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Multinomial Theorem Procured From Partial Differential Equation*

Syed Abbas†

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Abstract

In this article, we give a new alternate proof of the multinomial theorem using a simple first order partial differential equation. Although the multinomial theorem is a combinatorial result, our proof is different than usual methods and may be interesting for a student familiar with the basics of partial differential equations. The binomial theorem can be obtained as a special case.

1 Main Result

Multinomial theorem, as the name indicates is the result that applies to multiple variables. It is more general than binomial theorem which holds for only two variables. There are several applications of this theorem, it gives the number of ways of putting n distinct things into m distinct bins, with k_1 things in the 1st bin, k_2 things in the 2nd bin, and so on. In statistical mechanics, given a number distribution $\{n_i\}$ on a set of N total items, n_i represents the number of items to be given the label i, where i is the label of the energy state. This coefficient also gives the number of distinct ways to permute a multiset of n elements.

This theorem provides a method of evaluating an nth degree expression of the form $(x_1+x_2+\cdots+x_m)^n$, where n is an integer. The theorem states as follows:

Theorem 1 *Let m be any positive integer and n be any nonnegative integer, then the multinomial formula is as follows:*

$$(x_1+x_2+\cdots+x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \frac{n!}{k_1!\,k_2!\cdots k_m!} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m},$$

where k_i are nonnegative integers.

There are several methods to prove this theorem, for example Kuldeep Kataria in [1] proved this theorem using probabilistic method, the other method is using induction. Interested readers may see [2] for some more interpretation. In this note, we develop a new method by using the solution of first order partial differential equation. To achieve this, let us consider a first order partial differential equation $\frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0$ with the following initial condition $u_1(0, x_2) = x_2^n$. In order to get the integral (characteristic) curves, we solve $\frac{dx_1}{1} = \frac{dx_2}{-1}$. The solution is $x_1 + x_2 = c$, hence the solution of partial differential equation is given by $f(x_1 + x_2)$. After using initial condition, the solutions is given by $u_1(x_1, x_2) = (x_1 + x_2)^n$. Now, let us assume $y_1 = x_1 + x_2$ and solve $\frac{\partial u_2}{\partial y_1} - \frac{\partial u_2}{\partial x_3} = 0$ with the initial condition $u_2(0, x_3) = x_3^n$. The solution in this case is $u_2(y_1, x_3) = (y_1 + x_3)^n$, which implies $u_2(y_1, x_3) = (x_1 + x_2 + x_3)^n$. Following the same manner, after defining $y_{m-2} = x_1 + x_2 + \cdots + x_{m-1}$, we obtain $u_{m-1}(y_{m-1}, x_m) = (x_1 + x_2 + \cdots + x_m)^n$. Hence, the function $u(x_1, x_2, \cdots, x_m) = (x_1 + x_2 + \cdots + x_m)^n$ is solution of the above equations with the given initial conditions.

Next, we apply series method to solve the same set of partial differential equations. In order to achieve this, let us consider again our first equation $\frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0$ with $u_1(0, x_2) = x_2^n$. To check for uniqueness,

*Mathematics Subject Classifications: 65M25.
†School of Basic Sciences, Indian Institute of Technology Mandi, 175005, H. P., India

let us define a function $\phi(x_2) = x_2^n$. It is not difficult to see that this function is analytic. Moreover, $\frac{\partial u_1}{\partial x_2}(0,0) = \phi^{'}(0) = 0$. So, if we compare our equation with the equation $\frac{\partial u_1}{\partial x_1} = F(x_1, x_2, u_1, p)$, then $F = p$, which is analytic at $(0,0,0,0)$. Hence, Cauchy-Kovalevsky theorem ([3]) ensures the existence of unique analytic solution which can be obtain as a series. In order to get the series, we need to compute the values of coefficients, which are value of the derivatives of u_1 at $(0,0)$. One can easily derive the following values

$$\frac{\partial u_1}{\partial x_2}(0,0) = 0, \ \cdots, \ \frac{\partial^{n-1} u_1}{\partial x_2^{n-1}}(0,0) = 0, \ \frac{\partial^n u_1}{\partial x_2^n}(0,0) = n!.$$

We denote $u_{1x_1} = \frac{\partial u_1}{\partial x_1}$ and so on. Since $u_{1x_1} = u_{1x_2}$, we obtain $u_{1x_1x_2} = u_{1x_2x_2}$ and $u_{1x_1x_1} = u_{1x_1x_2} = u_{1x_2x_2}$. Similarly, we have

$$\frac{\partial^2 u_1}{\partial x_1 \partial x_2}(0,0) = 0 \ \text{ but } \ \frac{\partial^n u_1}{\partial x_1 \partial x_2 \cdots \partial x_2}(0,0) = n!$$

and so on. One can observe that when $k_1 + k_2 = n$, the value of $D_{x_1}^{k_1} D_{x_2}^{k_2} u_1(0,0) = n!$, otherwise it is zero. Hence, the series solution is given by

$$u_1(x_1, x_2) = \sum_{(k_1, k_2)} \frac{D_{x_1}^{k_1} D_{x_2}^{k_2} u(0,0)}{k_1! k_2!} x_1^{k_1} x_2^{k_2},$$

where D denotes the partial derivative and $D_{x_1}^{k_1} D_{x_2}^{k_2} u = \frac{\partial^{k_1+k_2} u}{\partial x_1^{k_1} \partial x_2^{k_2}}$, and $k_i \geq 0, i = 1, 2$. Putting values, we obtain

$$u_1(x_1, x_2) = \sum_{(k_1+k_2=n)} \frac{n!}{k_1! k_2!} x_1^{k_1} x_2^{k_2}.$$

So, using uniqueness, we obtain

$$(x_1 + x_2)^n = \sum_{(k_1+k_2=n)} \frac{n!}{k_1! k_2!} x_1^{k_1} x_2^{k_2}.$$

Now, define $y_1 = x_1 + x_2$ and other point is x_3, we solve the following partial differential equations $\frac{\partial u_2}{\partial y_1} - \frac{\partial u_2}{\partial x_3} = 0$ with $u_2(0, x_3) = x_3^n$. Using the same analysis above, we get the solution

$$u_2(y_1, x_3) = \sum_{(k_1+k_2=n)} \frac{n!}{k_1! k_2!} y_1^{k_1} x_3^{k_2} = \sum_{(k_1+k_2=n)} \frac{n!}{k_1! k_2!} (x_1 + x_2)^{k_1} x_3^{k_2}.$$

Using the formula for $(x_1 + x_2)^{k_1}$, we obtain

$$u_2(x_1, x_2, x_3) = \sum_{(k_1+k_2=n)} \frac{n!}{k_1! k_2!} \sum_{(r_1+r_2=k_1)} \frac{k_1!}{r_1! r_2!} x_1^{r_1} x_2^{r_2} x_3^{k_2},$$

which after simplification is

$$\sum_{(r_1+r_2+k_2=n)} \frac{n!}{r_1! r_2! k_2!} x_1^{r_1} x_2^{r_2} x_3^{k_2}.$$

In order to make notation uniform, we can write

$$u_2(x_1, x_2, x_3) = \sum_{(k_1+k_2+k_3=n)} \frac{n!}{k_1! k_2! k_3!} x_1^{k_1} x_2^{k_2} x_3^{k_3}.$$

Following the similar manner after defining $y_{m-2} = x_1 + x_2 + \cdots + x_{m-1}$, we obtain

$$u_{m-1}(y_{m-1}, x_m) = \sum_{(k_1+k_2+k_3+\cdots+k_m=n)} \frac{n!}{k_1!k_2!k_3!\cdots k_m!} x_1^{k_1} x_2^{k_2} x_3^{k_3} \cdots x_m^{k_m},$$

where $k_i \geq 0, i = 1, 2, \cdots, m$. Since the solution is unique, hence both obtained solution should coincides. Thus, we obtain the multinomial theorem

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{(k_1+k_2+k_3+\cdots+k_m=n)} \frac{n!}{k_1!k_2!k_3!\cdots k_m!} x_1^{k_1} x_2^{k_2} x_3^{k_3} \cdots x_m^{k_m}.$$

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