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Local Stability Analysis Of Epidemic Models Using A Corollary Of Gershgorin's Circle Theorem*

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Abstract

The techniques and methods that help to obtain necessary and sufficient conditions to determine the local stability of linearized systems are paramount. In this paper, a corollary of the Gershgorin's circle theorem was used to establish the local stability of different epidemic models with three or more states including, a Tuberculosis model, a SEIRS model, a vector-host model and Staged HIV/AIDS Model. It was observed that no matter the state or the dimension of the system or matrix, this corollary can be used to analyse the local stability for both disease-free and endemic equilibria, by establishing that when $\mathcal{R}_0 < 1$, the Jacobian matrix evaluated at the disease free equilibrium will have negative eigenvalues or negative real part eigenvalues where $\mathcal{R}_0$ is the basic reproduction number of the tuberculosis model. Thus, the disease-free equilibrium is stable but when $\mathcal{R}_0 > 1$, the Jacobian matrix evaluated at the endemic equilibrium will have negative eigenvalues or negative real part eigenvalues making the endemic equilibrium is stable.

1 Introduction

The stability of an equilibrium point (stationary state) of a mathematical model for an infectious disease helps to determine whether the solutions remain near the equilibrium point or get further away or not. The equilibrium point can be either stable or unstable or a saddle point [9, 12].

The main method used to analyze the local stability of the equilibrium points of epidemic models is the Lyapunov's indirect method that is, to determine whether the eigenvalues of the Jacobian matrix evaluated at the equilibrium points of the system are negative or have negative real part (that is, equilibrium points lie in the left half of the complex plane). Since the characteristic equation for an n-dimensional system is a polynomial equation of degree n for which it may be difficult or impossible to find all roots explicitly, different methods such as the Routh-Hurwitz criterion gives necessary and sufficient conditions for the eigenvalues to lie in the left half of the complex plane. In this case, the reproduction number can be obtained from the constant term. Whether the reproduction number is greater or less than 1 determines the sign of the constant term [10]. In most of these methods, it is complicated to apply in problem of many dimensions [4].

The local asymptotic stability analysis using Routh-Hurwitz criteria involves calculation of the characteristic equation and the determinants of the n Hurwitz matrices. The Routh-Hurwitz condition also becomes intractable especially for large n [1].

In [2], the authors investigated the local stability of the disease-free equilibrium point of different epidemic models with four states.

In this study, we investigate the local stability of the disease-free and endemic equilibrium points of some selected epidemic model with four or more states using a corollary of Gershgorin's circle theorem. The Gershgorin's theorem also provides sufficient conditions for the eigenvalues to lie in the left half of the complex plane [1, 13, 11]. That is, the local stability can be established without the need to calculate the eigenvalues, instead the basic reproduction number which describes averagely the number of secondary

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infections produced by one infected individual in a susceptible population during their whole infectious period gives a condition for an equilibrium point to be stable is used for the analysis. Informally, the theorem says that if the off-diagonal entries of a square matrix over the complex numbers have small norms then its eigenvalues are similar in norm to the diagonal entries of the matrix. This theorem is a very useful tool in numerical analysis, particularly in perturbation theory [8].

Corollary 1 (Corollary of Gershgorin Circle Theorem) *Let A be an $n \times n$ matrix with real entries. If the diagonal elements a_{ii} of A satisfy*

$$a_{ii} < -r_i$$

where

$$r_i = \sum_{j=1, j\neq i}^{n} \mid a_{ij} \mid \tag{1}$$

for $i = 1, ..., n$, then the eigenvalues of A are negative or have negative real parts [1].

The following theorems will be used in the analysis of the equilibrium points of the selected models.

Theorem 2 *The disease-free equilibrium is locally asymptotically stable if $\mathcal{R}_0 < 1$.*

Theorem 3 *The endemic equilibrium is locally asymptotically stable if $\mathcal{R}_0 > 1$.*

2 Tuberculosis Model

Using the tuberculosis model in [5], the population under consideration is sub-divided into three epidemiological classes: susceptible individuals S, latent or exposed infected individuals E, and infectious individuals I. The incidence rate given by βSI (using the mass action law). A portion $p\beta SI$ gives rise to immediate active cases (fast progression), while the rest $(1-p)\beta SI$ gives rise to latent-TB cases with a low risk of progressing to active TB (slow progression). The progression rate from latent TB to active TB is assumed to be proportional to the number of latent-TB cases, that is, it is given by κE. The total incidence rate is $p\beta SI + \kappa E$. The model is given by the following system:

$$\dot{S} = \Lambda - \beta SI - \mu S, \qquad \dot{E} = (1-p)\beta SI - \kappa E - \mu E, \qquad \dot{I} = p\beta SI + \kappa E - \mu I - \delta I. \tag{2}$$

Let Λ be recruitment rate of susceptible individuals, μ be the natural death rate, β be transmission rate of active TB, κ be progression rate from latent TB to active TB (Rate of slow progression), δ death rate due to TB infection and p rate of fast progression.

2.1 The Equilibrium Points

The equilibrium points of model (2) are

1. disease-free equilibrium point (P^0) given as

$$\left(S^0, E^0, I^0\right) = \left(\frac{\Lambda}{\mu}, 0, 0\right).$$

The basic reproduction number $\mathcal{R}_0$ was computed using the Next Generation Matrix approach in [7] and it is given as

$$\mathcal{R}_0 = \frac{\beta\Lambda(\kappa + \mu p)}{\mu(\kappa+\mu)(\mu+\delta)}. \tag{3}$$

2. The endemic equilibrium point (P^*) can now be expressed in terms of $\mathcal{R}_0$ as

$$(P^*) = (S^*, E^*, I^*) = \left(\frac{\Lambda}{\mu\mathcal{R}_0}, -\frac{(p-1)(\mathcal{R}_0-1)\mu(\mu+\delta)}{(\mu p+\kappa)\beta}, \frac{(\mathcal{R}_0-1)\mu}{\beta}\right).$$

2.2 Local Stability Analysis of the Equilibrium Points

The Jacobian matrix J of the system (2) is

$$J = \begin{bmatrix} -\beta I - \mu & 0 & -\beta S \\ (1-p)\beta I & -(\kappa+\mu) & (1-p)\beta S \\ p\beta I & \kappa & p\beta S - \delta - \mu \end{bmatrix}. \tag{4}$$

2.3 Local Stability Analysis of the Disease-Free Equilibrium

Evaluating the matrix (4) at the disease-free equilibrium gives

$$J_0 = \begin{bmatrix} -\mu & 0 & -\dfrac{\beta\Lambda}{\mu} \\ 0 & -(\kappa+\mu) & \dfrac{(1-p)\beta\Lambda}{\mu} \\ 0 & \kappa & \dfrac{p\beta\Lambda}{\mu} - (\mu+\delta) \end{bmatrix}. \tag{5}$$

The matrix J_0 has one eigenvalue which is negative that is, $-\mu$. The remaining sub-matrix is given by

$$J_r = \begin{bmatrix} -(\kappa+\mu) & \dfrac{(1-p)\beta\Lambda}{\mu} \\ \kappa & \dfrac{p\beta\Lambda}{\mu} - (\mu+\delta) \end{bmatrix}.$$

According to Corollary 1, the matrix (J_r) will have negative eigenvalues if the following inequalities are satisfied:

$$(\kappa+\mu) > \frac{(1-p)\beta\Lambda}{\mu}, \tag{6a}$$

$$-\frac{p\beta\Lambda}{\mu} + (\mu+\delta) > \kappa. \tag{6b}$$

Dividing (6a) through by $(\kappa+\mu)$ and (6b) by κ yields

$$1 > \frac{(1-p)\beta\Lambda}{\mu(\kappa+\mu)}, \tag{7}$$

$$\frac{-p\beta\Lambda + \mu(\mu+\delta)}{\mu\kappa} > 1. \tag{8}$$

From (7) and (8). After expanding and simplifying yields,

$$\begin{aligned} & \frac{-p\beta\Lambda + \mu(\mu+\delta)}{\mu\kappa} > 1 > \frac{(1-p)\beta\Lambda}{\mu(\kappa+\mu)} \\ \Rightarrow \quad & 1 > \frac{(\kappa+\mu p)\beta\Lambda}{\mu(\delta+\mu)(\kappa+\mu)} \\ \Rightarrow \quad & 1 > \mathcal{R}_0 \\ \Rightarrow \quad & \mathcal{R}_0 < 1. \end{aligned}$$

Therefore, we conclude that from the above proof the disease-free equilibrium (E_0) is locally asymptotically stable.

2.4 Local Stability Analysis of the Endemic Equilibrium Point

Evaluating the matrix (4) at the endemic equilibrium gives

$$J_e^s = \begin{bmatrix} -(\mathcal{R}_0-1)\mu-\mu & 0 & -\dfrac{\beta\Lambda}{\mathcal{R}_0\mu} \\ -(1-p)(1-\mathcal{R}_0)\mu & -(\kappa+\mu) & \dfrac{(1-p)\beta\Lambda}{\mathcal{R}_0\mu} \\ -p(1-\mathcal{R}_0)\mu & \kappa & \dfrac{p\beta\Lambda}{\mathcal{R}_0\mu}-(\mu+\delta) \end{bmatrix}. \tag{9}$$

According to Corollary 1, the matrix (J_e^s) will have negative eigenvalues if the following inequalities are satisfied:

$$(\mathcal{R}_0-1)\mu+\mu > \frac{\beta\Lambda}{\mathcal{R}_0\mu}, \tag{10a}$$

$$(\kappa+\mu) > (1-p)(1-\mathcal{R}_0)\mu + \frac{(1-p)\beta\Lambda}{\mathcal{R}_0\mu}, \tag{10b}$$

$$-\frac{p\beta\Lambda}{\mathcal{R}_0\mu}+(\mu+\delta)-p(1-\mathcal{R}_0)\mu > \kappa. \tag{10c}$$

Dividing (10b) through by $(\kappa+\mu)$ and (10c) by κ gives

$$1 > \frac{(1-p)\mathcal{R}_0\mu(1-\mathcal{R}_0)\mu+(1-p)\beta\Lambda}{\mathcal{R}_0\mu(\kappa+\mu)}, \tag{11}$$

$$\frac{-p\beta\Lambda+(\mu+\delta)\mathcal{R}_0\mu-p\mathcal{R}_0\mu(1-\mathcal{R}_0)\mu}{\mathcal{R}_0\mu\kappa} > 1. \tag{12}$$

From (11) and (12),

$$\begin{aligned} \frac{-p\beta\Lambda+(\mu+\delta)\mathcal{R}_0\mu-p\mathcal{R}_0\mu(1-\mathcal{R}_0)\mu}{\mathcal{R}_0\mu\kappa} &> 1 \\ &> \frac{(1-p)\mathcal{R}_0\mu(1-\mathcal{R}_0)\mu+(1-p)\beta\Lambda}{\mathcal{R}_0\mu(\kappa+\mu)}. \end{aligned} \tag{13}$$

Expanding and simplifying (13) gives

$$0 > \mu(1-\mathcal{R}_0)(\mu p+\kappa) \;\Rightarrow\; \mathcal{R}_0 > 1.$$

This shows that the Endemic Equilibrium point is locally asymptotically stable since $\mathcal{R}_0 > 1$.

3 SEIRS Model

The SEIRS model consists of four compartments, but the individual loses immunity after some time and moves back into the S class (that is, the individual becomes susceptible again). Let S be the number of susceptible individuals, E, be the number of exposed individuals, (infected but are not yet infectious), I, be the number of infectious individuals, and R, is the number of recovered individuals, (with temporary immunity). Furthermore, let the contact rate be given by β, Λ is the recruitment rate, μ is the birth rate (equal to the natural death rate), κ be the progression rate from E to I, γ the recovery rate, δ is the additional rate of disease-induced mortality, ρ is the rate of loss of immunity, α is the vaccination rate.

$$\begin{aligned} \dot{S} &= \Lambda-\beta IS-(\mu+\alpha)S+\rho R, \quad \dot{E}=\beta IS-(\kappa+\mu)E, \\ \dot{I} &= \kappa E-(\mu+\gamma+\delta)I, \qquad \dot{R}=\gamma I-(\mu+\rho)R+\alpha S. \end{aligned} \tag{14}$$

The system (14) has two equilibrium points:

(i) a disease-free equilibrium point P^0, given by

$$P^0 = (S^0, E^0, I^0, R^0) = (\frac{\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)}, 0, 0, \frac{\Lambda\alpha}{\mu(\alpha+\mu+\rho)}).$$

The Basic Reproduction number $\mathcal{R}_0$ was computed using the Next Generation Matrix approach given as

$$\mathcal{R}_0 = \frac{\Lambda}{\mu}\frac{\kappa}{(\kappa+\mu)}\frac{(\mu+\rho)}{(\alpha+\mu+\rho)}\frac{\beta}{(\gamma+\mu+\delta)}.$$

(ii) endemic equilibrium point $P^* = (S^*, E^*, I^*, R^*)$ expressed in terms of $\mathcal{R}_0$ as

$$\begin{aligned}
S^* &= \frac{\Lambda(\mu+\rho)}{\mathcal{R}_0\mu(\alpha+\mu+\rho)},\\
E^* &= \frac{(\mu+\gamma+\delta)(\mathcal{R}_0-1)\mu[(\mu+\kappa)(\alpha+\mu+\rho)(\gamma+\mu+\delta)]}{\alpha\kappa(\delta+\mu)+\mu((\alpha+\mu+\rho)(\gamma+\mu+\delta))},\\
I^* &= \frac{(\mathcal{R}_0-1)\mu[(\mu+\kappa)(\alpha+\mu+\rho)(\gamma+\mu+\delta)]}{\alpha\kappa(\delta+\mu)+\mu((\alpha+\mu+\rho)(\gamma+\mu+\delta))},\\
R^* &= \frac{\mu\mathcal{R}_0\kappa\gamma A_1(\gamma+\mu+\delta)}{\mu(\alpha+\mu+\rho)\alpha\kappa(\delta+\mu)+\mu((\alpha+\mu+\rho)(\gamma+\mu+\delta))},
\end{aligned}$$

where $A_1 = (\alpha+\mu+\rho) - (-\delta\kappa\rho - \delta\mu\rho + \gamma\kappa\mu - \gamma\mu\rho - \kappa\mu\rho - \mu^2\rho)]\mu(\mu+\kappa)(\alpha+\mu+\rho)$.

3.1 Local Stability Analysis for the Equilibrium Points

The Jacobian matrix J for the system (14) give by

$$J = \begin{bmatrix} -\beta I - (\mu+\alpha) & 0 & -\beta S & \rho \\ \beta I & -(\kappa+\mu) & \beta S & 0 \\ 0 & \kappa & -(\gamma+\mu+\delta) & 0 \\ \alpha & 0 & \gamma & -(\mu+\rho) \end{bmatrix}. \tag{15}$$

3.2 Local Stability Analysis for the Disease-free Equilibrium Point

Evaluating the matrix J at the disease-free equilibrium gives

$$J_0 = \begin{bmatrix} -(\mu+\alpha) & 0 & -\frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)} & \rho \\ 0 & -(\kappa+\mu) & \frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)} & 0 \\ 0 & \kappa & -(\gamma+\mu+\delta) & 0 \\ \alpha & 0 & \gamma & -(\mu+\rho) \end{bmatrix}. \tag{16}$$

According to Corollary 1, the matrix (J_0) will have negative eigenvalues if the following inequalities are satisfied

$$(\mu+\alpha) > \frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)} + \rho, \tag{17a}$$

$$(\kappa+\mu) > \frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)}, \tag{17b}$$

$$(\gamma+\mu+\delta) > \kappa, \tag{17c}$$

$$(\mu+\rho) > (\alpha+\gamma). \tag{17d}$$

Combining (17b) and (17d) gives

$$1 > \frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)(\mu+\kappa)} \tag{18}$$

and

$$\frac{(\gamma+\mu+\delta)}{\kappa} > 1. \tag{19}$$

From (18) and (19) we have

$$\frac{(\gamma+\mu+\delta)}{\kappa} > 1 > \frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)(\mu+\kappa)}.$$

It follows that

$$1 > \frac{\beta\Lambda(\mu+\rho)\kappa}{\mu(\alpha+\mu+\rho)(\mu+\kappa)(\gamma+\mu+\delta)} = \mathcal{R}_0.$$

This shows that the Disease-free equilibrium point is locally asymptotically stable since $\mathcal{R}_0 < 1$.

3.3 Local Stability Analysis for the Endemic Equilibrium

The Jacobian matrix J evaluated at the endemic equilibrium gives

$$J_1 = \begin{bmatrix} b-(\alpha+\mu) & 0 & -\dfrac{(\mu+\kappa)(\mu+\gamma+\delta)}{\kappa} & \rho \\ -b & -(\kappa+\mu) & \dfrac{(\mu+\kappa)(\mu+\gamma+\delta)}{\kappa} & 0 \\ 0 & \kappa & -(\gamma+\mu+\delta) & 0 \\ \alpha & 0 & \gamma & -(\mu+\rho) \end{bmatrix}, \tag{20}$$

where

$$b = \frac{(\alpha+\mu+\rho)(\mu(\mu+\kappa))(\mu+\gamma+\delta)(1-\mathcal{R}_0)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))}.$$

According to Corollary 1, the matrix (J_0) will have negative eigenvalues if the following inequalities are satisfied

(i^*)

$$\frac{(\alpha+\mu+\rho)(\mu(\mu+\kappa))(\mu+\gamma+\delta)(\mathcal{R}_0-1)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))} + (\alpha+\mu) > \frac{(\mu+\kappa)(\mu+\gamma+\delta)}{\kappa} + \rho.$$

(ii^*)

$$(\kappa+\mu) > \frac{(\alpha+\mu+\rho)(\mu(\mu+\kappa))(\mu+\gamma+\delta)(1-\mathcal{R}_0)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))} + \frac{(\mu+\kappa)(\mu+\gamma+\delta)}{\kappa}.$$

(iii^*) $(\mu+\delta+\gamma) > \kappa$.

(iv^*) $(\mu+\rho) > \alpha+\gamma$.

Dividing (ii^*) through $(\kappa+\mu)$ and (iii^*) by κ, and simplifying gives

$$1 > \frac{(\alpha+\mu+\rho)(\mu)(\mu+\gamma+\delta)(1-\mathcal{R}_0)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))} + \frac{(\mu+\gamma+\delta)}{\kappa} \tag{21}$$

and

$$\frac{(\mu+\delta+\gamma)}{\kappa} > 1. \tag{22}$$

The above inequality holds if

$$0 > 1-\mathcal{R}_0,$$

that is,

$$\mathcal{R}_0 > 1.$$

The endemic equilibrium point is locally asymptotically stable since $\mathcal{R}_0 > 1$.

4 7-Staged States Vector-Host Model

In this section, a malaria modelled as an $SEIRS$ for the host population and SEI for the vector population similar to that of [6] and a Zika virus model in [3] are analyzed.

4.1 The malaria model

The malaria model divides the human population into 4 classes: Susceptible, S_h, then comes the latent or exposed, L_h, I_h, people who have been infected and are capable of spreading the disease to those in the susceptible class and finally, the recovered (immune), R_h, people who recover from the infection through clinical treatment with temporary immunity . The recovered humans have some immunity to the disease and do not get clinically ill, but after some period of time, they lose their immunity and return to the susceptible class.

The vector population is divided into 3 classes: Susceptible, S_v, Latent or Exposed, L_v and Infectious, I_v. Let μ_h be Humans birth rate = Humans death rate, κ be transition rate from Latent class to infectious class at time t, γ be recovery rate of human, β_h be transmission rate of host (bite rate plus probability of transmission of disease), δ be Disease-induced death rate for humans, ρ be Rate of loss of immunity for humans, μ_c be Vector birth rate = vector death rate.

$$\begin{aligned}
\dot{S_h} &= \mu_h - \beta_h S_h I_v - \alpha S_h + \rho R_h - \mu_h S_h,\\
\dot{E_h} &= \beta_h S_h I_v - (\kappa + \mu_h) E_h,\\
\dot{I_h} &= \kappa E_h - (\gamma + \mu_h + \delta) I_h,\\
\dot{R_h} &= \gamma I_h - \mu_h R_h + \alpha S_h - \rho R_h,\\
\dot{S_v} &= \mu_c - \beta_v S_v I_h - \mu_c S_v,\\
\dot{E_v} &= \beta_v S_v I_h - \theta E_v - \mu_c E_v,\\
\dot{I_v} &= \theta E_v - \mu_c I_v.
\end{aligned} \tag{23}$$

4.2 Equilibrium Points of the Model and Basic Reproduction Number ($\mathcal{R}_0$)

The system (23) has two equilibrium points namely;

- The Disease-Free equilibrium; and
- The Endemic equilibrium.

4.2.1 The Disease-Free Equilibrium

The disease-Free equilibrium of the system (23) is given by

$$p^0 = (S_h^0, E_h^0, I_h^0, R_h^0, S_v^0, E_v^0, I_v^0) = \left(\frac{(\mu_h + \rho)}{(\alpha + \mu_h + \rho)}, 0, 0, \frac{\alpha}{(\alpha + \mu_h + \rho)}, 1, 0, 0 \right).$$

4.2.2 The Basic Reproduction Number ($\mathcal{R}_0$)

The basic reproduction number ($\mathcal{R}_0^{NG}$) of the model was computed using the next generation matrix approach discussed in [7].The basic reproduction for (23) is given as

$$\mathcal{R}_0^{NG} = \sqrt{\frac{\beta_h(\mu_h + \rho)\kappa}{(\kappa + \mu_h)(\mu_h + \gamma + \delta)(\alpha + \mu_h + \rho)} . \frac{\beta_v \theta}{(\theta + \mu_c)\mu_c}}.$$

According to [10], the square of the reproduction number obtained via the next-generation approach [7]:

$$(\mathcal{R}_0^{NG})^2 = \mathcal{R}_0$$

is the basic reproduction number ($\mathcal{R}_0$) of the model. Therefore, the basic reproduction number for (23) is given as

$$\mathcal{R}_0 = \frac{\beta_h(\mu_h+\rho)\kappa}{(\kappa+\mu_h)(\mu_h+\gamma+\delta)(\alpha+\mu_h+\rho)}.\frac{\beta_v\theta}{(\theta+\mu_c)\mu_c}. \tag{24}$$

Now, for a population where there is no prevention effort, $\alpha = 0$, the basic reproduction number is represented by

$$\mathcal{R}_0' = \frac{\beta_h\kappa}{(\kappa+\mu_h)(\mu_h+\gamma+\delta)}.\frac{\beta_v\theta}{(\theta+\mu_c)\mu_c}. \tag{25}$$

4.2.3 Endemic Equilibrium

The solution for the endemic equilibrium is obtained in terms of the infected humans which is also expressed in terms of $\mathcal{R}_0$ and after some algebraic manipulation, we have

$$\begin{aligned}
S_h^* &= \frac{\mu_h(\theta+\mu_c)(I_h^*\beta_v+\mu_c)}{(\alpha+\mu_c)(\theta+\mu_c)(I_h^*\beta_v+\mu_c)+\theta I_h^*\beta_v\beta_h},\\
E_h^* &= \frac{\theta I_h^*\beta_v\beta_h\mu_h}{(\kappa+\mu_h)(\mu_c(\theta+\mu_c)(\alpha+\mu_h)+I_h^*\beta_v(\theta\beta_h(\theta+\mu_c)(\alpha+\mu_h))},\\
I_h^* &= \frac{\mathcal{R}_0-1}{\beta_v(\gamma+\delta+\mu_h)(\kappa+\mu_h)(\theta\beta_h+(\theta+\mu_c)(\alpha+\mu_h+\rho))},\\
R_h^* &= \frac{(\gamma I_h+\alpha+\mu_h)(\theta+\mu_c)(I_h^*\beta_v+\mu_c)+(\theta I_h^*\beta_v)}{\mu_h(\theta+\mu_c)(I_h^*\beta_v+\mu_c)},\\
S_v^* &= \frac{\mu_h}{(I_h^*\beta_v+\mu_c)}, E_v^* = \frac{I_h^*\beta_v\mu_c}{(\theta+\mu_c)(I_h^*\beta_v+\mu_c)},\\
I_v^* &= \frac{\theta I_h^*\beta_v}{(\theta+\mu_c)(I_h^*\beta_v+\mu_c)}.
\end{aligned}$$

4.3 Local Stability Analysis of the Vector-Host Model

The Jacobian matrix for the system (23) is given as

$$J = \begin{bmatrix}
n^* & 0 & 0 & \rho & 0 & 0 & -\beta_h S_h\\
\beta_h I_v & -(\kappa+\mu_h) & 0 & 0 & 0 & 0 & \beta_h S_h\\
0 & \kappa & -(\mu_h+\gamma+\delta) & 0 & 0 & 0 & 0\\
\alpha & 0 & \gamma & -(\mu_h+\rho) & 0 & 0 & 0\\
0 & 0 & -\beta_v S_v & 0 & -\beta_v I_h-\mu_c & 0 & 0\\
0 & 0 & \beta_v S_v & 0 & \beta_v I_h & -(\theta+\mu_c) & 0\\
0 & 0 & 0 & 0 & 0 & \theta & -\mu_c
\end{bmatrix} \tag{26}$$

where $n^* = \beta_h I_v - (\alpha+\mu_h)$.

4.4 Local Stability Analysis of the Disease-Free Equilibrium Point

Now, we investigate the local stability of the disease-free equilibrium point with $\alpha = 0$, since the basic reproduction for the model without prevention is always greater than the basic reproduction for the model

with prevention, that is, $\mathcal{R}_0 < \mathcal{R}_0^{'}$. Evaluating the Jacobian matrix at the disease equilibrium point (E_0) yields

$$J_0 = \begin{bmatrix} -\mu_h & 0 & 0 & \rho & 0 & 0 & -\beta_h \\ 0 & -(\kappa+\mu_h) & 0 & 0 & 0 & 0 & \beta_h \\ 0 & \kappa & -(\mu_h+\gamma+\delta) & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & -(\mu_h+\rho) & 0 & 0 & 0 \\ 0 & 0 & -\beta_v & 0 & -\mu_c & 0 & 0 \\ 0 & 0 & \beta_v & 0 & 0 & -(\theta+\mu_c) & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta & -\mu_c \end{bmatrix}. \tag{27}$$

From rows 1, 4 and 5, we get three negatives eigenvalues which are $\lambda_1 = -\mu_h$, $\lambda_2 = -(\mu_h + \rho)$ and $\lambda_3 = -\mu_c$. Eliminating rows 1, 4 and 5 from J_0, we end up with the sub-matrix

$$J_s = \begin{bmatrix} -(\kappa+\mu_h) & 0 & 0 & \beta_h \\ \kappa & -(\mu_h+\gamma+\delta) & 0 & 0 \\ 0 & \beta_v & -(\theta+\mu_c) & 0 \\ 0 & 0 & \theta & -\mu_c \end{bmatrix}. \tag{28}$$

The matrix J_s satisfies the corollary of Gershgorin's circle theorem, if the following inequalities hold;

$$(\kappa+\mu_h) > \beta_h, \tag{29}$$

$$(\mu_h+\gamma+\delta) > \kappa, \tag{30}$$

$$\theta + \mu_c > \beta_v, \tag{31}$$

$$\mu_c > \theta. \tag{32}$$

Combining Eq. (29) to Eq. (32) gives

$$1 > \frac{\kappa\beta_h\theta\beta_v}{\mu_c(\theta+\mu_c)(\kappa+\mu_h)(\mu_h+\gamma+\delta)}.$$

Hence, from (25), $\mathcal{R}_0^{'} < 1$. Since $\mathcal{R}_0^{'} < 1$, it implies $\mathcal{R}_0 < 1$. Therefore, $\mathcal{R}_0 < 1$, ensures local stability of the disease-free equilibrium.

4.5 Local Stability Analysis of Endemic Equilibrium Point

The Jacobian matrix evaluated at the Endemic equilibrium (E_0) gives

$$J_e = \begin{bmatrix} a^* - \alpha - \mu_h & 0 & 0 & \rho & 0 & 0 & -b^* \\ -a^* & -(\kappa+\mu_h) & 0 & 0 & 0 & 0 & b^* \\ 0 & \kappa & -(\mu_h+\gamma+\delta) & 0 & 0 & 0 & 0 \\ \alpha & 0 & \gamma & -(\mu_h+\rho) & 0 & 0 & 0 \\ 0 & 0 & -c^* & 0 & d^* - \mu_c & 0 & 0 \\ 0 & 0 & c^* & 0 & h^* & -\theta - \mu_c & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta & -\mu_c \end{bmatrix} \tag{33}$$

where $h^* = -d$, $a^* = \frac{a_1}{b_1}$, $b^* = \frac{a_2}{b_2}$, $c^* = \frac{a_3}{b_3}$, $d^* = \frac{a_4}{b_4}$, $a_1 = \beta_h(I_h^*\beta_v\beta_h)$, $b_1 = (\theta+\mu_c)(I_h^*\beta_v + \mu_c)$,

$$a_2 = \beta_h\mu_h, b_2 = (\alpha+\mu_c)(\theta+\mu_c)(I_h^*\beta_v + \mu_c) + \theta I_h^*\beta_v\beta_h,$$

$a_3 = \beta_v\mu_h$, $b_3 = aaI_h^*\beta_v + \mu_c$, $a_4 = \beta_v(\mathcal{R}_0 - 1)$ and

$$b_4 = \beta_v(\gamma+\delta+\mu_h)(\kappa+\mu_h)(\theta\beta_h + (\theta+\mu_c)(\alpha+\mu_h+\rho)).$$

Using Corollary 1, the matrix (J_e) will have negative eigenvalues if the following inequalities are satisfied. That is,

$$\mu_h + \alpha > \rho + a^* + b^*, \tag{34}$$

$$1 > \frac{a^*}{\mu_h + \kappa} + \frac{b^*}{\mu_h + \kappa}, \tag{35}$$

$$\frac{\delta + \mu_h + \gamma}{\kappa} > 1, \tag{36}$$

$$\frac{\alpha + \gamma}{\mu_h + \rho} < 1, \tag{37}$$

$$\mu_c > d^* + c^*, \tag{38}$$

$$\theta + \mu_c > h^* + c^*, \tag{39}$$

$$\mu_c > \theta. \tag{40}$$

From Eq. (35) and Eq. (36) we get

$$1 > \frac{a^*\kappa}{(\mu_h + \kappa)(\delta + \mu_h + \gamma)} + \frac{b^*\kappa}{(\mu_h + \kappa)(\delta + \mu_h + \gamma)}. \tag{41}$$

Let

$$a^{**} = \frac{a^*\kappa}{(\mu_h + \kappa)(\delta + \mu_h + \gamma)} \quad \text{and} \quad b^{**} = \frac{b^*\kappa}{(\mu_h + \kappa)(\delta + \mu_h + \gamma)}.$$

Then Eq. (41) becomes

$$1 > a^{**} + b^{**}. \tag{42}$$

Adding Eq. (38) and Eq. (39) yields

$$\theta + 2\mu_c > 2c^*.$$

Since $h^* = -d^*$, we see that

$$2\theta + 2\mu_c > 2c^*. \tag{43}$$

From Eq. (40) and Eq. (43), we have

$$1 > \frac{c^*\theta}{(\theta + \mu_c)\mu_c}. \tag{44}$$

But

$$\frac{c_1^*\theta}{(\theta + \mu_c)\mu_c} = \frac{1}{b^{**}}.$$

This implies Eq. (44) becomes

$$-1 > -b^{**}. \tag{45}$$

Adding Eq. (42) and Eq. (45) gives $a^{**} < 0$. Simplifying a^{**} gives $0 > (1 - \mathcal{R}_0)$ and $\mathcal{R}_0 > 1$.

Therefore, $\mathcal{R}_0 > 1$, ensures local stability of the endemic equilibrium p^*.

5 A-Staged HIV/AIDS Model

We consider a sexually active population $N(t)$, divided into six compartments: $S(t)$, $I_1(t)$, $I_2(t)$, $I_3(t)$, $A(t)$ and $T(t)$. $S(t)$ represents the number of susceptible individuals; $I_1(t)$ represents the number of HIV-positive individuals in the asymptomatic stage of HIV infection; $I_2(t)$ represents the number of HIV-positive individuals in the pre-AIDS stage but not receiving antiretroviral (ARV) treatment, $I_3(t)$, $A(t)$ represents the number of individuals with full-blown AIDS but not receiving ARV treatment; $T(t)$ represents the number of individuals who are receiving ARV treatment; and $R(t)$, the removed class, represents the number of individuals who have changed their sexual habits sufficiently such that they are, literarily, 'immune' to HIV infection by sexual contact. Note that the individuals in the R class are people who take up safe sexual habits and maintain the habits for the rest of their lives. The significance of the removed R class is that it emphasizes the importance of prevention for a disease, such as HIV, that has no cure. Increasing the members in this class is one of the keys to controlling the spread of the disease. The population dynamics is given by the following equations:

$$\begin{aligned}
\dot{S} &= \Lambda - \Big(\frac{\beta_1 I_1}{N} + \frac{\beta_2 I_2}{N} + \frac{\beta_3 I_3}{N}\Big)S - (\mu+\rho)S,\\
\dot{I_1} &= \Big(\frac{\beta_1 I_1}{N} + \frac{\beta_2 I_2}{N} + \frac{\beta_3 I_3}{N}\Big)S - (\mu+\alpha_1)I_1,\\
\dot{I_2} &= \alpha_1 I_1 - (\mu+\alpha_2+\gamma_2)I_2,\\
\dot{I_3} &= \alpha_2 I_2 - (\mu+\alpha_3+\gamma_3)I_3,\\
\dot{A} &= \alpha_3 I_3 - (\mu+\delta+\gamma_A)A,\\
\dot{T} &= \gamma_2 I_2 + \gamma_3 I_3 + \gamma_A A - \mu T,
\end{aligned} \tag{46}$$

where $\beta_i = cp_i$, $i = 1, 2, 3$ is the product of the average number of sexual partners (c) and the probability (p_i) of the infection per partner with an infected individual in I_1, I_2 and I_3 respectively. Let β_1 be infection rate of primary infectious individual, β_2 be infection rate of asymptomatic infectious individual, β_3 be infection rate of symptomatic infectious individual, Λ be recruitment rate, μ be natural death rate, α_1 be progression from the I_1 to I_2, α_2 be progression from the I_2 to I_3, α_3 be progression from the I_3 to A, δ be disease-induced death, ρ be rate of removal of susceptible, α_2 be treatment rate of asymptomatic infectious individual, α_3 be treatment rate of symptomatic infectious individual, α_A be treatment rate for advance AIDS individual.

The total population, $N(t)$ is given by

$$N = S(t) + I_1(t) + I_2(t) + I_3(t) + A(t) + T(t). \tag{47}$$

5.1 Equilibrium Points of A-Staged HIV/AIDS Model

The model is made up of two equilibrium points.

(a) Disease free Equilibrium point given by

$$(S^0 = \frac{\Lambda}{\mu+\rho}, \quad I_1^0 = 0, \quad I_2^0 = 0, \quad I_3^0 = 0, \quad A^0 = 0, \quad T^0 = 0).$$

(b) Endemic Equilibrium point given by

$$S^* = \frac{\Lambda}{\lambda+\mu+\rho}, \tag{48}$$

$$\begin{aligned}
I_1^* &= \frac{\lambda}{\mu+\alpha_1}S^*, \qquad (49)\\
I_2^* &= \frac{\alpha_1}{(\mu+\alpha_2+\gamma_2)}I_1^*,
\end{aligned}$$

$$
\begin{aligned}
I_3^* &= \frac{\alpha_1}{(\mu+\alpha_2+\gamma_2)}\frac{\alpha_2}{(\mu+\alpha_3+\gamma_3)}I_1^*,\\
A^* &= \frac{\alpha_1}{(\mu+\alpha_2+\gamma_2)}\frac{\alpha_2}{(\mu+\alpha_3+\gamma_3)}\frac{\alpha_3}{(\mu+\delta+\gamma_A)}I_1^*,\\
T^* &= \Big[\frac{\gamma_2}{\mu}\frac{\alpha_1}{(\mu+\alpha_2+\gamma_2)}+\frac{\gamma_3}{\mu}\frac{\alpha_1}{(\mu+\alpha_2+\gamma_2)}\frac{\alpha_2}{(\mu+\alpha_3+\gamma_3)}\\
&\quad +\frac{\gamma_A}{\mu}\frac{\alpha_1}{(\mu+\alpha_2+\gamma_2)}\frac{\alpha_2}{(\mu+\alpha_3+\gamma_3)}\frac{\alpha_3}{(\mu+\delta+\gamma_A)}\Big],
\end{aligned}
\tag{50}
$$

where $\lambda = \dfrac{\beta_1 I_1}{N}+\dfrac{\beta_2 I_2}{N}+\dfrac{\beta_3 I_3}{N}$.

5.2 Basic Reproduction Number $\mathcal{R}_0$ of HIV/AIDS Model

The basic reproduction number for the model is given as

$$
\mathcal{R}_0 = \frac{\beta_1}{(\mu+\alpha_1)}+\frac{\alpha_1\beta_2}{(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)}+\frac{\alpha_1\alpha_2\beta_3}{(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)(\mu+\alpha_3+\gamma_3)}
$$

where

(1) $\dfrac{\beta_1}{(\mu+\alpha_1)}$ is the probability of an infective that progresses from I_1 to I_2,

(2) $\dfrac{\beta_2}{(\mu+\alpha_2+\gamma_2)}$ is the probability of an infective that progresses from I_3 to I_3,

(3) $\dfrac{\beta_3}{(\mu+\alpha_3+\gamma_3)}$ is the probability of an infective that progresses from I_3 to A.

From (49) we have

$$
\begin{aligned}
\frac{I_1^*}{S^*} &= \frac{\lambda}{(\mu+\alpha_1)},\\
\frac{I_1^*}{S^*} &= \frac{1}{(\mu+\alpha_1)}\frac{\beta_1 I_1+\beta_2 I_2+\beta_3 I_3}{N},\\
\frac{N^*}{S^*} &= \frac{1}{(\mu+\alpha_1)}\Big(\beta_1+\frac{\beta_2 I_2}{I_1^*}+\frac{\beta_3 I_3}{I_1^*}\Big),\\
\frac{N^*}{S^*} &= \frac{1}{(\mu+\alpha_1)}\Big(\beta_1+\frac{\alpha_1\beta_2}{(\mu+\alpha_2+\gamma_2)}+\frac{\alpha_1\alpha_2\beta_3}{(\mu+\alpha_2+\gamma_2)(\mu+\alpha_3+\gamma_3)}\Big),\\
\frac{N^*}{S^*} &= \mathcal{R}_0,\\
\mathcal{R}_0 &= \frac{N^*}{S^*}\\
&= \frac{S^*+I_1^*+I_2^*+I_3^*+A^*+T^*}{S^*}\\
&= 1+\lambda\frac{1}{(\mu+\alpha_1)}+\lambda\frac{\alpha_1}{(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)}+\lambda\frac{\alpha_1\alpha_2}{(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)(\mu+\alpha_3+\gamma_3)}+\\
&\quad \lambda\frac{\alpha_1\alpha_2\alpha_3}{(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)(\mu+\alpha_3+\gamma_3)(\mu+\delta+\gamma_A)}+\\
&\quad \lambda\frac{\alpha_1}{(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)}\Big[\frac{\gamma_A}{\mu}+\frac{\alpha_2\gamma_3}{\mu(\mu+\alpha_3+\gamma_3)}+\frac{\alpha_2\alpha_3\gamma_A}{\mu(\mu+\alpha_3+\gamma_3)(\mu+\delta+\gamma_A)}\Big],\\
\mathcal{R}_0-1 &= \lambda\pi,
\end{aligned}
$$

$$\lambda = \frac{(\mathcal{R}_0 - 1)}{\pi},$$

where π is the mean infective period given by

$$\begin{aligned}\pi = {} & \frac{1}{(\mu+\alpha_1)} + \frac{\alpha_1}{(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)} + \frac{\alpha_1\alpha_2}{(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)(\mu+\alpha_3+\gamma_3)} \\ & + \frac{\alpha_1\alpha_2\alpha_3}{(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)(\mu+\alpha_3+\gamma_3)(\mu+\delta+\gamma_A)} \\ & + \frac{\alpha_1}{(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)}\Big[\frac{\gamma_A}{\mu} + \frac{\alpha_2\gamma_3}{\mu(\mu+\alpha_3+\gamma_3)} + \frac{\alpha_2\alpha_3\gamma_A}{\mu(\mu+\alpha_3+\gamma_3)(\mu+\delta+\gamma_A)}\Big]\end{aligned}$$

by substituting λ into the endemic equilibrium point expressed in (49)-(50), we will obtain the endemic equilibrium in term of $\mathcal{R}_0$ to be

$$\begin{aligned} S^* &= \Big[\frac{\Lambda\pi}{(\mathcal{R}_0-1)+\mu+\rho}\Big], \\ I_1^* &= \Big[\frac{(\mathcal{R}_0-1)}{\pi(\mu+\alpha_1)}\Big]S^*, \\ I_2^* &= \Big[\frac{\alpha_1}{\pi(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)}\Big](\mathcal{R}_0-1)S^*, \\ I_3^* &= \Big[\frac{\alpha_1\alpha_2}{\pi(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)(\mu+\alpha_3+\gamma_3)}\Big](\mathcal{R}_0-1)S^*, \\ A^* &= \Big[\frac{\alpha_1\alpha_2\alpha_3}{\pi(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)(\mu+\alpha_3+\gamma_3)(\mu+\delta+\gamma_A)}\Big](\mathcal{R}_0-1)S^*, \\ T^* &= \frac{\alpha_1}{\pi(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)} \\ &\quad \Big[\frac{\gamma_2}{\mu} + \frac{\alpha_2\gamma_3}{\mu(\mu+\alpha_3+\gamma_3)} + \frac{\alpha_2\alpha_3\gamma_A}{\mu(\mu+\alpha_3+\gamma_3)(\mu+\delta+\gamma_A)}\Big](\mathcal{R}_0-1)S^*. \end{aligned}$$

Jacobian matrix of the system (46) is given as

$$J = \begin{pmatrix} n^* & -\frac{\beta_1}{N}S & -\frac{\beta_2}{N}S & -\frac{\beta_3}{N}S & 0 & 0 \\ m^* & \frac{\beta_1}{N}S-(\alpha_1+\mu) & \frac{\beta_2}{N}S & \frac{\beta_3}{N}S & 0 & 0 \\ 0 & \alpha_1 & -(\alpha_2+\gamma_2+\mu) & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & -(\alpha_3+\gamma_3+\mu) & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & -(\mu+\delta+\gamma_A) & 0 \\ 0 & 0 & \gamma_2 & \gamma_3 & \gamma_A & -\mu \end{pmatrix} \tag{51}$$

where

$$n^* = -\frac{\beta_1 I_1}{N} - \frac{\beta_2 I_2}{N} - \frac{\beta_3 I_3}{N} - (\mu+\rho) \quad \text{and} \quad m^* = \frac{\beta_1 I_1}{N} + \frac{\beta_2 I_2}{N} + \frac{\beta_3 I_3}{N}.$$

5.3 Local Stability Analysis at the Disease Free Equilibrium Point

The Jacobian matrix J evaluated at the disease free equilibrium point is given as

$$J_0 = \begin{pmatrix} -(\mu+\rho) & -\frac{\beta_1\mu}{(\mu+\rho)} & -\frac{\beta_2\mu}{(\mu+\rho)} & -\frac{\beta_3\mu}{(\mu+\rho)} & 0 & 0 \\ 0 & p^* & \frac{\beta_2\mu}{(\mu+\rho)} & \frac{\beta_3\mu}{(\mu+\rho)} & 0 & 0 \\ 0 & \alpha_1 & -(\alpha_2+\gamma_2+\mu) & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & -(\alpha_3+\gamma_3+\mu) & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & -(\mu+\delta+\gamma_A) & 0 \\ 0 & 0 & \gamma_2 & \gamma_3 & \gamma_A & -\mu \end{pmatrix} \tag{52}$$

where

$$p^* = \frac{\beta_1\mu}{(\mu+\rho)} - (\alpha_1+\mu).$$

The matrix J_0 has three negative eigenvalue, $\lambda_1 = \mu$, $\lambda_2 = (\mu+\rho)$, $\lambda_3 = (\mu+\delta+\gamma_A)$. The sub-matrix is given as

$$J_0^s = \begin{pmatrix} \frac{\beta_1\mu}{(\mu+\rho)} - (\alpha_1+\mu) & \frac{\beta_2\mu}{(\mu+\rho)} & \frac{\beta_3\mu}{(\mu+\rho)} \\ \alpha_1 & -(\alpha_2+\gamma_2+\mu) & 0 \\ 0 & \alpha_2 & -(\alpha_3+\gamma_3+\mu) \end{pmatrix}. \tag{53}$$

We now determine the eigenvalue of the sub-matrix J_0^s by applying a corollary of Gershgorin's circle theorem.

$$\begin{aligned} R_1 \quad &: \quad 1 > \frac{\beta_1\mu}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_2\mu}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_3\mu}{(\mu+\rho)(\alpha_1+\mu)}, \\ R_2 \quad &: \quad 1 > \frac{\alpha_1}{(\alpha_2+\gamma_2+\mu)}, \\ R_3 \quad &: \quad 1 > \frac{\alpha_2}{(\alpha_3+\gamma_3+\mu)}, \end{aligned}$$

where $R_i, i = 1, 2, 3$ stands for the rows in the matrix.

By multiplying the second term of the right hand side (rhs) of R_1 by the term obtained in the rhs of R_2 and also by multiplying the third term of rhs of R_1 by the terms obtained in rhs of R_1 and R_3 gives

$$\begin{aligned} 1 \quad > \quad & \frac{\beta_1\mu}{(\mu+\rho)(\mu+\alpha_1)} + \frac{\alpha_1\beta_2\mu}{(\mu+\rho)(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)} \\ & + \frac{\alpha_1\alpha_2\beta_3\mu}{(\mu+\rho)(\mu+\alpha_1)(\mu+\alpha_2+\gamma_2)(\mu+\alpha_3+\gamma_3)}, \end{aligned}$$

which implies $\mathcal{R}_0(\rho) < 1$, hence the disease free equilibrium point is locally asymptotically stable.

5.4 Local Stability Analysis at the Endemic Equilibrium Point

The Jacobian matrix evaluated at the endemic equilibrium gives

$$J_E = \begin{pmatrix} \lambda^* - (\mu+\rho) & -\frac{\beta_1}{\mathcal{R}_0} & -\frac{\beta_2}{\mathcal{R}_0} & -\frac{\beta_3}{\mathcal{R}_0} & 0 & 0 \\ \lambda^* & b^* & \frac{\beta_2}{\mathcal{R}_0} & \frac{\beta_3}{\mathcal{R}_0} & 0 & 0 \\ 0 & \alpha_1 & -(\alpha_2+\gamma_2+\mu) & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & -(\alpha_3+\gamma_3+\mu) & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & -(\mu+\delta+\gamma_A) & 0 \\ 0 & 0 & \gamma_2 & \gamma_3 & \gamma_A & -\mu \end{pmatrix}$$

where

$$\lambda^* = \frac{\beta_1 I_1}{N} + \frac{\beta_2 I_2}{N} + \frac{\beta_3 I_3}{N}, \quad b^* = \frac{\beta_1}{\mathcal{R}_0} - (\alpha_1+\mu) \quad \text{and} \quad \mathcal{R}_0 = \frac{N^*}{S^*}.$$

Two of the eigenvalues of J_E are $\lambda_1 = \mu$ and $\lambda_2 = -(\delta+\gamma_A+\mu)$. **Furthermore,**

$$J_E^s = \begin{pmatrix} -\lambda^* - (\mu+\rho) & -\frac{\beta_1}{\mathcal{R}_0} & -\frac{\beta_2}{\mathcal{R}_0} & -\frac{\beta_3}{\mathcal{R}_0} \\ \lambda^* & \frac{\beta_1}{\mathcal{R}_0} - (\alpha_1+\mu) & \frac{\beta_2}{\mathcal{R}_0} & \frac{\beta_3}{\mathcal{R}_0} \\ 0 & \alpha_1 & -(\alpha_2+\gamma_2+\mu) & 0 \\ 0 & 0 & \alpha_2 & -(\alpha_3+\gamma_3+\mu) \end{pmatrix},$$

$$
\begin{aligned}
A_1 \quad &: \quad \lambda^* + (\mu+\rho) > \frac{\beta_1+\beta_2+\beta_3}{\mathcal{R}_0},\\
A_2 \quad &: \quad -\lambda^* + (\alpha_1+\mu) > \frac{\beta_1+\beta_2+\beta_3}{\mathcal{R}_0},\\
A_3 \quad &: \quad 1 > \frac{\alpha_1}{(\alpha_2+\gamma_2+\mu)},\\
A_4 \quad &: \quad 1 > \frac{\alpha_2}{(\alpha_3+\gamma_3+\mu)}.
\end{aligned}
$$

Adding inequalities A_1 and A_2 gives

$$(\mu+\rho) + (\alpha_1+\mu) > \frac{2(\beta_1+\beta_2+\beta_3)}{\mathcal{R}_0}.$$

Dividing through by $(\mu+\rho)(\alpha_1+\mu)$ gives

$$\frac{1}{(\alpha_1+\mu)} + \frac{1}{(\mu+\rho)} > \frac{2}{\mathcal{R}_0}\Big[\frac{\beta_1}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_2}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_3}{(\mu+\rho)(\alpha_1+\mu)}\Big].$$

Multiplying the above inequality by μ gives

$$\frac{\mu}{(\alpha_1+\mu)} + \frac{\mu}{(\mu+\rho)} > \frac{2}{\mathcal{R}_0}\Big[\frac{\beta_1\mu}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_2\mu}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_3\mu}{(\mu+\rho)(\alpha_1+\mu)}\Big].$$

Dividing through by 2 gives

$$1 > \frac{1}{2}\Big(\frac{\mu}{(\alpha_1+\mu)} + \frac{\mu}{(\mu+\rho)}\Big) > \frac{1}{\mathcal{R}_0}\Big[\frac{\beta_1\mu}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_2\mu}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_3\mu}{(\mu+\rho)(\alpha_1+\mu)}\Big]$$

or

$$1 > \frac{1}{\mathcal{R}_0}\Big[\frac{\beta_1\mu}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_2\mu}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_3\mu}{(\mu+\rho)(\alpha_1+\mu)}\Big]. \tag{54}$$

From the inequalities A_3 and A_4, it is obvious that

$$\mathcal{R}_0 > \Big[\frac{\beta_1\mu}{(\mu+\rho)(\alpha_1+\mu)} + \frac{\beta_2\mu}{(\mu+\rho)(\alpha_1+\mu)} * A_3 + \frac{\beta_3\mu}{(\mu+\rho)(\alpha_1+\mu)} * (A_3 * A_4)\Big].$$

Hence (54) holds provided $\mathcal{R}_0 > 1$.

6 Concluding Remarks

In this paper, we investigated the local stability of both the disease-free and endemic equilibria of a tuberculosis model, SEIRS, a vector-host model and a-Staged HIV/AIDS Model. The corollary of Gershgorin's circles theorem which is an essential tool that helps to determine the regions of the complex plane in which the eigenvalues of a matrix are located was used. This criterion is very practical given that the local stability of the equilibrium points can be established without the need to calculate the eigenvalues, instead the basic reproduction number which also gives a condition for an equilibrium point to be stable were obtained for each model after simple computations by establishing that if $\mathcal{R}_0 < 1$, the Jacobian matrix evaluated at the disease-free equilibrium will have negative or negative real part eigenvalues. Thus, disease-free equilibrium is stable but if $\mathcal{R}_0 > 1$, then the Jacobian matrix evaluated at the endemic equilibrium will have negative or negative real part eigenvalues making the endemic equilibrium stable.

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