^2(\Omega)}^4 \\ &\leq c\|\Delta u(t)\|^2 (4E(t))^2 \\ &\leq c\|\Delta u(t)\|^2 \left(\frac{4}{1-\ell}\varepsilon(t)\right)^2 \\ &\leq c\|\Delta u(t)\|^2 \left(\frac{4}{1-\ell}E(0)\right)^2. \end{aligned} \tag{25}$$

Applying this to (24) , we obtain

$$\int_\Omega \int_0^{+\infty} g(t-\gamma)([u(t), F(u(t))], u(t) - u(\gamma))d\gamma dx \leq \delta_2 C E(0)^2 \|\Delta u(t)\|^2 + \frac{\ell}{4\delta_2\lambda}(g \diamond \Delta u)(t), \tag{26}$$

fifth term in (20) can be estimated as

$$\begin{aligned} &\left|\int_\Omega \left(\int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x, t-s)ds\right)\left(\int_0^{+\infty} g(\gamma)(u(t) - u(t-\gamma))d\gamma\right) dx\right| \\ \leq\ & \frac{1}{\delta}\int_\Omega \left(\int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x, t-s)ds\right)^2 + \delta\left(\int_0^{+\infty} g(\gamma)(u(t) - u(t-\gamma))d\gamma\right)^2 dx \\ \leq\ & \frac{1}{\delta}\underbrace{\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds}_{<\mu_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx + \frac{\delta}{\lambda}(g \diamond \Delta u)(t). \end{aligned} \tag{27}$$

substituting these estimates into (20), we find that

$$\begin{aligned} \chi'(t) \leq\ & (\delta_2 + 2\delta_2(1-l)^2 + \delta_2 c(E(0))^2)\|\Delta u(t)\|_2^2 - ((1-\ell) - \delta_2)\|u_t\|_2^2 - \frac{g(0)}{4\delta} C_*^2 (g' \diamond \Delta u)(t) \\ &+ \left\{(2\delta + \frac{1}{4\delta})(1-\ell) + \frac{(1-\ell)}{4\delta} + \frac{\ell}{4\lambda\delta_2} + \frac{\delta}{\lambda}\right\}(g \diamond \Delta u)(t) + \frac{k_1}{\delta}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx. \end{aligned}$$

This completes the proof. ∎

Lemma 7 *The functional*

$$\varphi_1(t) := \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}(|k(s)| + \xi) z^2(x, \rho, s, t) ds d\rho dx, \tag{28}$$

satisfies, along the solution of system (3) and for all $\gamma_0 > 0$, the estimate

$$\varphi_1'(t) \leq c \int_\Omega u_t^2 dx - \gamma_0 \int_\Omega \int_{\tau_1}^{\tau_2} s(|\mu_2(s)| + \xi) z^2(x, q, \rho, t) ds d\rho dx.$$

Proof. After differentiating (28) , we use the third equation of (3) to find that

$$\begin{aligned}
\varphi_1'(t) &= -2 \int_\Omega \int_{\tau_1}^{\tau_2} (|\mu_2(s)| + \xi) \int_0^1 e^{-s\rho} z z_\rho d\rho ds dx \\
&= -\int_\Omega \int_{\tau_1}^{\tau_2} (|\mu_2(s)| + \xi) \int_0^1 e^{-s\rho} z z^2 d\rho ds dx \\
&\quad - \int_\Omega \int_{\tau_1}^{\tau_2} (|\mu_2(s)| + \xi) \left[e^{-s} z^2(x, 1, s, t) - z^2(x, 0, s, t) + s \int_0^1 e^{-s\rho} z^2 d\rho \right] ds dx \\
&\leq c \int_\Omega u_t^2 dx - \gamma_0 \int_\Omega \int_{\tau_1}^{\tau_2} s(|\mu_2(s)| + \xi) z^2 ds d\rho dx.
\end{aligned}$$

■

Now let us define the perturbed functional by

$$F(t) = \gamma\varepsilon(t) + \epsilon_1\psi(t) + \epsilon_2\chi(t) + \epsilon_1\varphi_1(t). \tag{29}$$

Lemma 8 *Assume that (A_0)–(A_1) hold. Then, for $M > 0$ large enough there exist positive constants α_1 and α_1 such that*

$$\alpha_1\varepsilon(t) \leq F(t) \leq \alpha_1\varepsilon(t). \tag{30}$$

Proof. Young's inequality, Holer's inequality and (5) give that

$$\begin{aligned}
|\psi(t)| &\leq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2\lambda}\|\Delta u(t)\|_2^2 \\
&\leq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2\lambda(1-l)} \left(1 - \int_0^\infty g(\gamma) d\gamma\right) \|\Delta u(t)\|_2^2 \\
&\leq C_1\varepsilon(t),
\end{aligned}$$

and from (6), we get

$$\begin{aligned}
|\chi(t)| &\leq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(\int_0^\infty g(\gamma)\|u(t-\gamma) - u(t)\| d\gamma\right)^2 \\
&\leq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2\lambda}(g \diamond \Delta u)(t) \\
&\leq C_2\varepsilon(t).
\end{aligned}$$

Also we have

$$\begin{aligned}
|\varphi_1(t)| &\leq c \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(|\mu_2(s)| + \xi) z^2(x, \rho, s, t) ds d\rho dx \\
&\leq C_3\varepsilon(t).
\end{aligned}$$

Consequently, $|F(t) - \gamma\varepsilon(t)| \leq c\varepsilon(t)$, which yields

$$(\gamma - c)\varepsilon(t) \leq F(t) \leq (\gamma + c)\varepsilon(t).$$

Choosing γ large enough, we obtain estimate . ■

Lemma 9 *Assume that (A_0)–(A_1) hold. Then there exist positive constants such that the functional (3) satisfies, for all $t \in R^+$,*

$$F'(t) \leq -\alpha\varepsilon(t) + \beta(g \diamond \Delta u)(t). \tag{31}$$

Proof. After differentiating (29) and using Lemmas 1–4, we get

$$\begin{aligned} F'(t) \quad \leq \quad & -\left\{\epsilon_2\left\{(1-\ell)-\delta_2\right\}-\epsilon_1(1-c)\right\}\|u_t\|_2^2 + \left\{\frac{\gamma}{2} - \epsilon_2\frac{g(0)}{4\delta_2}C_*^2\right\}(g' \diamond \Delta u)(t) \\ & -\left\{\frac{\epsilon_1}{2}(1-(1+\mu_1)(1-\ell)^2) - \epsilon_2\left\{\delta_2 + 2\delta_2(1-\ell)^2 + \delta_2 c(E(0))^2\right\}\right\}\|\Delta u\|_2^2 \\ & +\left\{\frac{\epsilon_1}{2}\left(1+\frac{1}{\mu_1}\right)(1-\ell) + \epsilon_2\left(2\delta_2 + \frac{1}{4\delta}\right)(1-\ell) + \frac{(1-\ell)}{4\delta_2} + \frac{1}{4\lambda\delta_2}\right\}(g \diamond \Delta u)(t) \\ & -\epsilon_1\|F(u(t))\|_2^2 - \epsilon_1\gamma_0\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(|\mu_2(s)|+\xi)z^2(x,\rho,s,t)\,ds\,d\rho\,dx. \end{aligned} \tag{32}$$

At this point we choose our constants carefully

$$\delta < (1-\ell), \tag{33}$$

and by choosing two positive constants ϵ_1 and ϵ_1 satisfying

$$\frac{\epsilon_2\left\{\delta_2 + 2\delta_2(1-\ell)^2 + \delta_2 c(E(0))^2\right\}}{(1-(1+\mu_1)(1-\ell)^2)} < \epsilon_1 < \epsilon_2(1-\ell) - \delta_2, \tag{34}$$

the above inequalities produces

$$\beta_1 = \left\{\epsilon_2\left\{(1-\ell)-\delta_2\right\} - \epsilon_1(1-c)\right\} > 0,$$

$$\beta_2 = \left\{\frac{\epsilon_1}{2}(1-(1+\mu_1)(1-\ell)^2) - \epsilon_2\left\{\delta_2 + 2\delta_2(1-\ell)^2 + \delta_2 c(E(0))^2\right\}\right\} > 0,$$

and for ϵ_2 small enough we have

$$\beta_3 = \left\{\frac{\gamma}{2} - \epsilon_2\frac{g(0)}{4\delta_2}C_*^2\right\} > 0.$$

This yields

$$\begin{aligned} F'(t) \quad \leq \quad & -\beta_1\|u_t\|_2^2 - \beta_2\|\Delta u\|_2^2 + \beta(g \diamond \Delta u)(t) - \epsilon_1\|F(u(t))\|_2^2 \\ & -\epsilon_1\gamma_0\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(|\mu_2(s)|+\xi)z^2(x,\rho,s,t)\,ds\,d\rho\,dx. \end{aligned}$$

We put $\alpha = \max\{\beta_1, \beta_2, \epsilon_1\gamma_0, \epsilon_1\}$ Therefore, there exist two positive constants α and β such that

$$F'(t) \leq -\alpha\varepsilon(t) + \beta(g \diamond \Delta u)(t),$$

which completes the proof. ■

Theorem 2 *Let $(u_0, u_1, f_0) \in (H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega) \times L^2((0,1) \times \Omega)$ be given. Assume that (A_0)–(A_1) hold. Then, for each $t_0 > 0$, there exist strictly positive constant K such that the solution of (3) satisfies*

$$\varepsilon(t) \leq K e^{-\alpha\int_{t_0}^t \sigma(s)ds} \quad \text{for } t \geq t_0. \tag{35}$$

Proof. We have from Lemma 9,

$$\frac{dF(t)}{dt} \leq -\alpha\varepsilon(t) + \beta(g \diamond \Delta u)(t), \tag{36}$$

using Lemma 4, we have

$$\begin{aligned}\frac{dF(t)}{dt} &\leq -\rho_1\varepsilon(t)+\rho_2(go\Delta u)(t)\leq -\rho_1\varepsilon(t)-\rho_2(g'o\Delta u)(t)\\ &\leq -\rho_1\varepsilon(t)-2\rho_2\varepsilon'(t), \end{aligned} \tag{37}$$

since $\sigma(t)$ is nonincreasing, multiplying the last line in (37) by $\sigma(t)$ to get

$$\sigma(t)\frac{dF(t)}{dt}\leq -\rho_1\sigma(t)\varepsilon(t)-2\rho_2\sigma(t)\varepsilon'(t). \tag{38}$$

Let

$$H(t)=\sigma(t)F(t)+2\lambda\varepsilon(t).$$

We can easily see that $H(t)$ is equivalent to $\varepsilon(t)$. Now, subtracting and adding $\sigma^{'}(t)F(t)$ in the right hand side of (38), using the fact that $\sigma'(t)\leq 0$ and (A_1) $\forall t\geq 0$, then

$$\frac{dH(t)}{dt}\leq \sigma^{'}(t)F(t)-\rho_1\sigma(t)\varepsilon(t)\leq -\rho_1\sigma(t)\varepsilon(t)\leq -\rho_3\sigma(t)H(t),\quad t\geq t_0.$$

Integrating this over (t_0,t), we conclude that

$$H(t)\leq H(t_0)e^{-\rho_3\int_{t_0}^{t}\sigma(s)ds}\quad \text{for } t\geq t_0,$$

where ρ_3 is a positive constant. Finally, we get

$$\varepsilon(t)\leq \varepsilon(t_0)e^{-c\int_{t_0}^{t}\sigma(s)ds}\quad \text{for } t\geq t_0.$$

This completes the proof. ■

Acknowledgments. The author would like to thank very much the anonymous referees for their careful reading and valuable comments on this work.

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