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Bounds For The Zeros Of A Polynomial*

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Abstract

In this paper we prove some results concerning the bounds for the zeros of polynomials and as a special case obtain a generalization of the well-known Eneström-Kakeya theorem. Our result mainly generalizes as well as refines the result of Joyal et al. [*Bull. Canad. Math.,* **10**(1967), 53-63].

1 Introduction

Let $P(z) := a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a polynomial of degree n where $a_0, a_1, ..., a_n \in \mathbb{C}$. Then by the Fundamental Theorem of Algebra, $P(z)$ has exactly n number of zeros. Thus "Fundamental Theorem of Algebra" gives only exact number of zeros of a polynomial but says nothing regarding their locations in the complex domain. Cauchy added much to it, by giving more exact bounds for the moduli of the zeros than those given by Gauss. In this concern, we prove some results concerning the bounds for the zeros of a polynomial and some generalizations of an elegant result known as Eneström-Kakeya theorem concerning the location of zeros of polynomials with restriction on coefficients. The following result in the theory of distribution of zeros of polynomials due to Cauchy [4] is well known:

Theorem 1 *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If* $M = \max_{0\le j\le n-1} |\frac{a_j}{a_n}|$, *then all the zeros of* $P(z)$ *lie in*

$$|z| \le 1 + M.$$

There exist several improvements and generalizations of this result. As an improvement Joyal, Labelle and Rahman [6] proved the following:

Theorem 2 *If* $B := \max_{0\le j\le n-1} |a_j|$, *then all the zeros of* $P(z) := z^n + \sum_{j=0}^{n-1} a_j z^j$ *are contained in the circle*

$$|z| \le \frac{1}{2}\left[1 + |a_{n-1}| + \left\{(1 - |a_{n-1}|)^2 + 4B\right\}^{\frac{1}{2}}\right].$$

Montel and Marty [7, p.138] on the other hand independently proved the following:

Theorem 3 *All zeros of the polynomial* $P(z) := a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$ *lie in a circle* $|z| \le \max\left(L, L^{\frac{1}{n}}\right)$, *where* L *is the length of the polygonal line joining in succession the points* $0, a_0, \ldots, a_{n-1}, 1$; *that is,* $L = |a_0| + |a_1 - a_0| + \cdots + |a_{n-1} - a_{n-2}| + |1 - a_{n-1}|$.

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If we restrict the coefficients of the polynomial $P(z) := \sum_{j=0}^{n} a_j z^j$ and assume that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0.$$

Then according to a famous result in the theory of distribution of zeros of polynomials known as the Eneström-Kakeya theorem [7, p.136], all the zeros of $P(z)$ lie in $|z| \leq 1$. Applying this result to the polynomial $z^n P(\frac{1}{z})$, one gets an equivalent form of the Eneström-Kakeya theorem, which states that:

If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$$a_0 \geq a_1 \geq \cdots a_{n-1} \geq a_n > 0,$$

then $P(z)$ has no zeros in $|z| < 1$.

Applying the above result to the polynomial $P(tz)$, we have the following more general version of the Eneström-Kakeya theorem :

Theorem 4 *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $t > 0$,*

$$t^n a_n \geq t^{n-1} a_{n-1} \geq \ldots \geq t a_1 \geq a_0 > 0.$$

Then all the zeros of $P(z)$ lie in $|z| \leq t$ and in case

$$a_0 t^n \geq a_1 t^{n-1} \geq \cdots a_{n-1} t \geq a_n > 0,$$

then $P(z)$ has all zeros in $|z| \geq \frac{1}{t}$.

Recently by using Holder's inequality, Aziz and Rather [3] proved the following result concerning the distribution of zeros of polynomials.

Theorem 5 *All zeros of the polynomial $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle*

$$|z| \leq (n+1)^{\frac{1}{q}} \left\{ \sum_{j=0}^{n} \left| \frac{t a_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}},$$

where $p > 1,\ q > 1$ with $p^{-1} + q^{-1} = 1,$ and $t > 0$.

2 Lemmas

We first prove the following lemma.

Lemma 1 *For any positive numbers $t_1 \neq 0$ and t_2, such that $t_1 \geq t_2 \geq 0$, all zeros of the polynomial $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle*

$$|z| \leq t_1 \max \left\{ N_{P,t_1,t_2}, N_{P,t_1,t_2}^{\frac{1}{n+2}} \right\}, a_{-1} = a_{-2} = a_{n+1} = 0,$$

where

$$N_{P,t_1,t_2} = (n+2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{n+1} \left| \frac{t_1 t_2 a_j + (t_1 - t_2) a_{j-1} - a_{j-2}}{a_n t_1^{n-j+2}} \right|^p \right\}^{\frac{1}{p}}, \tag{1}$$

$p > 0,\ q > 0$ and $p^{-1} + q^{-1} = 1$.

Proof. Consider the polynomial

$$\begin{aligned} F(z) &= (t_1 - z)(t_2 + z)P(z) \\ &= t_1 t_2 a_0 + (t_1 t_2 a_1 + (t_1 - t_2)a_0)z + \dots \\ &\quad + (t_1 t_2 a_n + (t_1 - t_2)a_{n-1} - a_{n-2})z^n + ((t_1 - t_2)a_n - a_{n-1})z^{n+1} - a_n z^{n+2}, \end{aligned}$$

so that

$$\begin{aligned} |F(z)| &\geq |a_n||z^{n+2}|\left\{1 - \sum_{j=0}^{n+1}\left|\frac{t_1 t_2 a_j + (t_1 - t_2)a_{j-1} - a_{j-2}}{a_n z^{n-j+2}}\right|\right\} \\ &= |a_n||z^{n+2}|\left\{1 - \sum_{j=0}^{n+1}\left(\left|\frac{t_1 t_2 a_j + (t_1 - t_2)a_{j-1} - a_{j-2}}{a_n}\right|\frac{1}{t_1^{n-j+2}}\right)\left(\frac{t_1}{|z|}\right)^{n-j+2}\right\}. \end{aligned}$$

This gives by inequality (1) using Holder's inequality, for every $p > 0,\ q > 0,\ p^{-1} + q^{-1} = 1$,

$$|F(z)| \geq |a_n||z^{n+2}|\left[1 - \frac{N_{p,t_1,t_2}}{(n+2)^{\frac{1}{q}}}\left\{\sum_{j=0}^{n+1}\left(\frac{t_1}{|z|}\right)^{q(n-j+2)}\right\}^{\frac{1}{q}}\right]. \tag{2}$$

Now if $N_{p,t_1,t_2} \geq 1$, then $\max\left\{N_{p,t_1,t_2}, N_{p,t_1,t_2}^{\frac{1}{n+2}}\right\} = N_{p,t_1,t_2}$. Also for $|z| \geq t_1$, $\left(\frac{t_1}{|z|}\right)^{n-j+2} \leq \frac{t_1}{|z|}$, $j = 0, 1, 2, \dots, n+1$. Therefore if $|z| > t_1 N_{p,t_1,t_2}$, then from inequality (2) we have

$$\begin{aligned} |F(z)| &\geq |a_n||z^{n+2}|\left[1 - \frac{N_{p,t_1,t_2}}{(n+2)^{\frac{1}{q}}}\left\{\sum_{j=0}^{n+1}\left(\frac{t_1}{|z|}\right)^{q}\right\}^{\frac{1}{q}}\right] \\ &= |a_n||z^{n+2}|\left[1 - N_{p,t_1,t_2}\left(\frac{t_1}{|z|}\right)\right] > 0. \end{aligned} \tag{3}$$

Again if $N_{p,t_1,t_2} \leq 1$, then $\max\left\{N_{p,t_1,t_2}, N_{p,t_1,t_2}^{\frac{1}{n+2}}\right\} = N_{p,t_1,t_2}^{\frac{1}{n+2}}$. In this case if $|z| \leq t_1$, then $\left(\frac{t_1}{|z|}\right)^{n-j+2} \leq \left(\frac{t_1}{|z|}\right)^{n+2}$, $j = 0, 1, 2, \dots, n+1$ and from inequality (2), we infer that if $|z| > t_1 N_{p,t_1,t_2}^{\frac{1}{n+2}}$, then

$$\begin{aligned} |F(z)| &\geq |a_n||z^{n+2}|\left[1 - \frac{N_{p,t_1,t_2}}{(n+2)^{\frac{1}{q}}}\left\{\sum_{j=0}^{n+1}\left(\frac{t_1}{|z|}\right)^{q(n+2)}\right\}^{\frac{1}{q}}\right] \\ &= |a_n||z^{n+2}|\left[1 - N_{p,t_1,t_2}\left(\frac{t_1}{|z|}\right)^{n+2}\right] > 0. \end{aligned} \tag{4}$$

Combining (3) and (4), it follows that $F(z)$ does not vanish for

$$|z| > t_1 \max\left\{N_{P,t_1,t_2}, N_{P,t_1,t_2}^{\frac{1}{n+2}}\right\}.$$

From this, we conclude that all the zeros of $F(z)$ and hence $P(z)$ lie in the disc defined by

$$|z| \leq t_1 \max\left\{N_{P,t_1,t_2}, N_{P,t_1,t_2}^{\frac{1}{n+2}}\right\}.$$

This completes the proof of the Lemma 1. ∎

Next lemma is a well known generalization of Schwarz's lemma.

Lemma 2 *If $P(z)$ is analytic in $|z| \leq 1$, with $|P(z)| \leq M$ on $|z| = 1$, and $p(0) = b$, then*

$$|P(z)| \leq M\frac{M|z| + b}{|b|z + M}.$$

In the literature (see [2], [8], [10]) there exist several generalizations of Eneström-Kakeya theorem. Given a lower bound for the modulus of the zeros of a polynomial, what is a possible upper bound, needs an attention as the study of such cases have recently been a concern due to their application in linear control system, electrical networking, signal processing, coding theory and several other areas of physical sciences. In this paper we first prove the following result where we find a region containing all the zeros of a polynomial given a lower bound for the zeros of that polynomial.

Theorem 6 *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n which does not vanish in $|z| < t$, for some $t > 0$. Then all the zeros of $P(z)$ lie in*

$$\left|z - \frac{ta_n - a_{n-1}}{a_n}\right| \leq n^{\frac{1}{q}} \left\{ \sum_{j=0}^{n-1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}, \qquad a_{-1} = 0.$$

Proof. Consider the polynomial,

$$\begin{aligned} F(z) &= (t - z)P(z) \\ &= (t - z)(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_o) \\ &= -a_n z^{n+1} + (ta_n - a_{n-1})z^n \cdots + (ta_1 - a_0)z + ta_0. \end{aligned}$$

This gives

$$|F(z)| \geq |a_n||z^n| \left[\left|z - \frac{ta_n - a_{n-1}}{a_n}\right| - \sum_{j=0}^{n-1} \left| \frac{ta_j - a_{j-1}}{a_n} \right| \frac{1}{|z|^{n-j}} \right].$$

If $|z| > t$, then by using Holder's inequality, we have for $p > 0,\ q > 0$ and $p^{-1} + q^{-1} = 1$,

$$|F(z)| \geq |a_n||z^n| \left[\left|z - \frac{ta_n - a_{n-1}}{a_n}\right| - n^{\frac{1}{q}} \left\{ \sum_{j=0}^{n-1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}} \right] > 0,$$

if

$$\left|z - \frac{ta_n - a_{n-1}}{a_n}\right| > n^{\frac{1}{q}} \left\{ \sum_{j=0}^{n-1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}.$$

This shows that for $|z| > t$, $F(z)$ does not vanish in

$$\left|z - \frac{ta_n - a_{n-1}}{a_n}\right| > n^{\frac{1}{q}} \left\{ \sum_{j=0}^{n-1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}.$$

Hence, we conclude that those zeros of $F(z)$ and therefore of $P(z)$, whose modulus is greater than t lie in

$$\left|z - \frac{ta_n - a_{n-1}}{a_n}\right| \leq n^{\frac{1}{q}} \left\{ \sum_{j=0}^{n-1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}.$$

This completes proof of Theorem 6. ■

Theorem 6 and the second part of Theorem 4 together gives the following result.

Corollary 1 *If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for some $t > 0$,*

$$a_0 t^n \geq a_1 t^{n-1} \geq \cdots \geq a_{n-1} t \geq a_n > 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z - \frac{a_n - t a_{n-1}}{t a_n} \right| \leq n^{\frac{1}{q}} \left\{ \sum_{j=0}^{n-1} \left| \frac{t^{n-j} a_j - t^{n-j+1} a_{j-1}}{t a_n} \right|^p \right\}^{\frac{1}{p}}.$$

From Theorem 6 and Corollary 1, the following results follow respectively by letting $p \to 1$, and $q \to \infty$

Corollary 2 *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n which does not vanish in $|z| < t$, $t > 0$. Then all the zeros of $P(z)$ lie in*

$$\left| z - \frac{t a_n - a_{n-1}}{a_n} \right| \leq \sum_{j=0}^{n-1} \left| \frac{t a_j - a_{j-1}}{a_n t^{n-j}} \right|, \qquad a_{-1} = 0.$$

Corollary 3 *If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for some $t > 0$,*

$$a_0 t^n \geq a_1 t^{n-1} \geq \cdots a_{n-1} t \geq a_n > 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z - \frac{a_n - t a_{n-1}}{t a_n} \right| \leq \sum_{j=0}^{n-1} \left| \frac{t^{n-j} a_j - t^{n-j+1} a_{j-1}}{t a_n} \right|, \qquad a_{-1} = 0.$$

In particular if we take $t = 1$, in Corollary 3 then we have:

Corollary 4 *If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$a_0 \geq a_1 \geq \cdots \geq a_{n-1} \geq a_n > 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z - \frac{a_n - a_{n-1}}{a_n} \right| \leq \frac{2a_0 - a_{n-1}}{a_n}.$$

Combining the Eneström-Kakeya theorem and Corollary 4, we get the following interesting result:

Corollary 5 *If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$a_0 \geq a_1 \geq \cdots a_{n-1} \geq a_n > 0.$$

Then all the zeros of $P(z)$ lie in the region

$$D := \left\{ z : 1 \leq |z| \bigcap \left| z - \frac{a_n - a_{n-1}}{a_n} \right| \leq \frac{2a_0 - a_{n-1}}{a_n} \right\}.$$

3 Example

The following example shows that the bounds for the zeros of the polynomial obtained from our results are comparatively better.

Consider the Polynomial

$$f(z) = z^7 + 3z^6 + 3.1z^5 + 3.2z^4 + 3.2z^3 + 3.3z^2 + 3.4z + 3.4.$$

Its zeros are approximately:
$-1.005059{+}059132i$, $-1.005059{-}059132i$, -2.00427, $-0.16765{+}1.067855i$, $-0.16765{-}1.067855i$, $0.6748544+0.7824664i$, $0.6748544 - 0.7824664i$.
The bounds for the zeros of the polynomial $f(z)$ obtained by using the above results are given in the following table :

S.No	Theorems	Bounds
1	Theorem 1	$\|z\| \leq 4.4$
2	Theorem 2	$\|z\| \leq 4.09$
3	Enestrom Kakeya Theorem	$\|z\| \leq 1$
4	Theorem 5 with $q \to \infty,\ p \to 1$ *and* $t = 1$	$\|z\| \leq 5.7$
5	Corollary 2 with $t = 1$	$\|z + 2\| \leq 3.7$
6	Corollary 5	$\{1 \leq \|z\|\} \cap \{\|z + 2\| \leq 3.8\}$

From the table it is clear that the bounds for the zeros of $P(z)$ obtained in Corollary 2 and Corollary 5 are comparatively better.

In the literature [1, 7–10] there exists some extensions and generalizations of the Eneström-Kakeya theorem. Egervary [5] (see also Aziz and Mohammed [2]) generalized the Eneström-Kakeya theorem in a different way and proved

Theorem 7 *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real positive coefficients. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \quad r = 1, 2, \ldots, n + 1, \quad a_{-1} = a_{n+1} = 0.$$

Then all the zeros of $P(z)$ *lie in* $|z| \leq t_1$.

Combining the techniques used in the proofs of Theorems 6 and 7, we next prove the following:

Theorem 8 *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real positive coefficients. If* $t_1 \neq 0$ *and* t_2 *can be found such that* $t_1 \geq t_2 \geq 0$*, then all the zeros of polynomial* $P(z)$ *lie in*

$$|z| \leq (n + 2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{n+1} \left| \frac{t_1 t_2 a_j + (t_1 - t_2) a_{j-1} - a_{j-2}}{a_n t_1^{n-j+1}} \right|^p \right\}^{\frac{1}{p}},$$

$a_{-2} = a_{-1} = a_{n+1} = 0$*, where* $p > 0,\ q > 0$ *and* $p^{-1} + q^{-1} = 1$.

Proof. Consider the polynomial

$$\begin{aligned} F(z) &= (t_1 - z)(t_2 + z)P(z) \\ &= t_1 t_2 a_0 + (t_1 t_2 a_1 + (t_1 - t_2)a_0)z + (t_1 t_2 a_2 + (t_1 - t_2)a_1 - a_0)z^2 + \ldots \\ &\quad + (t_1 t_2 a_n + (t_1 - t_2)a_{n-1} - a_{n-2})z^n + (t_1 - t_2)a_n z^{n+1} - a_n z^{n+2}. \end{aligned}$$

Applying Lemma 1 to the polynomial $F(z)$, it follows that for every $t_1 \neq 0$ and t_2 such that $t_1 \geq t_2 \geq 0$, all the zeros of $F(z)$ lie in

$$|z| \leq t_1 \max\left\{N_{p,t_1,t_2}, N_{p,t_1,t_2}^{\frac{1}{n+2}}\right\},$$

where

$$N_{p,t_1,t_2} = (n+2)^{\frac{1}{q}}\left\{\sum_{j=0}^{n+1}\left|\frac{t_1t_2a_j + (t_1-t_2)a_{j-1} - a_{j-2}}{a_nt_1^{n-j+2}}\right|^p\right\}^{\frac{1}{p}}.$$

Now by using Holder's inequality, we have

$$\begin{aligned}
1 &= \frac{\left|(t_1^{n+2}a_n - t_1^{n+1}t_2a_n) + \cdots + (t_1^2t_2a_1 + t_1^2a_0 - t_1t_2a_0) + t_1t_2a_0\right|}{|a_nt^{n+2}|} \\
&\leq t_1\sum_{j=0}^{n+1}\left|\frac{t_1t_2a_j + (t_1-t_2)a_{j-1} - a_{j-2}}{a_nt_1^{n-j+2}}\right| \\
&\leq (n+2)^{\frac{1}{q}}\left\{\sum_{j=0}^{n+1}\left|\frac{t_1t_2a_j + (t_1-t_2)a_{j-1} - a_{j-2}}{a_nt_1^{n-j+1}}\right|^p\right\}^{\frac{1}{p}} \\
&= t_1N_{p,t_1,t_2}.
\end{aligned}$$

This means that $\max\left\{N_{p,t_1,t_2}, N_{p,t_1,t_2}^{\frac{1}{n+2}}\right\} = N_{p,t_1,t_2}$. From this we conclude that the zeros of $F(z)$ and hence all the zeros of $P(z)$ lie in the circle

$$|z| \leq (n+2)^{\frac{1}{q}}\left\{\sum_{j=0}^{n+1}\left|\frac{t_1t_2a_j + (t_1-t_2)a_{j-1} - a_{j-2}}{a_nt_1^{n-j+1}}\right|^p\right\}^{\frac{1}{p}}.$$

This completes the proof of Theorem 8. ■

Letting $q \to \infty$, so that $p \to 1$ and noting that $(n+2)^{\frac{1}{q}} \to 1$, we get the following:

Corollary 6 *For any $t_1 \neq 0$ and t_2, such that $t_1 \geq t_2 \geq 0$, all the zeros of the polynomial $P(z)$ of degree n, lie in*

$$|z| \leq \sum_{j=0}^{n+1}\left|\frac{t_1t_2a_j + (t_1-t_2)a_{j-1} - a_{j-2}}{a_nt_1^{n-j+1}}\right|, \qquad a_{n+1} = a_{-1} = a_{-2} = 0.$$

Assuming $t_1t_2a_j + (t_1-t_2)a_{j-1} - a_{j-2} \geq 0$ and noting that

$$\begin{aligned}
&\sum_{j=0}^{n+1}\left|\frac{t_1t_2a_j + (t_1-t_2)a_{j-1} - a_{j-2}}{a_nt_1^{n-j+1}}\right| \\
= \;& \frac{1}{a_nt_1^{n+1}}\sum_{j=0}^{n+1}\left\{t_2t_1^{j+1}a_j + t_1^{j+1}a_{j-1} - t_2t_1^ja_{j-1} - t_1^ja_{j-2}\right\} \\
= \;& \frac{1}{a_nt_1^{n+1}}\left\{\sum_{j=0}^{n+1}(t_2a_j + a_{j-1})t_1^{j+1} - \sum_{j=0}^{n+1}(t_2a_{j-1} + a_{j-2})t_1^j\right\} \\
= \;& \frac{1}{a_nt_1^{n+1}}\left\{\sum_{j=1}^{n+2}(t_2a_{j-1} + a_{j-2})t_1^j - \sum_{j=0}^{n+1}(t_2a_{j-1} + a_{j-2})t_1^j\right\} \\
= \;& \frac{1}{a_nt_1^{n+1}}\left\{a_nt_1^{n+2} + \sum_{j=1}^{n+1}(t_2a_{j-1} + a_{j-2})t_1^j - \sum_{j=1}^{n+1}(t_2a_{j-1} + a_{j-2})t_1^j\right\} \\
= \;& t_1.
\end{aligned}$$

It follows that the result of Aziz and Mohammed ([2], Theorem 7) is a special case of Corollary 6. Similarly a result of Aziz and Rather [3, Corollary 3] follows from Corollary 6, if we take $t_2 = 0$. Further the result of Montel and Marty ([7],Theorem 3) is a special case of Corollary 6, if we take $a_n = 1 = t_1$ and $t_2 = 0$.

By taking $t_1 = t_2 = t > 0$ in Theorem 8, we have the following:

Corollary 7 *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. Then for some $t > 0$, all the zeros of $P(z)$ lie in*

$$|z| \leq (n+2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{n+1} \left| \frac{t^2 a_j - a_{j-2}}{a_n t^{n-j+1}} \right|^p \right\}^{\frac{1}{p}}, \qquad a_{n+1} = a_{-1} = a_{-2} = 0,$$

where $p > 0$, $q > 0$ and $p^{-1} + q^{-1} = 1$.

In particular letting $p \to 1$ and $q \to \infty$ in Corollary 7, it follows that all the zeros of $P(z)$ lie in

$$|z| \leq \sum_{j=0}^{n+1} \left| \frac{t^2 a_j - a_{j-2}}{a_n t^{n-j+1}} \right|.$$

Next if we assume in Corollary 5 that $t^j a_j - t^{j-2} a_{j-2} \geq 0$, then we have the following :

Corollary 8 *If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for some $t > 0$, either*

$$a_n t^n \geq a_{n-2} t^{n-2} \geq \cdots \geq a_3 t^3 \geq a_1 t > 0 \textit{ and } a_{n-1} t^{n-1} \geq a_{n-3} t^{n-3} \geq \cdots \geq a_2 t^2 \geq a_0 > 0 \quad \textit{if } n \textit{ is odd},$$

or

$$a_n t^n \geq a_{n-2} t^{n-2} \geq \cdots \geq a_2 t^2 \geq a_0 > 0 \textit{ and } a_{n-1} t^{n-1} \geq a_{n-3} t^{n-3} \geq \cdots \geq a_1 t > 0 \quad \textit{if } n \textit{ is even},$$

then all the zeros of $P(z)$ lie in the circle $|z| \leq t$.

Proof. By assumption $t^j a_j - t^{j-2} a_{j-2} \geq 0$, $j = 0, 1, \ldots, n$. This gives

$$a_n t^n \geq a_{n-2} t^{n-2} \geq \cdots \geq a_3 t^3 \geq a_1 t > 0 \text{ and } a_{n-1} t^{n-1} \geq a_{n-3} t^{n-3} \geq \cdots \geq a_2 t^2 \geq a_0 > 0 \quad \text{if } n \text{ is odd},$$

and

$$a_n t^n \geq a_{n-2} t^{n-2} \geq \cdots \geq a_2 t^2 \geq a_0 > 0 \text{ and } a_{n-1} t^{n-1} \geq a_{n-3} t^{n-3} \geq \cdots \geq a_1 t > 0 \quad \text{if } n \text{ is even},$$

we have

$$\begin{aligned} \sum_{j=0}^{n+1} \left| \frac{t^2 a_j - a_{j-2}}{a_n t^{n-j+1}} \right| &= \frac{1}{|a_n| t^{n-1}} \sum_{j=0}^{n+1} \left| t^j a_j - t^{j-2} a_{j-2} \right| \\ &= \frac{1}{a_n t^{n-1}} \sum_{j=0}^{n+1} \left(t^j a_j - t^{j-2} a_{j-2} \right) = \frac{1}{a_n t^{n-1}} a_n t^n = t. \end{aligned}$$

Hence by Corollary 7 with $q \to \infty$ and $p \to 1$, it follows that all the zeros of $P(z)$ lie in $|z| \leq t$. ■

This in particular shows that Corollary 8 is an improvement over Theorem 4, as we only assume the alternate coefficients to satisfy the given hypothesis. Very recently Rather et al. [8] proved the following generalization of Eneström-Kakeya Theorem.

Theorem 9 *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real coefficients such that for some* $k_j \geq 1, j = 1, 2, ..., r,$ *where* $1 \leq r \leq n$,

$$k_1 a_n \geq k_2 a_{n-1} \geq k_3 a_{n-2} ... \geq k_r a_{n-r+1} \geq a_{n-r} \geq ... \geq a_1 \geq a_0.$$

Then all the zeros of $P(z)$ *lie in*

$$|z + k_1 - 1 - (k_2 - 1)a_{n-1}/a_n| \leq \frac{1}{|a_n|}\bigg(k_1 a_n - (k_2 - 1)|a_{n-1}| + 2\sum_{j=2}^{r}(k_j - 1)|a_{n-j+1}| - a_0 + |a_0|\bigg).$$

For the study of further generalizations of Eneström-Kakeya Theorem (see [9]). In the next result we obtain a ring shaped region containing all the zeros of $P(z)$. In this case we prove the result which generalizes as well as improves the result of Joyal et al. [6].

Theorem 10 *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real coefficients, such that for some* $k_j \geq 1, j = 0, 1, 2, ..., r,$ *where* $0 \leq r \leq n - 1$,

$$k_0 a_n \geq k_1 a_{n-1} \geq ... \geq k_{r-1} a_{n-r+1} \geq k_r a_{n-r} \geq a_{n-r-1} ... \geq a_1 \geq a_0.$$

Then all the zeros of $P(z)$ *lie in the disk* $|z| \leq r_0$, *where*

$$r_0 = \frac{b}{2}\bigg(\frac{1}{|a_n|} - \frac{1}{M}\bigg) + \bigg(\frac{b^2}{4}(\frac{1}{|a_n|} - \frac{1}{M})^2 + \frac{M}{|a_n|}\bigg)^{\frac{1}{2}},$$

$$b = -(k_0 - 1)a_n + (k_0 a_n - k_1 a_{n-1}) + (k_1 - 1)a_{n-1}$$

and

$$M = (k_0 - 1)|a_n| + k_0 a_n + 2\sum_{j=1}^{r}(k_j - 1)|a_{n-j}| - a_n + |a_0|.$$

Proof. Define a function

$$\begin{aligned}
H(z) &= (1 - z)P(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{n-r} - a_{n-r-1})z^{n-r} + ... + (a_1 - a_2)z + a_0 \\
&= -a_n z^{n+1} + \big\{(k_0 a_n - k_1 a_{n-1}) - (k_0 - 1)a_n + (k_1 - 1)a_{n-1}\big\}z^n \\
&+ \big\{(k_1 a_{n-1} - k_2 c_{n-2}) - (k_1 - 1)a_{n-1} + (k_2 - 1)a_{n-2}\big\}z^{n-1} + ... \\
&+ \big\{(k_{r-1} a_{n-r+1} - k_r a_{n-r}) - (k_{r-1} - 1)a_{n-r+1} + (k_r - 1)a_{n-r}\big\}z^{n-r-1} \\
&+ \big\{(k_r a_{n-r} - a_{n-r-1}) - (k_r - 1)a_{n-r}\big\}z^{n-r} + (a_{n-r-1} - a_{n-r-2})z^{n-r-1} + ... \\
&+ (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0.
\end{aligned}$$

Thus

$$H(z) = -a_0 z^{n+1} + T(z), \tag{5}$$

where

$$\begin{aligned}
T(z) &= \big\{(k_0 a_n - k_1 a_{n-1}) - (k_0 - 1)a_n + (k_1 - 1)a_{n-1}\big\}z^n \\
&+ \big\{(k_1 a_{n-1} - k_2 c_{n-2}) - (k_1 - 1)a_{n-1} + (k_2 - 1)a_{n-2}\big\}z^{n-1} + ... \\
&+ \big\{(k_{r-1} a_{n-r+1} - k_r a_{n-r}) - (k_{r-1} - 1)a_{n-r+1} + (k_r - 1)a_{n-r}\big\}z^{n-r-1} \\
&+ \big\{(k_r a_{n-r} - a_{n-r-1}) - (k_r - 1)a_{n-r}\big\}z^{n-r} + (a_{n-r-1} - a_{n-r-2})z^{n-r-1} + ... \\
&+ (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0.
\end{aligned}$$

Therefore,

$$G(z) = z^n T(\frac{1}{z}) = \{(k_0 a_n - k_1 a_{n-1}) - (k_0 - 1)a_n + (k_1 - 1)a_{n-1}\}$$
$$+ \{(k_1 a_{n-1} - k_2 c_{n-2}) - (k_1 - 1)a_{n-1} + (k_2 - 1)a_{n-2}\}z + ...$$
$$+ \{(k_{r-1} a_{n-r+1} - k_r a_{n-r}) - (k_{r-1} - 1)a_{n-r+1} + (k_r - 1)a_{n-r}\}z^{r+1}$$
$$+ \{(k_r a_{n-r} - a_{n-r-1}) - (k_r - 1)a_{n-r}\}z^r + (a_{n-r-1} - a_{n-r-2})z^{r+1} + ...$$
$$+ (a_2 - a_1)z^{n-2} + (a_1 - a_0)z^{n-1} + a_0 z^n.$$

Clearly

$$G(0) = (k_0 a_n - k_1 a_{n-1}) + (k_1 - 1)a_{n-1} - (k_0 - 1)a_n = b \text{ (say)}.$$

Also for $|z| = 1$,

$$|G(z)| \leq (k_0 - 1)|a_n| + k_0 a_n + 2\sum_{j=1}^{r}(k_j - 1)|a_{n-j}| - a_n + |a_0| = M.$$

Therefore using Lemma 2 for $G(z)$, we get

$$|G(z)| = z^n T\big(\frac{1}{z}\big) \leq M\frac{M + b|z|}{b + M|z|}. \tag{6}$$

If $|z| > 1$, we get from (6)

$$|T(z)| \leq M|z|^n \frac{M + b|z|}{b + M|z|}.$$

Hence for $|z| = R > 1$, we get from (5)

$$\begin{aligned}|H(z)| &\geq |a_n|R^{n+1} - |T(z)|\\ &\geq |a_n|R^{n+1} - MR^n \frac{M + b|z|}{b + M|z|}\\ &= \frac{R^n}{MR + b}\Bigg\{M|a_n|R^2 - bR(M - |a_n|) - M^2\Bigg\}\\ &> 0.\end{aligned}$$

If

$$R > \frac{b}{2}\left(\frac{1}{|a_n|} - \frac{1}{M}\right) + \left(\frac{b^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M}\right)^2 + \frac{M}{|a_n|}\right)^{\frac{1}{2}} = r_0.$$

That is all zeros of $P(z)$ lie in a disk $|z| \leq r_0$. This completely proves the theorem. ■

Remark 1 *We assume* $\frac{M}{|a_n|} \geq r_0$, *that is*

$$\frac{M}{|a_n|} \geq \frac{b}{2}\left(\frac{1}{|a_n|} - \frac{1}{M}\right) + \left(\frac{b^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M}\right)^2 + \frac{M}{|a_n|}\right)^{\frac{1}{2}}.$$

If

$$2M^2 \geq b(M - |a_n|) + \Big(b^2(M - |a_n|)^2 + 4M^3|a_n|\Big)^{\frac{1}{2}}.$$

This holds if $(M-b)(M-|a_n|) \geq 0$. *Since above inequality holds under the hypothesis of given theorem. Thus* $r_0 \leq \frac{M}{|a_n|}$, *in particular taking all* $k_j = 1, j = 0, 1, 2...r$, *we get*

$$|z| \leq r_0 \leq \frac{a_n - a_0 + |a_0|}{|a_n|},$$

a refinement of the result of Joyal et al. ([6], Theorem 3).

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