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Entire Functions That Share Rational Functions With Their k-Th Derivatives[*]

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Abstract

With the idea of normal family, we intend to deal with the uniqueness problems of entire functions that share two rational functions with their k-th derivatives. The result obtained in the paper improves and generalizes the result due to Chen and Zhang [2]. Some relevant examples are exhibited to show that the conditions of our result are the best possible.

1 Introduction, Definitions and Results

In this paper, by a meromorphic (resp. entire) function we shall always mean meromorphic (resp. entire) function in the open complex plane $\mathbb{C}$. We use the standard notations of Nevanlinna theory e.g., $N(r,f)$, $m(r,f)$, $T(r,f)$, $N(r,a;f)$, $\overline{N}(r,a;f)$, $m(r,a;f)$ etc. (see [4, 12]). We denote by $S(r,f)$ a quantity, not necessarily the same at each of its occurrence, that satisfies the condition $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ except possibly a set of finite linear measure.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane $\mathbb{C}$ and $Q(z)$ be a rational function or a finite complex number. If $g(z)-Q(z)=0$ whenever $f(z)-Q(z)=0$, we write $f(z)=Q(z)\Rightarrow g(z)=Q(z)$. We say that a non-constant meromorphic function $f(z)$ "partially" shares $Q(z)$ with a non-constant meromorphic function $g(z)$ if either $f(z)=Q(z)\Rightarrow g(z)=Q(z)$ or $g(z)=Q(z)\Rightarrow f(z)=Q(z)$. If $f(z)=Q(z)\Rightarrow g(z)=Q(z)$ and $g(z)=Q(z)\Rightarrow f(z)=Q(z)$, we then write $f(z)=Q(z)\Leftrightarrow g(z)=Q(z)$ and we say that $f(z)$ and $g(z)$ share $Q(z)$ IM (ignoring multiplicity). If $f(z)-Q(z)$ and $g(z)-Q(z)$ have the same zeros with the same multiplicities, we write $f(z)=Q(z)\rightleftharpoons g(z)=Q(z)$ and we say that $f(z)$ and $g(z)$ share $Q(z)$ CM (counting multiplicity).

Let $R(z)=\frac{P(z)}{Q(z)}\not\equiv 0$ be a rational function, where $P(z)$ and $Q(z)$ are co-prime polynomials. We define the degree of R as $\deg(R)=\deg(P)-\deg(Q)$. If $R(z)\equiv 0$, then we define $\deg(R)=-\infty$. Thus if R is a non-zero polynomial, then $\deg(R)=\deg(R)$. It is easy to verify that $\deg\left(\frac{R'}{R}\right)=-1$, if $R(z)$ is a non-constant rational function. Therefore $\lim\limits_{z\to\infty}\frac{R'(z)}{R(z)}=0$, if $R(z)$ is a non-zero rational function. If R_1 and R_2 are two non-zero rational functions, then $\deg\left(\frac{R_1}{R_2}\right)=\deg(R_1)-\deg(R_2)$.

We recall that the order $\rho(f)$ of meromorphic function $f(z)$ is defined by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

Furthermore when $f(z)$ is an entire function, we have

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r},$$

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where $M(r,f)=\max_{|z|=r}|f(z)|$. Let f be an entire function. We know that f can be expressed by the power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$. We denote by

$$\mu(r,f)=\max_{n\in\mathbb{N},|z|=r}\{|a_nz^n|\},\ \ \nu(r,f)=\sup\{n:|a_n|r^n=\mu(r,f)\}.$$

Clearly for a polynomial $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_0$, $a_n\neq 0$, we have

$$\mu(r,P)=|a_n|r^n\ \text{ and }\ \nu(r,P)=n$$

for all r sufficiently large.

Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z)\leq M$ for $\forall\ z\in\mathbb{C}$, where

$$h^{\#}(z)=\frac{|h'(z)|}{1+|h(z)|^2}$$

denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D\subset\mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}\subseteq\mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [11]).

During the last four decades the uniqueness theory of entire and meromorphic functions has become a prominent branch of the value distribution theory (see [12]). In the uniqueness theory an important subtopic that a meromorphic function and it's derivative share some values or functions or set is well investigated.

Rubel and Yang [10] were the first to study the entire functions that share values with their derivatives. In 1977 they proved the following important theorem.

Theorem 1 ([10]) *Let a and b be complex numbers such that $b\neq a$ and let $f(z)$ be a non-constant entire function. If $f(z)=a\rightleftharpoons f'(z)=a$ and $f(z)=b\rightleftharpoons f'(z)=b$, then $f(z)\equiv f'(z)$.*

In 1980, G. G. Gundersen [3] improved Theorem 1 and obtained the following result.

Theorem 2 ([3]) *Let f be a non-constant meromorphic function, a and b be two distinct finite values. If $f(z)=a\rightleftharpoons f'(z)=a$ and $f(z)=b\rightleftharpoons f'(z)=b$, then $f(z)\equiv f'(z)$.*

Mues and Steinmetz [8] generalized Theorem 1 from sharing values CM to IM and obtained the following result.

Theorem 3 ([8]) *Let a and b be complex numbers such that $b\neq a$ and let $f(z)$ be a non-constant entire function. If $f(z)=a\Leftrightarrow f'(z)=a$ and $f(z)=b\Leftrightarrow f'(z)=b$, then $f(z)\equiv f'(z)$.*

In general the condition that $f(z)$ and $f'(z)$ have two shared values in the above theorems is necessary. This may be seen by the example

$$f(z)=e^{e^z}\int_0^z e^{-e^t}\left(1-e^t\right)\,dt.$$

In this case, we have

$$\frac{f'(z)-1}{f(z)-1}=e^z$$

so that $f(z)=1\Leftrightarrow f'(z)=1$, but $f(z)\not\equiv f'(z)$.

In 2006, Li and Yi [6] improved Theorem 1 with the idea of "partially" sharing values. Actually they proved the following result.

Theorem 4 ([6]) *Let a and b be two complex numbers such that $b \neq a, 0$ and let $f(z)$ be a non-constant entire function. If $f(z) = a \rightleftharpoons f'(z) = a$ and $f'(z) = b \Rightarrow f(z) = b$, then $f(z) \equiv f'(z)$.*

Remark 1 *Since $b \neq a$, one may assume that $b \neq 0$ in Theorem 1. So Theorem 4 improves Theorem 1 with the idea of "partially" sharing values.*

In the same paper, Li and Yi [6] exhibited the following example to show that the condition "$b \neq 0$" can not be omitted in Theorem 4. Let

$$f(z) = Ce^{\frac{a}{C}z} + a - C, \text{ where } C \in \mathbb{C} \setminus \{0\}.$$

Note that $f'(z) \neq 0$. Clearly $f(z) = a \rightleftharpoons f'(z) = a$ and $f'(z) = 0 \Rightarrow f(z) = 0$, but $f(z) \not\equiv f'(z)$. In 2009, Qi, Lü and Chen [9] asked the following question.

Question 1 *What will happen if the sharing values a and b are replaced by sharing two non-zero polynomials Q_1 and Q_2 in Theorem 4 ?*

To give an affirmative answer of the above Question 1, Qi, Lü and Chen [9] obtained the following result.

Theorem 5 ([9]) *Let $Q_1(z) = a_1 z^p + a_{1,p-1} z^{p-1} + \ldots + a_{1,0}$ and $Q_1(z) = a_2 z^p + a_{2,p-1} z^{p-1} + \ldots + a_{2,0}$ be two polynomials such that $\deg(Q_1) = \deg Q_2 = p \in \mathbb{N} \cup \{0\}$ and a_1, $a_2 (a_2 \neq 0)$ are two distinct complex numbers. Let $f(z)$ be a transcendental entire function. If $f(z) = Q_1(z) \rightleftharpoons f'(z) = Q_1(z)$ and $f'(z) = Q_2(z) \Rightarrow f(z) = Q_2(z)$, then $f(z) \equiv f'(z)$.*

In the same paper, Qi, Lü and Chen [9] exhibited the following example to show that the hypothesis that $f(z)$ is transcendental can not be omitted in Theorem 5.

Let

$$f(z) = z^3, \ \ Q_1(z) = 2z^3 - 3z^2 \ \text{ and } \ Q_2(z) = z^3.$$

Then

$$\frac{f'(z) - Q_1(z)}{f(z) - Q_1(z)} = 2 \ \text{ and } \ f'(z) = Q_2(z) \Rightarrow f(z) = Q_2(z), \ \text{ but } \ f(z) \not\equiv f'(z).$$

Now observing Theorem 5, Chen and Zhang [2] emerged the following question in 2011.

Question 2 *Does Theorem 5 hold when Q_1 and Q_2 are respectively replaced by two rational functions R_1 and R_2 ?*

Now taking the possible answer of Question 2 into back ground, Chen and Zhang [2] obtained the following result.

Theorem 6 ([2]) *Let $R_1(z)$ and $R_2(z)$ be two non-zero rational functions such that $\lim\limits_{z \to \infty} \frac{R_2(z)}{R_1(z)} \neq 1$ and $\deg(R_1) = \deg(R_2)$ and let $f(z)$ be a transcendental entire function. If $f(z) = R_1(z) \rightleftharpoons f'(z) = R_1(z)$ and $f'(z) = R_2(z) \Rightarrow f(z) = R_2(z)$, then one of the following cases must occur:*

- *(i) $f(z) \equiv f'(z)$;*
- *(ii) $f'(z) = R_2(z) + C\lambda e^{\lambda z}$ and $(\lambda - 1)R_1'(z) = \lambda R_2(z) - R_2'(z)$, where C and $\lambda \neq 1$ are two non-zero constants. In fact, $R_1(z)$ and $R_2(z)$ are two polynomials.*

In the same paper, Chen and Zhang [2] exhibited the following example to show that the hypothesis that f is transcendental can not be omitted in Theorem 6.

Let

$$f(z) = z^4, \ \ R_1(z) = 2z^4 - 4z^3 \ \text{ and } \ R_2(z) = z^4.$$

Then

$$\frac{f'(z)-R_1(z)}{f(z)-R_1(z)}=2 \ \text{ and } \ f'(z)=R_2(z)\Rightarrow f(z)=R_2(z),$$

but neither $f(z)\not\equiv f'(z)$ nor $f'(z)=R_2(z)+C\lambda e^{\lambda z}$ and $(\lambda-1)R_1'(z)=\lambda R_2(z)-R_2'(z)$.

From Theorem 6, it is clear that Theorem 6 is the improvement of Theorem 5. Also the result obtained in Theorem 6 is no doubt a useful contribution in the field of uniqueness problems of entire functions that share two small functions with their derivatives. But unfortunately there is an error in the proof of Theorem 6 although the techniques of the proof of Theorem 6 are so novelty. Now we explicitly point out the error that occurred in the proof of Theorem 6.

In the last section of the proof of Theorem 6, the authors have concluded that

$$f'(z)=R_2(z)+C\lambda e^{\lambda z}$$

and

$$(\lambda-1)R_1'(z)=\lambda R_2(z)-R_2'(z),$$

where $R_1(z)$ and $R_2(z)$ are two non-zero polynomials. But the equality $(\lambda-1)R_1'(z)=\lambda R_2(z)-R_2'(z)$ does not hold for two non-zero polynomials of same degree. Therefore at a glance from the proof of Theorem 6, we solely have $f^{(k)}\equiv f$ as the conclusion of Theorem 6.

On the other hand in the same paper authors [2] have exhibited the following example to show that the conclusion (ii) of Theorem 6 can not be deleted.

Let

$$f(z)=2e^{\frac{z}{2}}+\frac{1}{2}z^2,\ \ R_1(z)=2z-\frac{1}{2}z^2 \ \text{ and } \ R_2(z)=z.$$

Note that $f'(z)\neq R_2(z)$. Then

$$\frac{f'(z)-R_1(z)}{f(z)-R_1(z)}=\frac{1}{2} \ \text{ and } \ f'(z)=R_2(z)\Rightarrow f(z)=R_2(z),$$

but neither $f(z)\not\equiv f'(z)$ nor $f'(z)=R_2(z)+C\lambda e^{\lambda z}$ and $(\lambda-1)R_1'(z)=\lambda R_2(z)-R_2'(z)$.

The above example has no relevancy to fortify the argument "conclusion (ii) of Theorem 6 can not be deleted" because in the above example we have $\deg(R_1)\neq\deg(R_2)$.

Therefore our first objective to write this paper is to find out the correct form of Theorem 6 without imposing any other conditions.

Our second objective to write this paper is to solve the following questions.

Question 3 *What will happen if the first derivative $f'(z)$ in Theorem 6 is replaced by the general derivative $f^{(k)}(z)$?*

Question 4 *What will happen if one replace the condition "$\deg(R_1)=\deg(R_2)$" by "$\deg(R_1)\neq\deg(R_2)$" in Theorem 6 ?*

Now in this paper taking the possible answers of the above questions into back ground, we obtain the following result which not only rectify Theorem 6 but also improves and generalizes Theorem 6 in a more compact way.

Theorem 7 *Let $R_1(z)$ and $R_2(z)$ be two non-zero rational functions such that $\lim\limits_{z\to\infty}\frac{R_2(z)}{R_1(z)}\neq 1$ and let $f(z)$ be a non-constant entire function. Suppose $f(z)=R_1(z)\rightleftharpoons f^{(k)}(z)=R_1(z)$ and $f^{(k)}(z)=R_2(z)\Rightarrow f(z)=R_2(z)$. Now*

- *(I) when $\deg(R_1)=\deg(R_2)$, then $f(z)\equiv f^{(k)}(z)$.*
- *(II) when $\deg(R_1)\neq\deg(R_2)$, then one of the following two cases must occur:*

(II1) $f(z) \equiv f^{(k)}(z)$;

(II2) $f^{(k)}(z) = R_2(z) + Ce^{cz}$ *and* $(\lambda - 1)R_1^{(k)}(z) = \lambda R_2(z) - R_2^{(k)}(z)$, *where* C, c *and* $\lambda(\neq 1)$ *are non-zero constants such that* $c^k = \lambda$ *and* $R_1(z), R_2(z)$ *reduce to polynomials.*

Remark 2 *Let us take* $k = 1$. *Then from Theorem 7 we can easily get a theorem which is the improvement as well as the generalization of Theorem 6.*

Remark 3 *Let us take* $k = 1$ *and* $\deg(R_1) = \deg(R_2)$. *Then from Theorem 7 we can easily get a theorem which is the rectification of Theorem 6.*

Remark 4 *The following example shows that the condition "*$\lim\limits_{z\to\infty} \frac{R_2(z)}{R_1(z)} \neq 1$*" in Theorem 7 is sharp.*

Example 1 *Let* $f(z) = \frac{1}{4}\left(e^{2z} + e^{-2z}\right) + \frac{3}{4}z$, $k = 2$ *and* $R_1(z) = R_2(z) = z$. *Note that* $f''(z) - R_2(z) = e^{2z} + e^{-2z} - z$ *and* $f(z) - R_2(z) = \frac{1}{4}\left(e^{2z} + e^{-2z} - z\right)$ *and so* $f''(z) = R_2(z) \Rightarrow f(z) = R_2(z)$. *Also we see that* $\deg(R_1) = \deg(R_2)$ *and* $\lim\limits_{z\to\infty} \frac{R_2(z)}{R_1(z)} = 1$. *On the other hand we have*

$$f(z) - R_1(z) = \frac{1}{4}\left(e^{2z} + e^{-2z} - z\right), \quad f''(z) - R_1(z) = e^{2z} + e^{-2z} - z$$

and so $f(z) = R_1(z) \rightleftharpoons f''(z) = R_1(z)$, *but* $f''(z) \not\equiv f(z)$.

Remark 5 *The following examples shows that the condition "*$f^{(k)} = R_2 \Rightarrow f = R_2$*" in Theorem 7 is sharp.*

Example 2 *Let* $f(z) = z + Ce^{cz}$, $k = 2$, $c = \sqrt{2}$, $\lambda = 2$, $R_1(z) = 2z$ *and* $R_2(z) = z$. *Then* $f''(z) - R_2(z)$ *has infinitely many zeros whereas* $f(z) - R_2(z)$ *has no zeros and so* $f''(z) = R_2(z) \not\Rightarrow f(z) = R_2(z)$. *Also we see that* $\deg(R_1) = \deg(R_2)$ *and* $\lim\limits_{z\to\infty} \frac{R_2(z)}{R_1(z)} = \frac{1}{2} \neq 1$. *On the other hand we have* $f(z) - R_1(z) = Ce^{cz} - z$, $f''(z) - R_1(z) = 2\left(Ce^{cz} - z\right)$ *and so* $f(z) = R_1(z) \rightleftharpoons f''(z) = R_1(z)$, *but* $f''(z) \not\equiv f(z)$.

Example 3 *Let* $f(z) = e^{\sqrt{2}z} - e^{-\sqrt{2}z} + z$, $k = 2$, $R_1(z) = 2z$ *and* $R_2(z) = 1$. *It is clear that* $f''(z) = R_2(z) \not\Rightarrow f(z) = R_2(z)$. *Also we see that* $\deg(R_1) \neq \deg(R_2)$ *and* $\lim\limits_{z\to\infty} \frac{R_2(z)}{R_1(z)} = 0 \neq 1$. *On the other hand we have*

$$f(z) - R_1(z) = e^{\sqrt{2}z} - e^{-\sqrt{2}z} - z, \quad f''(z) - R_1(z) = 2\left(e^{\sqrt{2}z} - e^{-\sqrt{2}z} - z\right)$$

and so $f(z) = R_1(z) \rightleftharpoons f''(z) = R_1(z)$, *but neither* $f''(z) \equiv f(z)$ *nor* $f''(z) = R_2(z) + Ce^{cz}$ *and* $(\lambda - 1)R_1''(z) = \lambda R_2(z) - R_2''(z)$.

Example 4 *Let* $f(z) = \frac{1}{4}e^{2z} + \frac{3}{4}z$, $k = 2$, $c = 2$, $\lambda = 4$, $R_1(z) = z$ *and* $R_2(z) = 1$. *It is clear that* $f''(z) = R_2(z) \not\Rightarrow f(z) = R_2(z)$. *Also we see that* $\deg(R_1) \neq \deg(R_2)$ *and* $\lim\limits_{z\to\infty} \frac{R_2(z)}{R_1(z)} = 0 \neq 1$. *On the other hand we have*

$$f(z) - R_1(z) = \frac{1}{4}\left(e^{2z} - z\right), \quad f''(z) - R_1(z) = e^{2z} - z$$

and so $f(z) = R_1(z) \rightleftharpoons f''(z) = R_1(z)$, *but neither* $f''(z) \equiv f(z)$ *nor* $f''(z) = R_2(z) + Ce^{cz}$ *and* $(\lambda - 1)R_1''(z) = \lambda R_2(z) - R_2''(z)$.

Remark 6 *We give an example to show that the conclusion* $(II2)$ *of Theorem 7can not be deleted.*

Example 5 *Let* $f(z) = z + Ce^{cz}$, $c = \lambda = \frac{1}{2}$, $R_1(z) = 2 - z$ *and* $R_2(z) = 1$. *Then* $f'(z) - R_2$ *has no zeros and so* $f'(z) = R_2(z) \Rightarrow f(z) = R_2(z)$. *Also we see that* $\deg(R_1) \neq \deg(R_2)$ *and* $\lim\limits_{z\to\infty} \frac{R_2(z)}{R_1(z)} = 0 \neq 1$. *On the other hand we have*

$$f(z) - R_1(z) = Ce^{cz} + 2z - 2, \quad f'(z) - R_1(z) = \frac{1}{2}\left(Ce^{cz} + 2z - 2\right)$$

and so $f(z) = R_1(z) \rightleftharpoons f'(z) = R_1(z)$. *Thus* $f(z)$ *satisfies all the conditions of Theorem 7, where* $f'(z) = R_2(z) + Ce^{cz}$ *and* $(\lambda - 1)R_1'(z) = \lambda R_2(z) - R_2'(z)$.

2 Lemmas

In this section we introduce the following lemmas which will be needed in the paper.

Lemma 1 (**[5, Lemma 1.3.1]**) *$P(z) = \sum_{i=1}^{n} a_i z^i$ where $a_n \neq 0$. Then for all $\varepsilon > 0$, there exists $r_0 > 0$ such that $\forall\ r = |z| > r_0$ the inequalities $(1-\varepsilon)|a_n|r^n \leq |P(z)| \leq (1+\varepsilon)|a_n|r^n$ hold.*

Lemma 2 (**[5, Theorem 3.1]**) *If $f(z)$ is an entire function of order $\rho(f)$, then*

$$\rho(f) = \limsup_{r \longrightarrow \infty} \frac{\log \nu(r, f)}{\log r}.$$

Lemma 3 (**[5, Theorem 3.2]**) *Let $f(z)$ be a transcendental entire function and $0 < \delta < \frac{1}{4}$. Suppose that at the point z with $|z| = r$ the inequality*

$$|f(z)| > M(r, f)\nu(r, f)^{-\frac{1}{4}+\delta}$$

holds. Then there exists a set $F \subset \mathbb{R}^+$ of finite logarithmic measure, i.e., $\int_F \frac{1}{t} dt < +\infty$ such that

$$f^{(m)}(z) = \left(\frac{\nu(r, f)}{z}\right)^m (1 + o(1)) f(z)$$

holds for all $m \in \mathbb{N} \cup \{0\}$ and $r \notin F$.

Lemma 4 (**[13]**) *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ such that all zeros of functions in $\mathcal{F}$ have multiplicity greater than or equal to l and all poles of functions in $\mathcal{F}$ have multiplicity greater than or equal to j and α be a real number satisfying $-l < \alpha < j$. Then $\mathcal{F}$ is not normal in any neighborhood of $z_0 \in \Delta$, if and only if there exist*

(i) points $z_n \in \Delta$, $z_n \to z_0$,

(ii) positive numbers ρ_n, $\rho_n \to 0^+$ and

(iii) functions $f_n \in \mathcal{F}$,

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \to g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where $g(z)$ is a non-constant meromorphic function. The function $g(z)$ may be taken to satisfy the normalisation $g^{\#}(\zeta) \leq g^{\#}(0) = 1 (\zeta \in \mathbb{C})$.

Remark 7 *Clearly if all functions in $\mathcal{F}$ are holomorphic (so that the condition on the poles is satisfied vacuously for arbitrary j), we may take $-1 < \alpha < \infty$.*

Lemma 5 (**[7]**) *Let $f(z)$ be a meromorphic function of infinite order on $\mathbb{C}$. Then there exist points $z_n \to \infty$ such that for every $N > 0$, $f^{\#}(z_n) > |z_n|^N$, if n is sufficiently large.*

Lemma 6 (**[1]**) *Let $f(z)$ be a meromorphic function on $\mathbb{C}$ with finitely many poles. If $f(z)$ has bounded spherical derivative on $\mathbb{C}$, then $f(z)$ is of order at most 1.*

3 Proof of Theorem 7

Suppose $R_1(z) = \frac{Q_1(z)}{Q_2(z)}$ and $R_2(z) = \frac{Q_3(z)}{Q_4(z)}$, where $Q_i(z)$ $(i = 1, 2, 3, 4)$ are polynomials such that $Q_1(z)$, $Q_2(z)$ and $Q_3(z)$, $Q_4(z)$ are co-prime respectively. Also we define $P_1(z) = Q_1(z)Q_4(z)$ and $P_2(z) = Q_2(z)Q_3(z)$. Since $\lim\limits_{z \to \infty} \frac{R_2(z)}{R_1(z)} \neq 1$, it follows that $R_1(z) \not\equiv R_2(z)$. If $f(z)$ is a polynomial, then $f(z) - R_1(z)$ and $f^{(k)}(z) - R_1(z)$ cannot have same zeros with same multiplicities, which contradicts the fact that $f(z) - R_1(z)$ and $f^{(k)}(z) - R_1(z)$ share 0 CM. Therefore $f(z)$ is a transcendental entire function. We now discuss the following two cases.

Case 1. Suppose that $\rho(f) < +\infty$. Let

$$\alpha(z) = \frac{f^{(k)}(z) - R_1(z)}{f(z) - R_1(z)}.$$

We claim that $\alpha(z) \not\equiv 0$. If not, suppose $\alpha(z) \equiv 0$. Then we have $f^{(k)}(z) = R_1(z)$. Since $f(z)$ is a transcendental entire function, it follows that $R_1(z)$ is a polynomial. Then by integration, we have $f(z) = \mathcal{P}(z)$, where $\mathcal{P}(z)$ is a polynomial of degree $\deg(R_1) + k$. This shows that $f(z)$ is a polynomial, which is a contradiction. Hence $\alpha(z) \not\equiv 0$. If $\alpha(z) \equiv 1$, then we have $f^{(k)}(z) \equiv f(z)$, which is one of the conclusion of our result. Next we suppose $\alpha(z) \not\equiv 1$. Consequently $f^{(k)}(z) \not\equiv f(z)$. Since $f(z) - R_1(z)$ and $f^{(k)}(z) - R_1(z)$ share 0 CM, one can easily deduce that $\alpha(z)$ has no zeros and poles. Note that

$$\rho\left(\frac{f^{(k)} - R_1}{f - R_1}\right) \leq \max\left\{\rho\left(f^{(k)} - R_1\right), \rho\left(f - R_1\right)\right\} = \max\left\{\rho\left(f^{(k)}\right), \rho(f)\right\} = \rho(f) < +\infty$$

and so $\rho(\alpha) < +\infty$. Therefore we can assume that $\alpha(z) = e^{\gamma(z)}$, where $\gamma(z)$ is a polynomial. Hence

$$\frac{f^{(k)}(z) - R_1(z)}{f(z) - R_1(z)} = e^{\gamma(z)}. \tag{1}$$

We claim that $\gamma(z)$ is a constant polynomial. If not, suppose that $\gamma(z)$ is a non-constant polynomial. Let $\deg(\gamma) = m \geq 1$ and $\gamma(z) = c_m z^m + c_{m-1} z^{m-1} + \ldots + c_0$, where $c_i \in \mathbb{C}$ for $i = 0, 1, \ldots, m$ and $c_m \neq 0$. Now from (1), we have

$$e^{\gamma(z)} = \frac{\frac{f^{(k)}(z)}{f(z)} - \frac{R_1(z)}{f(z)}}{1 - \frac{R_1(z)}{f(z)}}, \text{ i.e., } \gamma(z) = \log \frac{\frac{f^{(k)}(z)}{f(z)} - \frac{R_1(z)}{f(z)}}{1 - \frac{R_1(z)}{f(z)}},$$

where $\log h$ is the principle branch of the logarithm. Therefore by Lemma 1, we have

$$|c_m| r^m (1 + o(1)) = |\gamma(z)| = \left|\log \frac{\frac{f^{(k)}(z)}{f(z)} - \frac{R_1(z)}{f(z)}}{1 - \frac{R_1(z)}{f(z)}}\right|. \tag{2}$$

Since f is a transcendental entire function, $M(r, f) \to \infty$ as $r \to \infty$. Again let $M(r, f) = |f(z_r)|$, where $z_r = re^{i\theta_r}$ and $\theta_r \in [0, 2\pi)$. We see that $\lim\limits_{r\to\infty} \frac{1}{|f(z_r)|} = \lim\limits_{r\to\infty} \frac{1}{M(r,f)} = 0$. Now from Lemma 3, there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure such that for some point $z_r = re^{i\theta_r}(\theta_r \in [0, 2\pi))$, $r \notin E$ and $M(r, f) = |f(z_r)|$, we have

$$\frac{f^{(k)}(z_r)}{f(z_r)} = \left(\frac{\nu(r, f)}{z_r}\right)^k (1 + o(1)). \tag{3}$$

Since $f(z)$ is a transcendental entire function, it follows that $\frac{R_1(z)}{f(z)} \to 0$ as $|z| \to \infty$. Therefore from (2), (3) and Lemma 2, we get

$$|c_m| r^m (1 + o(1)) = |\gamma(z_r)| = \left|\log \frac{\frac{f^{(k)}(z_r)}{f(z_r)} - \frac{R_1(z_r)}{f(z_r)}}{1 - \frac{R_1(z_r)}{f(z_r)}}\right| = O(\log r),$$

for $|z| = r \to +\infty$, $r \notin E$, which is impossible. Hence γ is a constant polynomial. Without loss of generality we assume that

$$f^{(k)}(z) - R_1(z) \equiv \lambda(f(z) - R_1(z)), \text{ i.e., } f^{(k)}(z) \equiv \lambda f(z) + (1 - \lambda) R_1(z). \tag{4}$$

Since $f^{(k)}(z) \not\equiv f(z)$, it follows from (4) that $\lambda \neq 1$. Again since $f(z)$ is an entire function and $R_1(z)$ is a rational function, from (4) one can easily conclude that $R_1(z)$ is an entire function. Therefore $R_1(z)$ is a polynomial. Let z_0 be a zero of $f^{(k)}(z)-R_2(z)$. By the assumption, we have $f(z_0)=R_2(z_0)$. Putting $z=z_0$ into (4), we get $R_1(z_0)=R_2(z_0)$. Since $R_1(z)\not\equiv R_2(z)$, it follows that z_0 must be a zero of $R_1(z)-R_2(z)$. Therefore one can easily deduce that all the zeros of $f^{(k)}(z)-R_2(z)$ are the zeros of $R_1(z)-R_2(z)$ and so $f^{(k)}(z)-R_2(z)$ has finitely many zeros. Since $f(z)$ is a transcendental entire function of finite order, we can take

$$f^{(k)}(z)=R_2(z)+P(z)e^{Q(z)}, \tag{5}$$

where $P(z)$ is a non-zero rational function and $Q(z)$ is a non-constant polynomial. Now from (4) and (5), we have

$$\lambda f(z)=R_2(z)-(1-\lambda)R_1(z)+P(z)e^{Q(z)}. \tag{6}$$

Differentiating k-times, we obtain from (6) that

$$\lambda f^{(k)}(z)=R_2^{(k)}(z)-(1-\lambda)R_1^{(k)}(z)+\left(P(z)Q'^k+P_3(z)\right)e^{Q(z)}, \tag{7}$$

where $P_3(z)$ is a rational function. Now from (5) and (7), we have

$$R_2^{(k)}(z)-\lambda R_2(z)-(1-\lambda)R_1^{(k)}(z)+\left((Q'^k-\lambda)P(z)+P_3(z)\right)e^{Q(z)}\equiv 0. \tag{8}$$

Clearly from (8), one can easily conclude that

$$(\lambda-1)R_1^{(k)}(z)=\lambda R_2(z)-R_2^{(k)}(z) \tag{9}$$

and

$$\left(\lambda-Q'^k\right)P(z)=P_3(z). \tag{10}$$

Since $R_1(z)$ is a polynomial, from (9), we conclude that $R_2(z)$ is also a polynomial. Therefore from (5), we observe that $P(z)$ is a non-zero polynomial and so from (7) we conclude that $P_3(z)$ is a polynomial and $\deg(P_3)<\deg\left(PQ'^k\right)$. Now (10) gives $Q'^k=\lambda$ and $P_3(z)\equiv 0$. Thus $Q'(z)$ is a constant, say $Q'(z)=\mu$. Then $Q(z)=\mu z+b$ and $\mu^k=\lambda$, where $b\in\mathbb{C}$. So from (5), we have

$$f^{(k)}(z)=R_2(z)+P(z)e^{\mu z+b}.$$

It is easy to deduce that $\deg(P_3)=\deg(P')$. From $P_3(z)\equiv 0$, one can easily conclude that $P(z)$ is a non-zero constant. Let $P(z)e^b=C$. Finally we have

$$f^{(k)}(z)=R_2(z)+Ce^{cz} \tag{11}$$

and

$$(\lambda-1)R_1^{(k)}(z)=\lambda R_2(z)-R_2^{(k)}(z), \tag{12}$$

where $C\in\mathbb{C}\setminus\{0\}$ and $\lambda\in\mathbb{C}\setminus\{0,1\}$ such that $c^k=\lambda$. If $\deg(R_1)=\deg(R_2)$, from (12) we must have $\lambda=0$ and so $c=0$, which is impossible. Consequently (11) is valid only when $\deg(R_1)\neq\deg(R_2)$.

Case 2. Suppose that $\rho(f)=+\infty$. Let $F(z)=\frac{f(z)}{R_1(z)}$. Since $\rho(R_1)=0$, it follows that $\rho(F)=+\infty$. Now by Lemma 5, there exists $\{w_j\}_j\to\infty(j\to\infty)$ such that for every $N>0$

$$F^{\#}(w_j)>|w_j|^N,\ \text{ if } j \text{ is sufficiently large.} \tag{13}$$

Since $R_1(z)$ has finitely many poles and zeros, there exists a $r>0$ such that $F(z)$ is analytic and $R_1(z)\neq 0,\infty$ in $D=\{z:|z|\geq r\}$. Also since $\omega_j\to\infty$ as $j\to\infty$, without loss of generality we may assume that $|\omega_j|\geq r+1$ for all j. Let $D_1=\{z:|z|<1\}$. Note that

$$F_j(z)=F(\omega_j+z)=\frac{f(\omega_j+z)}{R_1(\omega_j+z)}.$$

Since $|\omega_j+z|\geq|\omega_j|-|z|$, it follows that $\omega_j+z\in D$ for all $z\in D_1$. Also since $F(z)$ is analytic in D, it follows that $F_j(z)$ is analytic in D_1 for all j. Thus we have structured a family $(F_j)_j$ of holomorphic functions. Note that $F^{\#}(0)=F^{\#}(w_j)\to\infty$ as $j\to\infty$. Now it follows from Marty's criterion that $(F_j)_j$ is not normal at $z=0$. Then by Lemma 4, there exist

(i) points $z_j \in D_1$ such that $z_j \to 0$ as $j \to \infty$,

(ii) positive numbers ρ_j, $\rho_j \to 0^+$,

(iii) a subsequence $\{F(\omega_j + z_j + \rho_j\zeta) = F_j(z_j + \rho_j\zeta)\}$ of $\{F(\omega_j + z)\}$

such that

$$g_j(\zeta) = F_j(z_j + \rho_j\zeta) = \frac{f(\omega_j + z_j + \rho_j\zeta)}{R_1(\omega_j + z_j + \rho_j\zeta)} \to g(\zeta) \tag{14}$$

spherically locally uniformly in $\mathbb{C}$, where $g(\zeta)$ is a non-constant entire function such that $g^{\#}(\zeta) \leq g^{\#}(0) = 1$. Now from Lemma 6, we see that $\rho(g) \leq 1$. Also in the proof of Zalcman's lemma, we have

$$\rho_j \leq \frac{M}{F^{\#}(w_j)} \tag{15}$$

for a positive number M. Now from (13) and (15), we deduce that for every $N > 0$,

$$\rho_j \leq M|w_j|^{-N} \tag{16}$$

for sufficiently large values of j. We now want to prove that

$$\rho_j^k \frac{f^{(k)}(\omega_j + z_j + \rho_j\zeta)}{R_1(\omega_j + z_j + \rho_j\zeta)} \to g^{(k)}(\zeta). \tag{17}$$

From (14), we see that

$$\begin{aligned}
\rho_j \frac{f'(\omega_j + z_j + \rho_j\zeta)}{R_1(\omega_j + z_j + \rho_j\zeta)} &= g_j'(\zeta) + \rho_j \frac{R_1'(\omega_j + z_j + \rho_j\zeta)}{R_1^2(\omega_j + z_j + \rho_j\zeta)} f(\omega_j + z_j + \rho_j\zeta) \\
&= g_j'(\zeta) + \rho_j \frac{R_1'(\omega_j + z_j + \rho_j\zeta)}{R_1(\omega_j + z_j + \rho_j\zeta)} g_j(\zeta).
\end{aligned} \tag{18}$$

Also we see that

$$\frac{R_1'(\omega_j + z_j + \rho_j\zeta)}{R_1(\omega_j + z_j + \rho_j\zeta)} \to 0 \ \ (\text{as } \ j \to \infty). \tag{19}$$

Now from (14), (18) and (19), we observe that

$$\rho_j \frac{f'(\omega_j + z_j + \rho_j\zeta)}{R_1(\omega_j + z_j + \rho_j\zeta)} \to g'(\zeta).$$

Suppose

$$\rho_j^p \frac{f^{(p)}(\omega_j + z_j + \rho_j\zeta)}{R_1(\omega_j + z_j + \rho_j\zeta)} \to g^{(p)}(\zeta).$$

Let

$$G_j(\zeta) = \rho_j^p \frac{f^{(p)}(\omega_j + z_j + \rho_j\zeta)}{R_1(\omega_j + z_j + \rho_j\zeta)}.$$

Then $G_j(\zeta) \to g^{(p)}(\zeta)$. Note that

$$\begin{aligned}
\rho_j^{p+1} \frac{f^{(p+1)}(\omega_j + z_j + \rho_j\zeta)}{R_1(\omega_j + z_j + \rho_j\zeta)} &= G_j'(\zeta) + \rho_j^{p+1} \frac{R_1'(\omega_j + z_j + \rho_j\zeta)}{R_1^2(\omega_j + z_j + \rho_j\zeta)} f^{(p)}(\omega_j + z_j + \rho_j\zeta) \\
&= G_j'(\zeta) + \rho_j \frac{R_1'(\omega_j + z_j + \rho_j\zeta)}{R_1(\omega_j + z_j + \rho_j\zeta)} G_j(\zeta).
\end{aligned} \tag{20}$$

Now from (19) and (20), we see that

$$\rho_j^{p+1}\frac{f^{(p+1)}(\omega_j+z_j+\rho_j\zeta)}{R_1(\omega_j+z_j+\rho_j\zeta)}\to G_j'(\zeta),\ \text{ i.e., }\ \rho_j^{p+1}\frac{f^{(p+1)}(\omega_j+z_j+\rho_j\zeta)}{R_1(\omega_j+z_j+\rho_j\zeta)}\to g^{(p+1)}(\zeta).$$

Then by mathematical induction we get the desired result (17).
First prove that $g=1\Rightarrow g^{(k)}=0$. Suppose that $g(\zeta_0)=1$. Then by Hurwitz's Theorem there exists a sequence $\{\zeta_j\}$, $\zeta_j\to\zeta_0$ such that (for sufficiently large j)

$$g_j(\zeta_j)=\frac{f(\omega_j+z_j+\rho_j\zeta_j)}{R_1(\omega_j+z_j+\rho_j\zeta_j)}=1,\ \text{ i.e., }\ f(\omega_j+z_j+\rho_j\zeta_j)=R_1(\omega_j+z_j+\rho_j\zeta_j).$$

By the given condition, we have $f^{(k)}(\omega_j+z_j+\rho_j\zeta_j)=R_1(\omega_j+z_j+\rho_j\zeta_j)$. Now from (17), we see that

$$g^{(k)}(\zeta_0)=\lim_{j\to\infty}g^{(k)}(\zeta_j)=\lim_{j\to\infty}\rho_j^k\frac{f^{(k)}(\omega_j+z_j+\rho_j\zeta_j)}{R_1(\omega_j+z_j+\rho_j\zeta_j)}=\lim_{j\to\infty}\rho_j^k\frac{R_1(\omega_j+z_j+\rho_j\zeta_j)}{R_1(\omega_j+z_j+\rho_j\zeta_j)}=0.$$

Thus $g(\zeta)=1\Rightarrow g^{(k)}(\zeta)=0$. Next we prove that $g^{(k)}=0\Rightarrow g=1$. Now from (17), we see that

$$\rho_j^k\frac{f^{(k)}(\omega_j+z_j+\rho_j\zeta)-R_1(\omega_j+z_j+\rho_j\zeta)}{R_1(\omega_j+z_j+\rho_j\zeta)}\to g^{(k)}(\zeta). \tag{21}$$

Suppose that $g^{(k)}(\eta_0)=0$. Then by (21) and Hurwitz's Theorem there exists a sequence $\{\eta_j\}$, $\eta_j\to\eta_0$ such that (for sufficiently large j) $f^{(k)}(\omega_j+z_j+\rho_j\eta_j)=R_1(\omega_j+z_j+\rho_j\eta_j)$. By the given condition we have $f(\omega_j+z_j+\rho_j\eta_j)=R_1(\omega_j+z_j+\rho_j\eta_j)$. Therefore

$$g(\eta_0)=\lim_{j\to\infty}\frac{f(\omega_j+z_j+\rho_j\eta_j)}{R_1(\omega_j+z_j+\rho_j\eta_j)}=1.$$

Thus $g^{(k)}=0\Rightarrow g=1$. Consequently we have $g=1\Leftrightarrow g^{(k)}=0$. Note that

$$\left|\frac{R_2(w_j+z_j+\rho_j\xi)}{R_1(w_j+z_j+\rho_j\xi)}\right|=\left|\frac{P_2(w_j+z_j+\rho_j\xi)}{P_1(w_j+z_j+\rho_j\xi)}\right|=\begin{cases}O(1), & \text{if } \deg(P_2)\leq\deg(P_1),\\ O(|w_j|^t), & \text{if } \deg(P_2)>\deg(P_1),\end{cases} \tag{22}$$

where $t=\deg(P_2)-\deg(P_1)>0$. Now let $kN>t$. Therefore from (16) we have

$$\lim_{j\to\infty}\rho_j^k|w_j|^t\leq\lim_{j\to\infty}M^k|w_j|^{t-kN}=0. \tag{23}$$

Since $\rho_j\to0$ as $j\to\infty$, from (22) and (23), we have

$$\rho_j^k\left|\frac{R_2(w_j+z_j+\rho_j\xi)}{R_1(w_j+z_j+\rho_j\xi)}\right|\to0\ \ (\text{as }\ j\to\infty). \tag{24}$$

By the given condition, we have $\lim\limits_{z\to\infty}\frac{P_2(z)}{P_1(z)}=\lim\limits_{z\to\infty}\frac{R_2(z)}{R_1(z)}\neq1$. Also we see that

$$\lim_{j\to\infty}\frac{R_2(w_j+z_j+\rho_j\xi)}{R_1(w_j+z_j+\rho_j\xi)}=\lim_{j\to\infty}\frac{P_2(w_j+z_j+\rho_j\xi)}{P_1(w_j+z_j+\rho_j\xi)}=\begin{cases}\infty, & \text{if } \deg(P_2)>\deg(P_1),\\ 0, & \text{if } \deg(P_2)<\deg(P_1),\\ c_0, & \text{if } \deg(P_2)=\deg(P_1),\end{cases} \tag{25}$$

where $c_0\in\mathbb{C}\setminus\{0,1\}$. Now from (17) and (24), we see that

$$\rho_j^k\frac{f^{(k)}(\omega_j+z_j+\rho_j\xi)-R_2(\omega_j+z_j+\rho_j\xi)}{R_1(\omega_j+z_j+\rho_j\xi)}\to g^{(k)}(\xi). \tag{26}$$

Suppose that $g^{(k)}(\xi_0) = 0$. Then by (26) and Hurwitz's Theorem there exists a sequence $(\xi_j)_j$, $\xi_j \to \xi_0$ such that (for sufficiently large j) $f^{(k)}(\omega_j + z_j + \rho_j\xi_j) = R_2(\omega_j + z_j + \rho_j\xi_j)$. By the given condition we have $f(\omega_j + z_j + \rho_j\xi_j) = R_2(\omega_j + z_j + \rho_j\xi_j)$. Therefore from (14) and (25), we have

$$g(\xi_0) = \lim_{j\to\infty} \frac{f(\omega_j + z_j + \rho_j\xi_j)}{R_1(\omega_j + z_j + \rho_j\xi_j)} = \lim_{j\to\infty} \frac{R_2(\omega_j + z_j + \rho_j\xi_j)}{R_1(\omega_j + z_j + \rho_j\xi_j)} = \begin{cases} \infty, & \text{if } \deg(P_2) > \deg(P_1), \\ 0, & \text{if } \deg(P_2) < \deg(P_1), \\ c_0, & \text{if } \deg(P_2) = \deg(P_1). \end{cases}$$

Since g is an entire function, it follows that $g(\xi_0) \neq \infty$. Consequently we have $g(\xi_0) = c_1 \in \mathbb{C}\backslash\{1\}$. Therefore we have $g^{(k)} = 0 \Rightarrow g = c_1$. Since $g = 1 \Leftrightarrow g^{(k)} = 0$, we arrive at a contradiction.

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