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Convergence And Convexity Of Two Families Of Third-Order Methods For Computing Simple Roots Of Nonlinear Equations*

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Abstract

In this work, conditions over the logarithmic convexity functions of a function and its derivative are presented for the convergence of two families of third-order methods. The families of methods to be considered are the family obtained as a convex combination of Halley and Chebyshev methods and the Chebyshev Halley family. Some graphic examples are also presented.

1 Introduction

It is well known that non-linear equations are in general not solvable analytically and the solutions of such equations must be approached by means of iterative methods, [1, 2]. Newton-Raphson's method is the most used iterative method for solving non-linear equations, it represents a second order convergent iterative method for finding simple roots of non-linear equations.

The construction of iterative methods of third order of convergence can be obtained from Gander's theorem, [7]. Usually when the methods depend on one or more parameters they are called method families. Three of the methods of third-order of convergence that satisfy the hypotheses of Gander's theorem are those of Chebyshev [4, 12, 13], Halley [9, 5] and Super-Halley [5, 6, 11]. These methods are part of the Chebyshev-Halley family.

This paper is mainly based on a study made by Miguel Á. Hernández Verón who in 1992 presented an article [10] in which he introduces two important concepts as the log-degree of convexity and the logarithmic convexity function which measure the convexity of a function. By using these concepts, he also provides a technique to determine the convergence of Newton-Raphson's method.

In the current work, we establish sufficient conditions over third order families of iterative methods that will allow us to determine the convergence of such families by using the analysis over the convexity of a function introduced by Miguel Á. Hernández Verón together with an approach that involves a well-known result due to Robert Brouwer.

In order to provide a structure to be followed, we introduce the following list of upcoming sections:

- Section 2: We introduce the basic terminology and the preliminary results that are needed to prove the main results regarding convergence of families of iterative methods.
- Section 3: We state and prove the first main result regarding the convergence of a convex combination of Chebyshev and Halley iterative methods by means of the influence of the convexity of a continuous real valued function.
- Section 4: We generalize the analysis made in the previous section in order to determine sufficient conditions that will allow us to assure the convergence of the family of methods given by Chebyshev-Halley Family. As a consequence of this result, we establish sufficient conditions to determine the convergence of well-known iterative methods such as Chebyshev, Halley and Super-Halley methods.

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- Section 5: We introduce some graphic examples that will show us the applications of the results exposed in this paper. This examples will give us a clear image of the influence of the convexity of a real valued function to the convergence of an iterative method.

2 Preliminaries

Families of methods used throughout the document are the family called Convex Combination of Halley and Chebyshev's methods and the Chebyshev-Halley family, see [3] and [8]. These two families are given by the iteration equations:

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}\left(\frac{A}{1-\frac{1}{2}L_f(z_n)} + (1-A)\left(1+\frac{1}{2}L_f(z_n)\right)\right), \quad A \in [0,1],\ n = 0,1,2,\cdots \tag{1}$$

and

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}\left(1+\frac{L_f(z_n)}{2(1-AL_f(z_n))}\right), \quad A \in \mathbb{R},\ n = 0,1,2,\cdots \tag{2}$$

respectively, where L_f is the logarithmic convexity function given by

$$L_f(z) = \frac{f(z)f''(z)}{[f'(u)]^2}.$$

Chebyshev-Halley Family includes among its methods those of Chebyshev, Halley and Super-Halley when the value of A is equal to 0, $\frac{1}{2}$ and 1, respectively.

The following lemma is taken from the text [1] and will be fundamental for our study of convergence.

Lemma 1 (R. Brouwer) *Let g be a continuous real valued function and α a fixed point of g. Suppose that g is a differentiable function in a neighborhood O around α. If $|g'(x)| \leq M < 1$ in the neighborhood defined by O and the interval $I = [\alpha-\epsilon, \alpha+\epsilon]$ is such that $I \subseteq O$, where $\epsilon > 0$, then g has a unique fixed point in I defined by α and the iteration $x_{n+1} = g(x_n)$ converges to α for each initial point $x_0 \in I$.*

3 Convergence and Convexity of the First Family

We are going to start the analysis of convergence for the family introduced in [3] that goes by the name of a convex combination of Halley and Chebyshev methods, but first we are going to provide a short introduction of the methods, notation and conventions that are going to be used in this section.

Let f denote a continuous real valued function defined in an interval $[a,b] \subseteq \mathbb{R}$ and $z \in [a,b]$. Let us consider the following iterative functions given by the next rules:

$$F(z) = z - \frac{f(z)}{f'(z)}\frac{1}{1-\frac{1}{2}L_f(z)} \quad \text{and} \quad G(z) = z - \frac{f(z)}{f'(z)}\left(1+\frac{1}{2}L_f(z)\right)$$

where the first iterative function given by $F(z)$ represents Halley's method and the second one given by $G(z)$ represents Chebyshev's method. Let A be a real number such that $A \in [0,1]$ and let us consider the convex combination of the previous iterative functions provided by:

$$R(z) = AF(z) + (1-A)G(z). \tag{3}$$

Then, having into account the previous remarks and that $L_{f'}(z) = \frac{f'(z)f'''(z)}{(f'(z))^2}$, we propose the following:

Theorem 1 *Suppose that f denotes a continuous real valued function defined in an interval $[a,b] \subseteq \mathbb{R}$ for any two different real numbers a and b. Let $z \in [a,b]$ and assume that*

$$\phi(L_f(z)) = (1-A)(2-L_f(z))^2.$$

Thus, if the following conditions hold:

1. $|L_f(z) - 4| > 2\sqrt{2}$, *and*
2. $\left| L_{f'}(z) - \dfrac{6A + 3\phi(L_f(z))}{4A + \phi(L_f(z))} \right| < \dfrac{1}{4A + \phi(L_f(z))}$,

then the family of iterative methods (1) obtained from the iteration function given by (3) converges to a unique simple root of f for any initial point z_0 that satisfies the conditions proposed.

Proof. It is clear to see that the derivatives of the iterative functions defined by $F(z)$ and $G(z)$ are given by the following expressions:

$$F^{'}(z) = \frac{L_f^2(z)(3 - 2L_{f'}(z))}{(2 - L_f(z))^2} \quad \text{and} \quad G^{'}(z) = \frac{L_f^2(z)(3 - L_{f'}(z))}{2}.$$

Thus, having into account (3), then we have that the derivative of (3) is given by:

$$R^{'}(z) = \frac{L_f^2(z)}{2(2 - L_f(z))^2}[A(6 - 4L_{f'}(z)) + (1 - A)(3 - L_{f'}(z))(2 - L_f(z))^2].$$

Now, a simple calculation shows us that if $|L_f(z) - 4| > 2\sqrt{2}$, then we have

$$0 < \frac{L_f^2(z)}{2(2 - L_f(z))^2} < 1.$$

Lastly, if the second condition proposed in the hypothesis also holds, then we have the following relation:

$$\left| L_{f'}(z) - \frac{6A + 3\phi(L_f(z))}{4A + \phi(L_f(z))} \right| < \frac{1}{4A + \phi(L_f(z))}$$

if and only if

$$|A(6 - 4L_{f'}(z)) + (1 - A)(3 - L_{f'}(z))(2 - L_f(z))^2| < 1.$$

Therefore, having into account the latest two inequalities that were led by the hypothesis of this proposition, then $|R(z)| \leq M < 1$, where $M \in (0, 1)$. Thus, by Lemma 1 we assure that the iterative method given in equation (1) converges to a unique fixed point for any initial point $z_0 \in [a, b]$ that satisfies the two conditions proposed and consequently to a simple root of the function f for any real number $z_0 \in [a, b]$ that satisfies the conditions (1) and (2) of the hypothesis. ■

4 Convergence and Convexity of the Second Family

We are going to start the analysis of convergence for the Chebyshev-Halley Family given in [8]. Let f denote a continuous real valued function defined in an interval $[a, b] \subseteq \mathbb{R}$. Let $z \in [a, b]$ and A denote any real number. Let us consider the family of iterative functions given by:

$$H(z) = z - \frac{f(z)}{f'(z)}\left(1 + \frac{L_f(z)}{2(1 - AL_f(z))}\right). \tag{4}$$

This family of iterative functions is known as Chebyshev-Halley family of iterative methods for finding simple roots of non-linear equations. The conditions needed to assure the convergence of this family of iterative methods are provided by the following:

Theorem 2 *Suppose that f denotes a continuous real valued function defined in an interval $[a, b] \subseteq \mathbb{R}$ for any two different real numbers a and b. Let $z \in [a, b]$ and consider the following expressions*

$$B = A(2A - 1); \quad C = 3(1 - A); \quad \theta(L_f(z)) = 2\left(\frac{1 - AL_f(z)}{L_f(z)}\right)^2.$$

Thus, if the following condition holds

$$\left| L_{f'}(z) - (BL_f(z) + C) \right| < \theta(L_f(z)),$$

then, the family of iterative methods (2) obtained from the iteration function given by (4) converges to a unique simple root of f for any initial point z_0 that satisfies the conditions proposed.

Proof. Computing the derivative of the function $H(z)$, we have the following expression:

$$H^{'}(z) = \frac{L_f^2(z)(AL_f(z)(2A-1) + 3(1-A) - L_{f'}(z))}{2(1-AL_f(z))^2}.$$

Our main objective is to have $|H^{'}(z)| \leq M < 1$ in order to assure the convergence of $H(z)$ by means of using Lemma 1.

Now, having into account the condition proposed in the hypothesis, we have the following:

$$BL_f(z) + C - \theta(L_f(z)) < L_{f'}(z) < BL_f(z) + C + \theta(L_f(z)).$$

Then, if we subtract the expression $BL_f(z) + C$ from the previous inequality and then multiply by the inverse of $\theta(L_f(z))$, we have that:

$$\left| \frac{L_f^2(z)(AL_f(z)(2A-1) + 3(1-A) - L_{f'}(z))}{2(1-AL_f(z))^2} \right| < 1. \tag{5}$$

Therefore, under the conditions proposed in (1) and (2) we can assure that $|H'(z)| \leq M < 1$, where $M \in (0, 1)$. Thus, by Lemma 1 we have that the family of iterative methods given by $H(z)$ converges to a unique fixed point for any initial point $z_0 \in [a, b]$ that satisfies the two conditions proposed and consequently to a simple root of the function f for any real number $z_0 \in [a, b]$ that satisfies the conditions proposed in the hypothesis. ■

Remark 1 *Finding sufficient conditions over the logarithmic convexity function of f and f' that allow us to determine the convergence of the family of iterative methods given by $H(z)$ leads to some consequences when we compute different values of A. We highlight three different values of A that represent well-known methods for finding simple roots of non-linear equations, those values are $A = 0$, $A = \frac{1}{2}$ and $A = 1$. Then, we have the following:*

Corollary 1 *Let f represent a continuous real valued function defined in an interval $[a, b] \subseteq \mathbb{R}$ for two different real numbers a and b. The following conditions are sufficient to assure the convergence of Chebyshev's iterative method for finding simple roots of non-linear equations:*

$$L_f^2(z)|L_{f'}(z) - 3| < 2.$$

Corollary 2 *Let f represent a continuous real valued function defined in an interval $[a, b] \subseteq \mathbb{R}$ for two different real numbers a and b. The following conditions are sufficient to assure the convergence of Halley's iterative method for finding simple roots of non-linear equations:*

$$L_f^2(z) \left| 2L_{f'}(z) - 3 \right| < (2 - L_f(z))^2.$$

Corollary 3 *Let f represent a continuous real valued function defined in an interval $[a, b] \subseteq \mathbb{R}$ for two different real numbers a and b. The following conditions are sufficient to assure the convergence of Super-Halley's iterative method for finding simple roots of non-linear equations:*

$$L_f^2(z) \left| L_{f'}(z) - L_f(z) \right| < 2\left(1 - L_f(z)\right)^2. \tag{6}$$

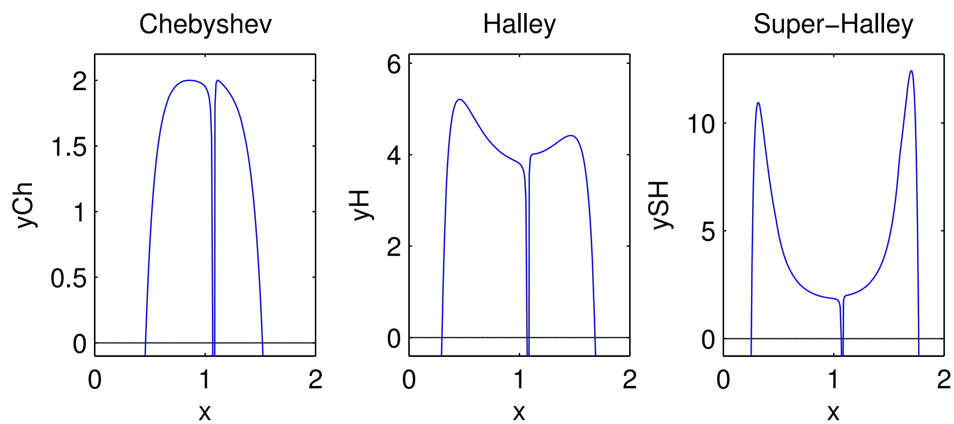

Figure 1: Graphs of yCh, yH and ySH with $f(x) = x\sin(x) - 1$.

5 Graphic Examples

In this last section we are going to show by means of an example how we can use the previous conditions on a particular function in order to determine what values should be taken to obtain a good approximation of a simple root of such functions.

Using the three corollaries presented above, we define the functions yCh, yH and ySH as follows:

$$\begin{aligned} yCh(x) &= 2 - L_f^2(x)|L_{f'}(x) - 3|, \\ yH(x) &= (2 - L_f(x))^2 - L_f^2(x)\left|2L_{f'}(x) - 3\right|, \\ ySH(x) &= 2(1 - L_f(x))^2 - L_f^2(x)\left|L_{f'}(x) - L_f(x)\right|. \end{aligned}$$

The idea of this section is to show the graphs of these three functions in certain intervals and observe where the conditions of these corollaries are fulfilled for the function f selected as an example, that is, where $yCh > 0$, $yH > 0$ and $ySH > 0$ for Chebyshev, Halley and Super Halley methods, respectively.

Let $f(x)$ be a real valued function defined by $f(x) = x\sin(x) - 1$. That function has a root which is approximately $x = 1.114157141$. Now, let us compute $L_f(z)$ and $L_{f'}(z)$. For that, we need to compute the derivatives of $f(x)$ of order one, two and three, respectively. Thus, we have the following:

- $f'(x) = \sin(x) + x\cos(x)$; $f''(x) = 2\cos(x) - x\sin(x)$; $f'''(x) = -3\sin(x) - x\cos(x)$.

- $L_f(z) = -\dfrac{(x\sin(x) - 1)(x\sin(x) - 2\cos(x))}{(\sin(x) + x\cos(x))^2}$ and $L_{f'}(x) = -\dfrac{(\sin(x) + x\cos(x))(x\cos(x) + 3\sin(x))}{(x\sin(x) - 2\cos(x))^2}$.

In the three graphs shown in Figure 1, we only exhibit two of the infinite number of intervals that satisfy the sufficient condition established in their respective corollary.

In all cases, it can be verified that the functions yCh, yH and ySH are even. If we denote these two intervals by (a_1, b_1) and (a_2, b_2), it follows that for each of these functions the values are given in Table 1.

For this example, based on Figure 1 and Table 1, it can be observed that larger convergence regions are obtained for Super Halley method more than for the other methods. It is also observed that the convergence region of the Halley method is broader than that of Chebyshev method. The afore mentioned is also valid for all the other intervals that were analyzed.

	a_1	b_1	a_2	b_2
yCh	0.4611	1.0701	1.0870	1.5198
yH	0.2983	1.0700	1.0871	1.6862
ySH	0.2532	1.0698	1.0871	1.7710

Table 1: Values of the extremes of convergence intervals with $f(x) = x\sin(x) - 1$.

6 Conclusion

In this work, sufficient conditions are presented to determine the convergence of two families of iterative methods by means of the concept of logarithmic convexity function and also by means of Brouwer's fixed point theorem. The families of iterative methods studied in the current paper are among the most used families of methods to solve non-linear equations, therefore, the importance of finding or establishing simple ways to assure the convergence of such methods is paramount for both theoretical and applied studies.

We have to remark that this work represents a different approach of that given by Miguel Á. Hernández Verón in [10] for finding conditions over the convexity of a function in order to determine the convergence of different iterative methods.

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