Applied Mathematics E-Notes, 22(2022), 561-565 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Functional Equations Characterizing Certain Determinants And Permanents*

Wuttichai Suriyacharoen†, Vichian Laohakosol‡

Received 11 June 2021

Abstract

General solution functions $g, h, f : \mathbb{R}^2 \to \mathbb{R}$ of the following three functional equations

$$\begin{aligned} g(ux - vy, uy + v(x + y)) &= g(x, y)g(u, v), \\ h(ux - vy, uy + v(x - y)) &= h(x, y)h(u, v), \\ f(vx + uy, vy - ux) &= f(x, y)f(u, v) \end{aligned}$$

are determined without any regularity assumptions on the unknown functions. These equations arise from the problem of characterizing certain determinants and permanents.

1 Introduction

It is easily seen that the 2×2 determinant function defined by

$$d(x, y) := \det \begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 - y^2$$

satisfies the functional equations

$$f(ux + vy, uy + vx) = f(x, y)f(u, v), \tag{1}$$

$$f(ux - vy, uy - vx) = f(x, y)f(u, v), \tag{2}$$

while the 2×2 permanent function defined by

$$p(x, y) := \operatorname{per} \begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 + y^2$$

satisfies the functional equation

$$f(ux + vy, uy - vx) = f(x, y)f(u, v). \tag{3}$$

The functional equation (1), as well as its pexiderized version, was solved by Chung and Sahoo [2] in 2002, while the functional equation (3) was solved first in 2003 by Jung and Bae [6]. The equation (3) was solved again in 2014 by Chung and Chang [3] using a different technique. The functional equation (1) was extended to

$$f(ux + vy, uy + vx, zw) = f(x, y, z)f(u, v, w)$$

*Mathematics Subject Classifications: 39B22, 15A15.

†Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Pathum Thani 12120, Thailand

‡Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand

and was solved by Chung and Sahoo [2] in 2002, while the functional equations (2) and (3) were, respectively, generalized to

$$f(ux - vy, uy - vx, zw) = f(x, y, z)f(u, v, w), \tag{4}$$
$$f(ux + vy, uy - vx, zw) = f(x, y, z)f(u, v, w), \tag{5}$$

and were solved by Choi et al. [1] in 2016; see also [4], [5] for related problems. Closer analysis reveals that earlier techniques used for solving (1)–(5) are not sufficient to solve the pexiderized version of (3) which is

$$f(ux + vy, uy - vx) = g(x, y)h(u, v). \tag{6}$$

In [8], the authors solved, among other things, (6) using the unit circle and trigonometric transformation. Indeed, to solve (6), it suffices to solve its non–pexiderized version, viz.,

$$f(vx + uy, vy - ux) = f(x, y)f(u, v). \tag{7}$$

In another direction, from the identity

$$\text{per}\begin{pmatrix} ux - vy & uy + v(x+y) \\ uy + v(x+y) & vx + u(x+y) \end{pmatrix} = \text{per}\begin{pmatrix} x & y \\ y & x+y \end{pmatrix}\text{per}\begin{pmatrix} u & v \\ v & u+v \end{pmatrix}.$$

If the 2×2 permanent function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$g(x, y) := \text{per}\begin{pmatrix} x & y \\ y & x+y \end{pmatrix} = x^2 + xy + y^2,$$

then this function g satisfies

$$g(ux - vy, uy + v(x + y)) = g(x, y)g(u, v), \tag{8}$$

while the functional equation

$$h(ux - vy, uy + v(x - y)) = h(x, y)h(u, v) \tag{9}$$

is satisfied by the 2×2 permanent function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h(x, y) := \text{per}\begin{pmatrix} x & y \\ y & x-y \end{pmatrix} = x^2 - xy + y^2.$$

It is thus a natural question to search for general solution functions $g, h : \mathbb{R}^2 \to \mathbb{R}$ of the two functional equations (8) and (9). The objective of this work is to solve both (8) and (9) by employing a new technique which can also be applied to give a simple treatment of the functional equation (7), solved in [8].

2 Main Results

Our analysis involves the use of the *exponential* and $E : \mathbb{R} \to \mathbb{R}$ and *multiplicative* function $M : \mathbb{R} \to \mathbb{R}$ which are generically defined [7, Chapter 1], respectively, as general solutions of the following two functional equations

$$E(x + y) = E(x)E(y), \quad M(xy) = M(x)M(y).$$

Theorem 1 *General solution functions $g, h : \mathbb{R}^2 \to \mathbb{R}$ of the functional equations (8) and (9) are given by*

- *for the functional equation (8): $g \equiv 0$, or $g \equiv 1$, or*

$$g(x, y) = M\left(\sqrt{x^2 + xy + y^2}\right)E(\theta_1);$$

- *for the functional equation* (9)*:* $h \equiv 0$, *or* $h \equiv 1$, *or*

$$h(x,y) = M\left(\sqrt{x^2-xy+y^2}\right)E(\theta_2),$$

where M *and* E *are the multiplicative and exponential functions defined above, and* θ_1 *and* θ_2 *are the angular coordinates of the polar coordinates of* $((2x+y)/2, -\sqrt{3}y)/2)$ *and* $((2x-y)/2, -\sqrt{3}y/2)$*, respectively.*

Proof. To solve (8), note first that if g is a constant function solution of (8), then either $g \equiv 0$ or $g \equiv 1$. Assume now that g is a non–constant solution of (8). Substituting x by $x + y/\sqrt{3}$, y by $-2y/\sqrt{3}$, u by $u + v/\sqrt{3}$ and v by $-2v/\sqrt{3}$ into (8), we have

$$g\left(ux - vy + (uy+vx)/\sqrt{3}, -2(uy+vx)/\sqrt{3}\right) = g\left(x + y/\sqrt{3}, -2y/\sqrt{3}\right) g\left(u + v/\sqrt{3}, -2v/\sqrt{3}\right). \tag{10}$$

Define the function $G : \mathbb{C} \to \mathbb{R}$ by

$$G(x+iy) = g(x + y/\sqrt{3}, -2y/\sqrt{3}) \qquad (x, y \in \mathbb{R}). \tag{11}$$

Then for $z = x + iy$, $w = u + iv \in \mathbb{C}$, using (10) and (11), we get

$$\begin{aligned} G(zw) &= G(ux - vy + i(uy+vx)) = g\left(ux - vy + (uy+vx)/\sqrt{3}, -2(uy+vx)/\sqrt{3}\right) \\ &= g\left(x + y/\sqrt{3}, -2y/\sqrt{3}\right) g\left(u + v/\sqrt{3}, -2v/\sqrt{3}\right) = G(x+iy)G(u+iv) \\ &= G(z)G(w). \end{aligned} \tag{12}$$

Writing complex numbers in polar form

$$z = r_1(\cos\theta_1 + i\sin\theta_1), \quad w = r_2(\cos\theta_2 + i\sin\theta_2),$$

where $r_1, r_2 \in \mathbb{R}_{\geq 0}$, and the angular coordinates $\theta_1, \theta_2 \in \mathbb{R}$ are considered mod 2π, and substituting into (12), we get

$$G(r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))) = G(r_1(\cos\theta_1 + i\sin\theta_1))G(r_2(\cos\theta_2 + i\sin\theta_2)). \tag{13}$$

Define the function $\tilde{G} : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{G}(r,\theta) = G(r(\cos\theta + i\sin\theta)), \tag{14}$$

where θ is considered mod 2π; note that the function $\tilde{G}$ is periodic mod 2π in the second variable θ. Using (13), we have

$$\tilde{G}(r_1r_2, \theta_1+\theta_2) = \tilde{G}(r_1,\theta_1)\tilde{G}(r_2,\theta_2). \tag{15}$$

Taking $\theta_1 = \theta_2 = 0$ in (15), we get

$$\tilde{G}(r_1r_2, 0) = \tilde{G}(r_1, 0)\tilde{G}(r_2, 0). \tag{16}$$

Define the function $M : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by

$$M(x) = \tilde{G}(x, 0). \tag{17}$$

Using (16) and (17), we deduce that M is multiplicative. In another direction, setting $r_1 = r_2 = 1$ in (15), we have

$$\tilde{G}(1, \theta_1+\theta_2) = \tilde{G}(1,\theta_1)\tilde{G}(1,\theta_2). \tag{18}$$

Define the periodic function $E : \mathbb{R} \to \mathbb{R}$, with period 2π, by

$$E(y) = \tilde{G}(1, y). \tag{19}$$

From (18) and (19), we see that E is an exponential function, which is periodic with period 2π. Taking $r_2 = 1,\ \theta_1 = 0$ in (15), and applying (14), (17), (19), we obtain

$$G(r_1(\cos\theta_2 + i\sin\theta_2)) = \tilde{G}(r_1, 0)\tilde{G}(1, \theta_2) = M(r_1)E(\theta_2),$$

and the first part follows.

To solve (9), if h is a constant function solution, then clearly either $h \equiv 0$ or $h \equiv 1$. Assume now that h is a non–constant solution of (9). Replacing x by $x - y/\sqrt{3}$, y by $-2y/\sqrt{3}$, u by $u - v/\sqrt{3}$ and v by $-2v/\sqrt{3}$ into (9), we have

$$h\left(ux - vy - (uy + vx)/\sqrt{3}, -2(uy + vx)/\sqrt{3}\right) = h\left(x - y/\sqrt{3}, -2y/\sqrt{3}\right) h\left(u - v/\sqrt{3}, -2v/\sqrt{3}\right), \quad (20)$$

Define the function $H : \mathbb{C} \to \mathbb{R}$ by

$$H(x + iy) = h\left(x - y/\sqrt{3}, -2y/\sqrt{3}\right) \qquad (x, y \in \mathbb{R}). \quad (21)$$

Putting $z = x + iy, w = u + iv \in \mathbb{C}$, using (20) and (21), we get

$$\begin{aligned} H(zw) &= H(ux - vy + i(uy + vx)) = h\left(ux - vy - (uy + vx)/\sqrt{3}, -2(uy + vx)/\sqrt{3}\right) \\ &= h\left(x - y/\sqrt{3}, -2y/\sqrt{3}\right) h\left(u - v/\sqrt{3}, -2v/\sqrt{3}\right) = H(x + iy)H(u + iv) \\ &= H(z)H(w), \end{aligned}$$

which is of the form of (12). Using similar arguments as in the first part, the second assertion follows. That the function solutions so obtained do indeed satisfy the corresponding functional equations are easily verified. ■

As a by-product, we now solve (7) using the technique presented in Theorem 1.

Theorem 2 *General solution functions $f : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation (7) are given by $f \equiv 0$, or $f \equiv 1$, or*

$$f(x, y) = M\left(\sqrt{x^2 + y^2}\right) E(\theta_0),$$

where M and E are the multiplicative and exponential functions, θ_0 is the angular coordinate of the polar coordinate of $(y, -x)$.

Proof. If f is a constant function solution of (7), then either $f \equiv 0$ or $f \equiv 1$. Assume henceforth that f is a non–constant solution of (7). Substituting x by $-x$ and u by $-u$ into (7), we have

$$f(-uy - vx, vy - ux) = f(-x, y)f(-u, v). \quad (22)$$

Interchanging x with y and u with v, (22) becomes

$$f(-uy - vx, ux - vy) = f(-y, x)f(-v, u). \quad (23)$$

Putting $F(x, y) := f(-y, x)$, the equation (23) becomes

$$F(ux - vy, uy + vx) = F(x, y)F(u, v). \quad (24)$$

Defining the function $\tilde{F} : \mathbb{C} \to \mathbb{R}$ via

$$\tilde{F}(x + iy) = F(x, y) \qquad (x, y \in \mathbb{R}), \quad (25)$$

and substituting into (24), we get

$$\begin{aligned} \tilde{F}(zw) &= \tilde{F}(ux - vy + i(uy + vx)) = F(ux - vy, uy + vx) = F(x, y)F(u, v) \\ &= \tilde{F}(x + iy)\tilde{F}(u + iv) = \tilde{F}(z)\tilde{F}(w), \end{aligned}$$

which is of the form of (12). Proceeding in the same manner as in the proof of Theorem 1, the desired result follows. It is easily verified that the functions so obtained satisfy (7). ■

References

[1] C. K. Choi, J. Kim and B. Lee, Stability of two functional equations arising from determinant of matrices, Commun. Korean Math. Soc., 31(2016), 495–505.

[2] J. K. Chung and P. K. Sahoo, General solution of some functional equations related to the determinant of symmetric matrices, Demonstratio Math., 35(2002), 539–544.

[3] J. Chung and J. Chang, On two functional equations originating from number theory, Proc. Indian Acad. Sci. (Math. Sci.), 125(2014), 563–572.

[4] K. B. Houston and P. K. Sahoo, On a functional equation related to the determinant of symmetric two-by-two matrices, Sarajevo J. Math., 3(2007), 1–12.

[5] K. B. Houston and P. K. Sahoo, On two functional equations and their solutions, Appl. Math. lett., 21(2008), 974–977.

[6] S. M. Jung, J. H. Bae: Some functional equations originating from number theory, Proc. Indian Acad. Sci. (Math. Sci.), 113(2003), 91–98.

[7] Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer, Dordrecht, 2009.

[8] V. Laohakosol and W. Suriyacharoen, Solution of a general pexiderized permanent functional equation, Proc. Indian Acad. Sci. (Math. Sci.), (2019), 129:6.