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A Class Of Orthogonal Polynomials Associated With The Legendre Polynomial*

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Abstract

It is known that the function $(1-2xt+t^2)^{-1/2}$ arose from the (electric or gravitational) potential theory. The series expansion of this function in powers of t generates the coefficients which are well known as the Legendre polynomials. These polynomials are orthogonal over $(-1,1)$ with respect to the weight function unity. The present work incorporates the class $\{P_n(x;M) : n \in \mathbb{N},\ M \in 2\mathbb{N}\}$ of orthogonal polynomials associated with the Legendre polynomial to which it would reduce when $M=2$. It is shown that the polynomial $\{P_n(x;M)\}$ is a solution of a generalized differential equation. Following this, it is shown that this class forms an orthogonal set with respect to the weight function x^{M-2} over the interval $(-1,1)$. Among the other properties derived include the Rodrigues formula, generating function relations and zeros. The graphs of $\{P_n(x;M)\}$ are plotted using MATLAB program, for the even and odd degree cases.

1 Introduction

It is known that the Legendre polynomials arise as the coefficients in the power series expansion of electric or gravitational potential function. If we consider an electric charge q placed on the x-axis at $x=a$, $a<r$ (Figure 1), then at the point A, the electrostatic potential V due to the charge q is given by $V \propto q/d$, where d is the length of the segment shown in the Figure 1. From this, we have $V = kq/d$, k is constant of proportionality. Since $a/r<1$, using cosine rule, we have [7, Ch. 11, p. 552–553]

$$\begin{aligned} V &= \frac{kq}{\sqrt{r^2+a^2-2ar\cos\theta}} = kq\left(r^2+a^2-2ar\cos\theta\right)^{-1/2} \\ &= \frac{kq}{r}\left(1+\frac{a^2}{r^2}-2\left(\frac{a}{r}\right)\cos\theta\right)^{-1/2}. \end{aligned}$$

If $a/r=t,\ \cos\theta = x$, then $t<1$, and the function rV/kq assumes the elegant form $(1-2xt+t^2)^{-1/2} = F(x,t)$, say. The function $F(x,t)$ when expanded in power series in powers of t, generates the coefficients which are nothing but the Legendre polynomials $P_n(x)$. Thus with $|t|<1$,

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

For the case $a>r$, see [7, Ch. 11, Ex. 11.1.3, p. 561] (also for Linear electric Multipoles and associated Legendre polynomials see [7, Ch. 11, p. 558]). Among many other physical phenomena, the Legerdre polynomials are also associated with one dimensional steady-state transport equation and neutron scattering functions for one-energy group (see [2] for the detailed account).

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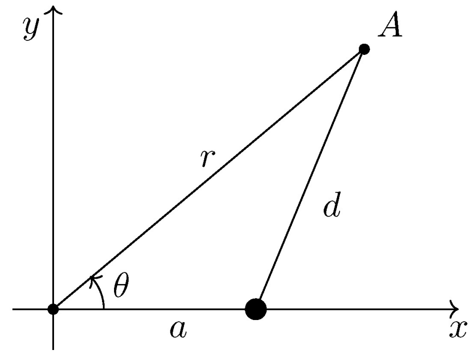

Figure 1: Electrostatic potential due to charge q.

The explicit representation of this polynomial is given by [4, p.157]

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k}}{(n-2k)!\ k!}\ (2x)^{n-2k}.$$

It satisfies the equation ([3, 4, 5, 6, 7]):

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0. \tag{1}$$

In the present work, we propose the class $\{P_n(x;M)\}$ of even and odd degree polynomials, as follows. For $M \in 2\mathbb{N}$,

$$P_{Mr}(x;M) = \sum_{k=0}^{r} \frac{(-1)^k \left(1-\frac{1}{M}\right)_{2r-k}}{k!\ (r-k)!\left(1-\frac{1}{M}\right)_{r-k}}\ x^{M(r-k)} = \sum_{k=0}^{r} \frac{(-1)^{r-k} \left(1-\frac{1}{M}\right)_{r+k}}{k!\ (r-k)!\left(1-\frac{1}{M}\right)_{k}}\ x^{Mk}, \tag{2}$$

and

$$P_{Mr+1}(x;M) = \sum_{k=0}^{r} \frac{(-1)^k \left(1+\frac{1}{M}\right)_{2r-k}}{k!\ (r-k)!\left(1+\frac{1}{M}\right)_{r-k}}\ x^{M(r-k)+1} = \sum_{k=0}^{r} \frac{(-1)^{r-k} \left(1+\frac{1}{M}\right)_{r+k}}{k!\ (r-k)!\left(1+\frac{1}{M}\right)_{k}}\ x^{Mk+1}. \tag{3}$$

Our objective is to derive certain properties of these polynomials; namely the orthogonality, Rodrigues formula, generating function relation and zeros.

Note 1. We notice that $P_{2r}(x;2) = P_{2r}(x)$ when $n = 2r$ whereas $P_{2r+1}(x;2) = P_{2r+1}(x)$ when $n = 2r+1$.

In what follows, we shall use the following notations and definitions ([1, 4]). The generalized factorial notation:

$$(a)_n = \begin{cases} a(a+1)(a+2)\cdots(a+n-1), & \text{if } n \in \mathbb{N}, \\ 1 & \text{if } n = 0. \end{cases}$$

The Gauss hypergeometric function is denoted and defined by ([1, 4])

$${}_2F_1\left[\begin{array}{ccc} a, & b; & z \\ & c; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}\ z^n,$$

where $c \neq 0, -1, -2, \ldots,$ and $|z| < 1$. If either $a \in \mathbb{Z}_{\leq 0}$ or $b \in \mathbb{Z}_{\leq 0}$ or both $a, b \in \mathbb{Z}_{\leq 0}$, then this function will represent a polynomial in z.

2 M-Legendre Polynomial

We first show that the polynomials in (2) and (3) are solutions of the equation (cf. (1) for $M = 2$):

$$(1 - x^M)y'' - Mx^{M-1}y' + n(n + M - 1)x^{M-2}y = 0. \tag{4}$$

We follow the method described in [5, Theorem A, p.180] of obtaining the power series solution of the second order homogeneous ordinary linear differential equation. Here $x = 0$ is an ordinary point, hence assuming the power series solution $y(x) = \sum_{p=0}^{\infty} a_p x^p$ along with its derivatives:

$$y'(x) = \sum_{p=1}^{\infty} pa_p x^{p-1} \text{ and } y''(x) = \sum_{p=2}^{\infty} p(p-1)a_p x^{p-2},$$

we are led to

$$a_2 = a_3 = \cdots a_{M-1} = 0, \qquad a_M = -\frac{n(n+M-1)}{M(M-1)}a_0, \quad a_{M+1} = -\frac{n(n+M-1) - M}{M(M+1)}a_1$$

and the recursion formula (cf. [5, eq. (9), p. 179] with $M = 2$):

$$a_p = -\frac{n(n+M-1) - (p-M)(p-1)}{p(p-1)}\, a_{p-M}. \tag{5}$$

From the equation (5) and the values of $a_i\ 's$ $(i = 2, 3, \cdots, M-1)$, for $k \in \mathbb{N}$ we have

$$a_{Mk+2} = a_{Mk+3} = \cdots a_{M(k+1)-1} = 0. \tag{6}$$

If we put $p = 2M,\ 3M,\ ...$ successively in (5), then for $k \in \mathbb{N}$, we get

$$a_{Mk} = \frac{(-1)^k}{k!M^k}\left\{\prod_{s=1}^{k} \frac{n(n+M-1) - M(s-1)(Ms-1)}{Ms-1}\right\}a_0 \tag{7}$$

and similarly, putting $p = 2M+1,\ 3M+1,\ ...$ in (5) successively, we get

$$a_{Mk+1} = \frac{(-1)^k}{k!M^k}\left\{\prod_{s=1}^{k} \frac{n(n+M-1) - Ms(M(s-1)+1)}{Ms+1}\right\}a_1. \tag{8}$$

Thus, from (6), (7) and (8), the series solution occurs in the form:

$$y(x) = \left[a_0 + \sum_{k=1}^{\infty} a_{Mk}\ x^{Mk}\right] + \left[a_1 x + \sum_{k=1}^{\infty} a_{Mk+1}\ x^{Mk+1}\right]. \tag{9}$$

These are linearly independent solutions of the differential equation (4) since neither series is a constant multiple of the other [5, p. 178].

When n is not an integer, both series have radii of convergence $R = 1$. If $n = Mr$, $r \in \{0\} \cup \mathbb{N}$, then the first series in (9) terminates and for $n = Mr + 1$, the second series terminates. With

$$a_0 = \frac{(-1)^r}{r!}\left(1 - \frac{1}{M}\right) \quad \text{and} \quad a_1 = \frac{(-1)^r}{r!}\left(1 + \frac{1}{M}\right),$$

we are led to the particular solution of (4) which are the proposed polynomials stated in (2) and (3).

3 Hypergeometric Function Forms

From first series in (2), we have

$$\begin{aligned} P_{Mr}(x;M) &= \sum_{k=0}^{r} \frac{(-1)^k \left(1-\frac{1}{M}\right)_{2r} \left(\frac{1}{M}-r\right)_k \ (-r)_k}{k!\ r!\ \left(\frac{1}{M}-2r\right)_k \left(1-\frac{1}{M}\right)_r} x^{M(r-k)} \\ &= \frac{\left(1-\frac{1}{M}\right)_{2r}\ x^{Mr}}{r!\ \left(1-\frac{1}{M}\right)_r}\ {}_2F_1\left[\begin{array}{ccc} -r, & \frac{1}{M}-r; & x^{-M} \\ & \frac{1}{M}-2r; & \end{array}\right] \end{aligned}$$

and from the second series in (2), we have

$$\begin{aligned} P_{Mr}(x;M) &= \frac{(-1)^r \left(1-\frac{1}{M}\right)_r}{r!} \sum_{k=0}^{r} \frac{(-r)_k \left(r+1-\frac{1}{M}\right)_k}{k! \left(1-\frac{1}{M}\right)_k}\ x^{Mk} \\ &= \frac{(-1)^r \left(1-\frac{1}{M}\right)_r}{r!}\ {}_2F_1\left[\begin{array}{ccc} -r, & r+1-\frac{1}{M}; & x^M \\ & 1-\frac{1}{M}; & \end{array}\right]. \end{aligned}$$

Similarly, from first series of (3), we get

$$\begin{aligned} P_{Mr+1}(x;M) &= \sum_{k=0}^{r} \frac{(-1)^{2k} \left(1+\frac{1}{M}\right)_{2r} \left(-\frac{1}{M}-r\right)_k \ (-r)_k}{k!\ r!\ (-1)^{2k} \left(-\frac{1}{M}-2r\right)_k \left(1+\frac{1}{M}\right)_r} x^{M(r-k)+1} \\ &= \frac{\left(1+\frac{1}{M}\right)_{2r}\ x^{Mr+1}}{\left(1+\frac{1}{M}\right)_r\ r!}\ {}_2F_1\left[\begin{array}{ccc} -r, & -r-\frac{1}{M}; & x^{-M} \\ & -2r-\frac{1}{M}; & \end{array}\right] \end{aligned}$$

and from the second series,

$$\begin{aligned} P_{Mr+1}(x;M) &= \frac{(-1)^r\ x \left(1+\frac{1}{M}\right)_r}{r!} \sum_{k=0}^{r} \frac{(-r)_k \left(r+1+\frac{1}{M}\right)_k}{k! \left(1+\frac{1}{M}\right)_k}\ x^{Mk} \\ &= \frac{(-1)^r x \left(1+\frac{1}{M}\right)_r}{r!}\ {}_2F_1\left[\begin{array}{ccc} -r, & r+1+\frac{1}{M}; & x^M \\ & 1+\frac{1}{M}; & \end{array}\right]. \end{aligned}$$

4 Orthogonality

We now derive the orthogonality of the polynomials $P_n(x;M)$, where n is any non negative integer and $M \in 2\mathbb{N}$.

Theorem 1 *For* $n,\ m \in \mathbb{N} \cup \{0\}$ with $n \neq m,$ and $M \in 2\mathbb{N}$,

$$\int_{-1}^{1} x^{M-2} P_n(x;M) P_m(x;M)\ dx = 0, \tag{10}$$

in which $n = Mr$ *or* $Mr+1,$ *and* $m = Ms$ *or* $Ms+1.$

Proof. We use the equation (4) and combine the first two terms to get

$$[(1-x^M)P_n'(x;M)]' + n(n+M-1)x^{M-2}P_n(x;M) = 0$$

for $Mr = n$ or $Mr+1 = n$. In this, replacing n by m, it becomes

$$[(1-x^M)P_m'(x;M)]' + m(m+M-1)x^{M-2}P_m(x;M) = 0.$$

If we multiply the last two equations by $P_m(x;M)$ and $P_n(x;M)$ respectively, and subtract one from the other, then we obtain

$$\begin{aligned}&[(1-x^M)P_n'(x;M)]'P_m(x;M)+n(n+M-1)x^{M-2}P_n(x;M)P_m(x;M)\\&-[(1-x^M)P_m'(x;M)]'P_n(x;M)-m(m+M-1)x^{M-2}P_m(x;M)P_n(x;M)=0.\end{aligned}$$

Now combining the second and fourth terms in this equation and introducing the term $(1-x^M)P_n'(x;M)$ $P_m'(x;M)$, it simplifies to

$$\begin{aligned}&\left[(1-x^M)\{P_n'(x;M)P_m(x;M)-P_m'(x;M)P_n(x;M)\}\right]'\\&+[n(n+M-1)-m(m+M-1)]\;x^{M-2}P_n(x;M)P_m(x;M)=0.\end{aligned}$$

Integrating this from a to b with respect to x, we have

$$\begin{aligned}&\left[(1-x^M)\{P_n'(x;M)P_m(x;M)-P_m'(x;M)P_n(x;M)\}\right]_a^b\\&+\;[n(n+M-1)-m(m+M-1)]\int_a^b x^{M-2}P_n(x;M)P_m(x;M)dx=0.\end{aligned}$$

Here if M is an even positive integer, then the first term vanishes for the choice $a=-1$ and $b=1$. This leads to the property (10). ■

5 Rodrigues Formula

We aim at representing $P_{Mr}(x;M)$ and $P_{Mr+1}(x;M)$ as the r^{th} derivative of certain function. This enables us to evaluate the integral (10) for $m=n$.

Theorem 2 *There holds the r^{th} derivative representation of $P_{Mr}(x;M)$ and $P_{Mr+1}(x;M)$ given by*

$$P_{Mr}(x;M)=\frac{x}{r!\;M^r}\mathcal{D}^r\left[x^{Mr-1}(x^M-1)^r\right],\;\text{ and }P_{Mr+1}(x;M)=\frac{1}{r!\;M^r}\mathcal{D}^r\left[x^{Mr+1}(x^M-1)^r\right],$$

where $\mathcal{D}=x^{-M+1}\frac{d}{dx}$ and $M\in\mathbb{N}$.

Proof. We note that

$$\begin{aligned}\frac{\left(1-\frac{1}{M}\right)_{r+k}}{\left(1-\frac{1}{M}\right)_k}&=\left(1-\frac{1}{M}+k\right)\left(1-\frac{1}{M}+k+1\right)\cdots\left(1-\frac{1}{M}+r+k-1\right)\\&=\frac{1}{M^r}(M(k+1)-1)(M(k+2)-1)\cdots(M(k+r)-1).\end{aligned}\tag{11}$$

Now, applying the differential operator: $x^{-M+1}\frac{d}{dx}=\mathcal{D}$ iteratively on $x^{M(k+r)-1}$, we obtain

$$\mathcal{D}x^{M(k+r)-1}=(M(k+r)-1)\;x^{M(k+r-1)-1},$$

$$\mathcal{D}^2x^{M(k+r)-1}=(M(k+r)-1)(M(k+r-1)-1)\;x^{M(k+r-2)-1},$$

and in general,

$$\mathcal{D}^rx^{M(k+r)-1}=(M(k+r)-1)(M(k+r-1)-1)\cdots(M(k+1)-1)\;x^{Mk-1}.$$

Thus the identity in (11) may be written as

$$\frac{\left(1-\frac{1}{M}\right)_{r+k}}{\left(1-\frac{1}{M}\right)_k}\;x^{Mk}=\frac{x}{M^r}\;\mathcal{D}^rx^{M(k+r)-1}.$$

Using this in (2), we finally obtain

$$\begin{aligned} P_{Mr}(x;M) &= \sum_{k=0}^{r} \frac{(-1)^{r-k} x \ \mathcal{D}^r x^{M(k+r)-1}}{k!(r-k)! \ M^r} \\ &= \frac{x}{r! \ M^r} \mathcal{D}^r \left(x^{Mr-1} \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} x^{Mk} \right) \\ &= \frac{x}{r! \ M^r} \mathcal{D}^r \left[x^{Mr-1} (x^M - 1)^r \right] \end{aligned} \tag{12}$$

which is the Rodrigues formula for $P_{Mr}(x;M)$. Similarly,

$$\frac{\left(1+\frac{1}{M}\right)_{r+k}}{\left(1+\frac{1}{M}\right)_k} \ x^{Mk+1} = \frac{1}{M^r} \ \mathcal{D}^r x^{M(k+r)+1}.$$

This in view of (3), leads us to

$$\begin{aligned} P_{Mr+1}(x;M) &= \sum_{k=0}^{r} \frac{(-1)^{r-k} \ \mathcal{D}^r x^{M(k+r)+1}}{k!(r-k)! \ M^r} \\ &= \frac{1}{r! \ M^r} \mathcal{D}^r \left(x^{Mr+1} \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} x^{Mk} \right) \\ &= \frac{1}{r! \ M^r} \mathcal{D}^r \left[x^{Mr+1} (x^M - 1)^r \right]. \end{aligned} \tag{13}$$

■

6 Evaluation of the Integral

It is natural to examine the integral in (10) when $m = n$. In doing this, we employ the Rodrigue's formula (12) in the integrand to replace $P_{Mr}(x;M)$, and then apply the method of integration by parts r times.

Theorem 3 *For $M \in 2\mathbb{N}$,*

$$\int_{-1}^{1} x^{M-2} \left(P_n(x;M)\right)^2 dx = \begin{cases} \dfrac{2}{2Mr+M-1} & \text{if } n = Mr, \\ \dfrac{2}{2Mr+M+1} & \text{if } n = Mr+1. \end{cases}$$

Proof. With regard to the operator $\mathcal{D}$ of preceding section, we adopt the notation $\mathfrak{S}$ and write

$$\int_{-1}^{1} x^{M-1} f(x) dx = \mathfrak{S}_{-1}^{1} f(x) \ dx,$$

then using the notation g_n or g_{Mr}, we have

$$\begin{aligned}
g_n &= g_{Mr} = \int_{-1}^{1} x^{M-2}\left(P_{Mr}(x;M)\right)^2 dx \\
&= \int_{-1}^{1} x^{M-2} P_{Mr}(x;M)\ \left(\frac{x}{r!\ M^r}\mathcal{D}^r\left(x^{Mr-1}(x^M-1)^r\right)\right) dx \\
&= \mathfrak{S}_{-1}^{1}\ x^{-1} P_{Mr}(x;M)\frac{x}{r!\ M^r}\mathcal{D}^r\left(x^{Mr-1}(x^M-1)^r\right) dx \\
&= \frac{1}{r!\ M^r}\ \mathfrak{S}_{-1}^{1}\ P_{Mr}(x;M)\ \mathcal{D}^r\left(x^{Mr-1}(x^M-1)^r\right) dx \\
&= \frac{1}{r!\ M^r}\left[P_{Mr}(x;M)\ \mathcal{D}^{r-1}\left(x^{Mr-1}(x^M-1)^r\right)\right]_{-1}^{1} \\
&\quad - \frac{1}{r!\ M^r}\ \mathfrak{S}_{-1}^{1}\ \left[\mathcal{D}P_{Mr}(x;M)\right]\left[\mathcal{D}^{r-1}\left(x^{Mr-1}(x^M-1)^r\right)\right] dx \\
&= \frac{1}{r!\ M^r}\ \mathfrak{S}_{-1}^{1}\ \left[\mathcal{D}P_{Mr}(x;M)\right]\left[\mathcal{D}^{r-1}\left(x^{Mr-1}(x^M-1)^r\right)\right] dx.
\end{aligned}$$

Proceeding similarly using the method of integration by parts $(r-1)$-times, we finally obtain

$$g_{Mr} = \frac{(-1)^r}{r!\ M^r}\ \mathfrak{S}_{-1}^{1}\ \left[\mathcal{D}^r P_{Mr}(x;M)\right]\left[x^{Mr-1}(x^M-1)^r\right] dx.$$

But

$$\begin{aligned}
\mathcal{D}^r P_{Mr}(x;M) &= \mathcal{D}^r\left(\sum_{k=0}^{r}\frac{(-1)^{r-k}\left(1-\frac{1}{M}\right)_{r+k}}{k!(r-k)!\left(1-\frac{1}{M}\right)_k}x^{Mk}\right) \\
&= \mathcal{D}^r\left(\frac{\left(1-\frac{1}{M}\right)_{2r}}{r!\left(1-\frac{1}{M}\right)_r}x^{Mr}\right) + \mathcal{D}^r\left(\sum_{k=0}^{r-1}\frac{(-1)^{r-k}\left(1-\frac{1}{M}\right)_{r+k}}{k!(r-k)!\left(1-\frac{1}{M}\right)_k}x^{Mk}\right) \\
&= \frac{\left(1-\frac{1}{M}\right)_{2r}}{r!\left(1-\frac{1}{M}\right)_r}\ \mathcal{D}^r x^{Mr} \\
&= \frac{M^r\ \left(1-\frac{1}{M}\right)_{2r}}{\left(1-\frac{1}{M}\right)_r},
\end{aligned}$$

hence

$$\begin{aligned}
g_{Mr} &= \frac{(-1)^r}{r!\ M^r}\ \mathfrak{S}_{-1}^{1}\ \frac{M^r\ \left(1-\frac{1}{M}\right)_{2r}}{\left(1-\frac{1}{M}\right)_r}\left(x^{Mr-1}(x^M-1)^r\right) dx \\
&= \frac{(-1)^r}{r!}\frac{\left(1-\frac{1}{M}\right)_{2r}}{\left(1-\frac{1}{M}\right)_r}\ \int_{-1}^{1}\ x^{M(r+1)-2}\ (x^M-1)^r dx.
\end{aligned}$$

Since M is an even positive integer,

$$\begin{aligned}
g_{Mr} &= \frac{2(-1)^r}{r!}\frac{\left(1-\frac{1}{M}\right)_{2r}}{\left(1-\frac{1}{M}\right)_r}\ \int_{0}^{1}\ x^{M(r+1)-2}\left[\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}x^{Mk}\right] dx \\
&= \frac{2\ \left(1-\frac{1}{M}\right)_{2r}}{r!\ \left(1-\frac{1}{M}\right)_r}\ \sum_{k=0}^{r}\binom{r}{k}(-1)^k\left[\frac{1}{M(r+1+k)-1}\right].
\end{aligned}$$

But

$$\begin{aligned}
\frac{1}{M(r+1)-1}\ \frac{\left(r+1-\frac{1}{M}\right)_k}{\left(r+2-\frac{1}{M}\right)_k} &= \frac{1}{M\left(r+1-\frac{1}{M}\right)}\ \frac{\left(r+1-\frac{1}{M}\right)\left(r+1-\frac{1}{M}+1\right)\cdots\left(r+1-\frac{1}{M}+k-1\right)}{\left(r+2-\frac{1}{M}\right)\left(r+2-\frac{1}{M}+1\right)\cdots\left(r+2-\frac{1}{M}+k-1\right)} \\
&= \frac{1}{M(r+1+k)-1},
\end{aligned}$$

hence, we have

$$g_{Mr} = \frac{2\ \left(1-\frac{1}{M}\right)_{2r}}{r!\ \left(1-\frac{1}{M}\right)_r} \sum_{k=0}^{r} \binom{r}{k} (-1)^k \frac{1}{M(r+1)-1} \frac{\left(r+1-\frac{1}{M}\right)_k}{\left(r+2-\frac{1}{M}\right)_k}.$$

For the sake of simplicity, let us put $1-\frac{1}{M}=\alpha$, then we find that

$$g_{Mr} = \frac{2\ (\alpha)_{2r}}{r!\ (\alpha)_r} \frac{1}{(M(r+1)-1)} \sum_{k=0}^{r} \frac{(-r)_k (r+\alpha)_k}{k!\ (r+1+\alpha)_k}.$$

Here the finite sum represents the function ${}_2F_1(-r, r+\alpha; r+1+\alpha; 1)$, hence substituting its value (see [4, Theorem 18, p.49]), we have

$$\begin{aligned} g_{Mr} &= \frac{2\ (\alpha)_{2r}}{r!\ (\alpha)_r} \frac{1}{(M(r+1)-1)} \frac{\Gamma(r+1+\alpha)\Gamma(r+1)}{\Gamma(2r+1+\alpha)\Gamma(1)} \\ &= \frac{2\ \Gamma(\alpha+2r)\ \Gamma(\alpha)}{r!\ (Mr+M-1)\ \Gamma(\alpha)\ \Gamma(\alpha+r)} \frac{(r+\alpha)\ \Gamma(r+\alpha)\ (1)_r}{(2r+\alpha)\ \Gamma(2r+\alpha)} \\ &= \frac{2\ (r+\alpha)}{(Mr+M-1)(2r+\alpha)} \\ &= \frac{2}{2Mr+M-1}. \end{aligned}$$

Finally, we obtain

$$\int_{-1}^{1} x^{M-2} \left(P_{Mr}(x;M)\right)^2 dx = \frac{2}{2Mr+M-1}.$$

In case of $P_{Mr+1}(x;M)$, we have

$$\begin{aligned} g_{Mr+1} &= \int_{-1}^{1} x^{M-2} \left(P_{Mr+1}(x;M)\right)^2 dx \\ &= \int_{-1}^{1} x^{M-2} \left(P_{Mr+1}(x;M)\right) \left(\frac{1}{r!\ M^r} \mathcal{D}^r \left(x^{Mr+1}(x^M-1)^r\right)\right) dx. \end{aligned}$$

Proceeding similarly using the method of integration by parts $(r-1)$-times, we finally obtain the value as stated in the theorem. ■

Note 2. For $M=2$, we get from theorem 3 ([4, eq.(12), p.175]),

$$\int_{-1}^{1} (P_n(x))^2\ dx = \frac{2}{2n+1}.$$

7 Generating Function Relations

In the following theorem, the generating function relations are obtained.

Theorem 4 *If* $|t|<1$, $\left|\frac{4(xt)^M}{(1+t^M)^2}\right| < 1$, *then*

$$\sum_{r=0}^{\infty} P_{Mr}(x;M) t^{Mr} = (1+t^M)^{\frac{1}{M}-1}\ {}_2F_1\left[\begin{array}{cc} \frac{1}{2}-\frac{1}{2M},\quad 1-\frac{1}{2M}; & \frac{4(xt)^M}{(1+t^M)^2} \\ 1-\frac{1}{M}; & \end{array}\right],$$

and

$$\sum_{r=0}^{\infty} P_{Mr+1}(x;M) t^{Mr} = xt\left(1+t^M\right)^{-1-\frac{1}{M}}\ {}_2F_1\left[\begin{array}{cc} \frac{1}{2}+\frac{1}{2M},\quad 1+\frac{1}{2M}; & \frac{4(xt)^M}{(1+t^M)^2} \\ 1+\frac{1}{M}; & \end{array}\right].$$

Proof. We begin with

$$\begin{aligned}
\sum_{r=0}^{\infty} P_{Mr}(x;M)t^{Mr} &= \sum_{r=0}^{\infty}\sum_{k=0}^{r} \frac{(-1)^k \left(1-\frac{1}{M}\right)_{2r-k}}{k!(r-k)!\left(1-\frac{1}{M}\right)_{r-k}} x^{M(r-k)} t^{Mr} \\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \left(1-\frac{1}{M}\right)_{2r+k}}{k!\ r!\left(1-\frac{1}{M}\right)_{r}} x^{Mr} t^{M(r+k)} \\
&= \sum_{r=0}^{\infty} \frac{(1-\frac{1}{M})_{2r}}{(1-\frac{1}{M})_r\ r!}(xt)^{Mr} \sum_{k=0}^{\infty} \frac{(-1)^k(1-\frac{1}{M}+2r)_k}{k!} t^{Mk}.
\end{aligned}$$

As before, taking $(1-\frac{1}{M})=\alpha$ and assuming $|t|<1$ and $\left|\frac{4(xt)^M}{(1+t^M)^2}\right|<1$, we have

$$\begin{aligned}
\sum_{r=0}^{\infty} P_{Mr}(x;M)t^{Mr} &= \sum_{r=0}^{\infty} \frac{(\alpha)_{2r}}{(\alpha)_r\ r!}(xt)^{Mr} \sum_{k=0}^{\infty} \frac{(-1)^k(\alpha+2r)_k}{k!} t^{Mk} \\
&= \sum_{r=0}^{\infty} \frac{2^{2r}\left(\frac{\alpha}{2}\right)_r \left(\frac{\alpha}{2}+\frac{1}{2}\right)_r}{(\alpha)_r\ r!}(xt)^{Mr}(1+t^M)^{-\alpha-2r} \\
&= (1+t^M)^{-\alpha} \sum_{r=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_r \left(\frac{\alpha}{2}+\frac{1}{2}\right)_r}{(\alpha)_r\ r!} \frac{4^r(xt)^{Mr}}{(1+t^M)^{2r}} \\
&= (1+t^M)^{\frac{1}{M}-1}\ {}_2F_1\left[\begin{array}{c} \frac{1}{2}-\frac{1}{2M},\quad 1-\frac{1}{2M};\quad \frac{4(xt)^M}{(1+t^M)^2} \\ 1-\frac{1}{M}; \end{array}\right],
\end{aligned}$$

which is first generating function relation. For the odd degree polynomial, we assume $|t|<1$, and consider

$$\begin{aligned}
\sum_{r=0}^{\infty} P_{Mr+1}(x;M)t^{Mr+1} &= \sum_{r=0}^{\infty}\sum_{k=0}^{r} \frac{(-1)^k \left(1+\frac{1}{M}\right)_{2r-k}}{k!\ (r-k)!\left(1+\frac{1}{M}\right)_{r-k}} x^{M(r-k)+1}\ t^{Mr+1} \\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \left(1+\frac{1}{M}\right)_{2r+k}}{k!\ r!\left(1+\frac{1}{M}\right)_{r}} x^{Mr+1}\ t^{M(r+k)+1} \\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \left(1+\frac{1}{M}\right)_{2r}\left(1+\frac{1}{M}+2r\right)_k}{k!\ r!\left(1+\frac{1}{M}\right)_{r}} (xt)^{Mr+1}\ t^{Mk} \\
&= \sum_{r=0}^{\infty} \frac{\left(1+\frac{1}{M}\right)_{2r}\ (xt)^{Mr+1}}{r!\left(1+\frac{1}{M}\right)_r} \sum_{k=0}^{\infty} \frac{(-1)^k\left(1+\frac{1}{M}+2r\right)_k}{k!}\ t^{Mk} \\
&= \sum_{r=0}^{\infty} \frac{\left(1+\frac{1}{M}\right)_{2r}\ (xt)^{Mr+1}}{r!\left(1+\frac{1}{M}\right)_r} \left(1+t^M\right)^{-1-\frac{1}{M}-2r} \\
&= xt\left(1+t^M\right)^{-1-\frac{1}{M}} \sum_{r=0}^{\infty} \frac{2^{2r}\left(\frac{1}{2}+\frac{1}{2M}\right)_r\left(1+\frac{1}{2M}\right)_r}{r!\left(1+\frac{1}{M}\right)_r} \frac{(xt)^{Mr}}{\left(1+t^M\right)^{2r}}.
\end{aligned}$$

Finally, assuming $\left|\frac{4(xt)^M}{(1+t^M)^2}\right|<1$, we obtain the second generating function relation of the theorem. ■

8 MATLAB Programming to Compute Zeros

It is quite natural to ask: how one can find the zeros? To answer this, we provide here the MATLAB platform for computing zeros of $P_n(x;M)$ for $n=Mr$ or $n=Mr+1$.

```
b = input('If the degree of polynomial is odd then enter 1 else 2:');
r= input('Enter the value of r:');
m= input('Enter the value of M:');
if b==2
t = 1 - 1/m;
p1=1;
for k= 1:r
p1=p1*(t+k-1);
end
f(1)=p1*(-1)^r/gamma(r+1);
for k=1:r
a = (-1)^k *gamma(r+1-k)* gamma(k+1);
q1=1;
for i=1:k
q1=q1*(t+r+i-1)/(t+i-1);
end
f(k+1)=(p1*q1)/a;
end
for i=1:r
g((i-1)*m+1)=f(i);
for j=2:m
g((i-1)*m+j)=0;
end
end
g(r*m+1)=f(r+1);
prm=flip(g)
x=roots(prm)
else
t = 1 + 1/m;
p1=1;
for k= 1:r
p1=p1*(t+k-1);
end
f(1)=p1*(-1)^r/gamma(r+1);
for k=1:r
a = (-1)^k *gamma(r+1-k)* gamma(k+1);
q1=1;
for i=1:k
q1=q1*(t+r+i-1)/(t+i-1);
end
f(k+1)=(p1*q1)/a;
end
g(1)=0;
for i=1:r
g((i-1)*m+2)=f(i);
for j=3:m+1
g((i-1)*m+j)=0;
end
end
g(r*m+2)=f(r+1);
prm1=flip(g)
x=roots(prm1)
end
```

Example 1 *This program is illustrated by choosing $r = 2$ and $M = 4$ for both even and odd cases. The zeros of $P_8(x; 4)$ are*

$$\begin{array}{rr} -0.9360 + 0.0000i & 0.0000 - 0.9360i \\ 0.9360 + 0.0000i & 0.0000 + 0.9360i \\ 0.6381 + 0.0000i & 0.0000 + 0.6381i \\ -0.6381 + 0.0000i & 0.0000 - 0.6381i \end{array}$$

and the zeros of $P_9(x;4)$ *are*

$$\begin{array}{ccc} 0.9476+0.0000i & -0.9476+0.0000i & 0.0000+0.0000i \\ 0.7089+0.0000i & -0.7089+0.0000i & 0.0000+0.9476i \\ 0.0000+0.7089i & 0.0000-0.7089i & 0.0000-0.9476i \end{array}$$

We observe that $P_8(x;4)$ has four real zeros and four complex zeros and $P_9(x;4)$ has five real zeros and four complex zeros; unlike the nature of zeros of the Legendre polynomial which are all real.

9 Graphical Behavior

It is well known that the graphs of $P_n(x)$, for $n = 0, 1, 2, \ldots$ intersect the x-axis between $x = -1$ and $x = 1$ (see Figure 2). Hence, it would be interesting to examine the graphs of $P_{Mr}(x;M)$ and $P_{Mr+1}(x;M)$ for $r = 0, 1, 2, \ldots$ and for fixed M. In Figure 3, the graphs are plotted for $M = 4$ and $r = 0, 1, 2$. Since, the zeros of $P_{Mr}(x;M)$ and $P_{Mr+1}(x;M)$ for $M = 4, 6, \ldots$ are not all real, hence the observation is that for these values of M, not all the graphs will show the intersections with the x-axis.

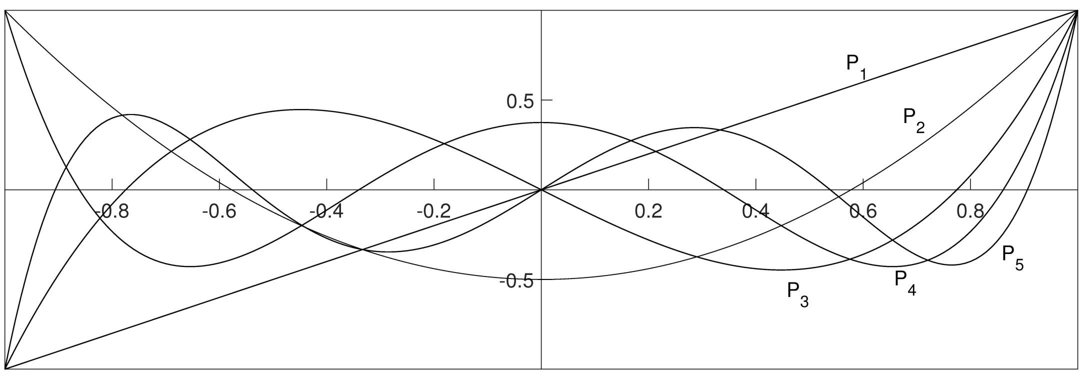

Figure 2: $P_n(x;2)$

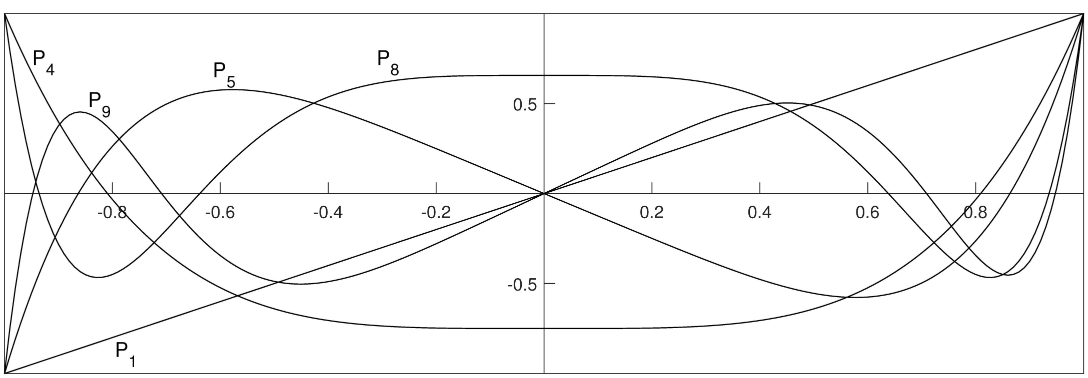

Figure 3: $P_n(x;4)$

9.1 Observation

From Figure 3, it may be seen that the graph of P_8 intersects x-axis 4 times whereas the graph of P_9 intersects x-axis 5 times (Example 1). We further observe that the graphs of even degree polynomials are symmetric about y-axis whereas the graphs of the odd degree polynomials are symmetric about the origin.

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