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On Inequalities For The Derivative Of A Polynomial With Restricted Zeros*

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Abstract

If $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ is a complex polynomial of degree n having all its zeros in $|z| \leq K$ where $K \geq 1$, then Kumar [8] proved that

$$\max_{|z|=1} |P'(z)| \geq \left(\frac{2}{1+K^n} + \frac{(|a_n|K^n - |a_0|)(K-1)}{(1+K^n)(|a_n|K^n + |a_0|K)} \right) \sum_{j=1}^{n} \frac{K}{K+|z_j|} \max_{|z|=1} |P(z)|. \tag{A}$$

In this paper we first extend inequality (A) to the class of polynomials having s-fold zero at origin and then establish the polar derivative analogue of the result obtained.

1 Introduction

A well known inequality due to Bernstein [5] states that if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

In connection with inequality (1), P. Erdös conjectured and later Lax [9] proved that if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

The inequality (2) is best possible and equality holds if $P(z) = \alpha + \beta z$, where $|\alpha| = |\beta|$. On the other hand Turan's classical inequality [14] provides the lower bound estimate to the size of derivative of a polynomial on the unit circle relative to the size of polynomial itself when zeros lie in $|z| \leq 1$. It states that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{3}$$

Equality in (3) holds for polynomials having all zeros on $|z| = 1$. As a generalisation of (3) to the polynomials having all their zeros in $|z| \leq K$ where $K \geq 1$, Govil [6] proved if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K,\ K \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |P(z)|. \tag{4}$$

The inequality (4) is sharp and equality holds for the polynomial $P(z) = z^n + K^n$. While considering the modulus of each zero of $P(z)$ in inequality (3), Aziz [1] established the following generalisation of

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inequality (3) to the class of polynomials having all their zeros in $|z| \leq K$ where $K \geq 1$ by proving that if $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ is a complex polynomial of degree n with $|z_j| \leq K,\ K \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{2}{1+K^n} \sum_{j=1}^{n} \frac{K}{K+|z_j|} \max_{|z|=1} |P(z)|. \tag{5}$$

Very recently Kumar [8] while preserving the modulus of each zero in the inequality (5) sharpened the inequality by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K,\ K \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \left(\frac{2}{1+K^n} + \frac{(|a_n|K^n - |a_0|)(K-1)}{(1+K^n)(|a_n|K^n + |a_0|K)} \right) \sum_{j=1}^{n} \frac{K}{K+|z_j|} \max_{|z|=1} |P(z)|. \tag{6}$$

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial of degree n with respect to a real or complex number α. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polar derivative $D_\alpha P(z)$ is a polynomial of degree at most $n-1$. Furthermore, it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R,\ R > 0$.

For more information about the polar derivative of a polynomial one can refer monographs by Rahman and Schmeisser or Milovanovic et al. [10]. The analogue of inequality (4) for the polar derivative of a polynomial was established by Aziz and Rather [3] who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K,\ K \geq 1$, then for every $\alpha \in C$ with $|\alpha| \geq K$

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - K}{1+K^n} \right) \max_{|z|=1} |P(z)|. \tag{7}$$

Several refinements of inequality (7) can be found in the literature (see [4], [12] and [13]). For the class of polynomials having s-fold zero at origin, inequality (7) was recently refined by Govil and Kumar [7] by establishing that if $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s} z^{n-s}),\ 0 \leq s \leq n$ is a polynomial of degree n having all its zeros in $|z| \leq K,\ K \geq 1$, then for every $\alpha \in C$ with $|\alpha| \geq K$

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{|\alpha| - K}{1+K^n} \left(n + s + \frac{|a_{n-s}|K^{n-s} - |a_0|}{|a_{n-s}|K^{n-s} + |a_0|} \right) \max_{|z|=1} |P(z)|.$$

2 Main Results

In this paper we generalize the inequality (6) to the class of polynomials having s-fold zero at origin. In fact we prove

Theorem 1 *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s} z^{n-s}) = a_{n-s} z^s \prod_{j=1}^{n-s}(z - z_j),\ 0 \leq s \leq n$ with $z_j \neq 0$ for $1 \leq j \leq n-s$ is a polynomial of degree n which has all its zeros in $|z| \leq K$ with $K \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \left(\frac{2}{1+K^{n-s}} + \frac{(K-1)(|a_{n-s}|K^{n-s} - |a_0|)}{(1+K^{n-s})(|a_{n-s}|K^{n-s} + |a_0|K)} \right) \left(s + \sum_{j=1}^{n-s} \frac{K}{K+|z_j|} \right) \max_{|z|=1} |P(z)|. \tag{8}$$

Remark 1 *If we take $s = 0$ in Theorem 1, we obtain inequality (6).*

If we take $K = 1$ in Theorem 1, we obtain the following refinement of inequality (3) for the polynomials having s-fold zero at origin.

Corollary 1 *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s}) = a_{n-s}z^s \prod_{j=1}^{n-s}(z - z_j),\ 0 \leq s \leq n$ with $z_j \neq 0$ for $1 \leq j \leq n - s$ is a polynomial of degree n which has all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \left(s + \sum_{j=1}^{n-s} \frac{1}{1 + |z_j|} \right) \max_{|z|=1} |P(z)|.$$

We next prove the following extension of Theorem 1 to the polar derivative of a polynomial having s-fold zero at origin.

Theorem 2 *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s}) = a_{n-s}z^s \prod_{j=1}^{n-s}(z - z_j),\ 0 \leq s \leq n$ with $z_j \neq 0$ for $1 \leq j \leq n - s$ is a polynomial of degree n having all its zeros in $|z| \leq K,\ K \geq 1$, then for any complex number α with $|\alpha| \geq K$*

$$\begin{aligned} \max_{|z|=1} |D_\alpha P(z)| \quad &\geq \quad \left(\frac{2}{1 + K^{n-s}} + \frac{(K-1)(|a_{n-s}|K^{n-s} - |a_0|)}{(1 + K^{n-s})(|a_{n-s}|K^{n-s} + |a_0|K)} \right) \\ &\times \left(s(|\alpha| - K) + \sum_{j=1}^{n-s} \frac{K(|\alpha| - K)}{K + |z_j|} \right) \max_{|z|=1} |P(z)|. \end{aligned} \tag{9}$$

Remark 2 *If we divide both sides to inequality (9) by $|\alpha|$ and let $|\alpha| \to \infty$ in (9), we get (8) and thus Theorem 1 is a special case of Theorem 2.*

Remark 3 *If we take $s = 0$ in Theorem 2, we obtain Theorem 1.4 due to Kumar [8].*

3 Lemmas

The first lemma is the generalization of Schwarz Lemma due to Osserman [11].

Lemma 1 *Let $f(z)$ be analytic in $|z| < 1$ such that $|f(z)| < 1$ for $|z| < 1$ and $f(0) = 0$. Then*

$$|f(z)| \leq |z| \frac{|z| + |f'(0)|}{1 + |f'(0)|\,|z|} \quad \text{for } |z| < 1.$$

The next lemma is due to Aziz and Mohammad [2].

Lemma 2 *If $P(z)$ is a polynomial of degree n, then for any $R \geq 1$ and $0 \leq \theta \leq 2\pi$*

$$|P(Re^{i\theta})| + |Q(Re^{i\theta})| \leq (1 + R^n) \max_{|z|=1} |P(z)|$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Lemma 3 *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s}),\ 0 \leq s \leq n$ is a polynomial of degree $n \geq 1$ having s-fold zero at origin and all other zeros in $|z| \geq 1$, then for any $R \geq 1$*

$$\max_{|z|=R} |P(z)| \leq \frac{(1 + R^n)(|a_0| + R|a_{n-s}|)}{(1 + R)(|a_0| + |a_{n-s}|)} \max_{|z|=1} |P(z)|.$$

Proof. Let $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s}) = z^s A(z)$, where $A(z) = a_0 + a_1 z + ... + a_{n-s}z^{n-s}$ is a polynomial of degree $n-s$. Then $A(z)$ has no zero in $|z| < 1$. Therefore the conjugate polynomial $B(z) := z^{n-s}\overline{A(1/\overline{z})}$ of $A(z)$ has all its zeros in $|z| \leq 1$. It follows that the polynomial $F(z) = \frac{zB(z)}{A(z)}$ satisfies the hypothesis of Lemma 1 and hence we obtain for $|z| < 1$,

$$|F(z)| \leq |z| \frac{|z| + |F'(0)|}{1 + |F'(0)|\,|z|}$$

which is equivalent to

$$|B(z)| \leq \frac{|z|\,|a_0| + |a_{n-s}|}{|a_0| + |a_{n-s}|\,|z|}|A(z)| \quad \text{for} \quad |z| < 1. \tag{10}$$

Replacing z by $1/z$ in (10), we get for $|z| > 1$

$$|z^s A(z)| \leq \frac{|a_0| + |a_{n-s}|\,|z|}{|a_0|\,|z| + |a_{n-s}|}|B(z)|. \tag{11}$$

Since the inequality (11) is already true for all z on $|z| = 1$. Therefore for any $R \geq 1$ and $0 \leq \theta < 2\pi$, we have

$$|P(Re^{i\theta})| \leq \frac{|a_0| + |a_{n-s}|R}{|a_0|R + |a_{n-s}|}|B(Re^{i\theta})|. \tag{12}$$

Inequality (12) in conjunction with Lemma 2 and the fact that $z^n\overline{P(1/\overline{z})} = B(z)$ yields the desired inequality. ■

Lemma 4 *If $P(z) = z^s(a_0 + a_1 z + ... + a_{n-s}z^{n-s}),\ 0 \leq s \leq n$ is a polynomial of degree n with all its zeros in $|z| \leq K$ and $K \geq 1$, then*

$$\max_{|z|=K}|P(z)| \geq \left(\frac{2K^n}{1+K^{n-s}} + \frac{K^n(K-1)(|a_{n-s}|K^{n-s} - |a_0|)}{(1+K^{n-s})(|a_{n-s}|K^{n-s} + |a_0|K)}\right)\max_{|z|=1}|P(z)|.$$

Proof. Since $P(z)$ has all its zeros in $|z| \leq K,\ K \geq 1$, the polynomial $G(z) = P(Kz)$ has all its zeros in the unit disc $|z| \leq 1$. Hence the $(n-s)$th degree polynomial $H(z) = z^n G(1/z)$ has no zero in $|z| < 1$. Therefore applying Lemma 3 to the polynomial $H(z)$ with $R = K,\ K \geq 1$, we have

$$\max_{|z|=K}|H(z)| \leq \frac{(1+K^{n-s})(|a_{n-s}|K^n + |a_0|K^{s+1})}{(1+K)(|a_{n-s}|K^n + |a_0|K^s)}\max_{|z|=1}|H(z)|,$$

which is equivalent to

$$\max_{|z|=1}|G(z)| \geq \frac{(1+K)(|a_{n-s}|K^n + |a_0|K^s)}{(1+K^{n-s})(|a_{n-s}|K^n + |a_0|K^{s+1})}\max_{|z|=K}|H(z)|. \tag{13}$$

But $H(z) = z^n G(1/z) = z^n P(K/z)$ so that

$$\max_{|z|=K}|H(z)| = K^n \max_{|z|=1}|P(z)|. \tag{14}$$

Using (14) in (13), we get

$$\max_{|z|=1}|G(z)| \geq K^n \frac{(1+K)(|a_{n-s}|K^n + |a_0|K^s)}{(1+K^{n-s})(|a_{n-s}|K^n + |a_0|K^{s+1})}\max_{|z|=1}|P(z)|. \tag{15}$$

Replacing $G(z)$ by $P(Kz)$ in (15) and simplifying we get

$$\max_{|z|=K}|P(z)| \geq \left(\frac{2K^n}{1+K^{n-s}} + \frac{K^n(K-1)(|a_{n-s}|K^{n-s} - |a_0|)}{(1+K^{n-s})(|a_{n-s}|K^{n-s} + |a_0|K)}\right)\max_{|z|=1}|P(z)|.$$

■

4 Proofs of Theorems

Proof of Theorem 1. Since $P(z) = a_{n-s}z^s \prod_{j=1}^{n-s}(z - z_j),\ 0 \leq s \leq n$ has all its zeros in $|z| \leq K$, the polynomial $G(z) = P(Kz) = K^n a_{n-s} z^s \prod_{j=1}^{n-s}(z - z_j/K)$ has all its zeros in $|z| \leq 1$. Hence for all z on $|z| = 1$ for which $G(z) \neq 0$, we have

$$\frac{zG'(z)}{G(z)} = s + \sum_{j=1}^{n-s} \frac{z}{z - \frac{z_j}{K}}.$$

This gives

$$Re\left(\frac{zG'(z)}{G(z)}\right) = s + Re\left(\sum_{j=1}^{n-s} \frac{z}{z - z_j/K}\right) \geq s + \sum_{j=1}^{n-s} \frac{K}{K + |z_j|}.$$

Which implies

$$\left|\frac{zG'(z)}{G(z)}\right| \geq s + \sum_{j=1}^{n-s} \frac{K}{K + |z_j|}$$

for all z on $|z| = 1$ for which $G(z) \neq 0$. Therefore

$$\max_{|z|=1} |G'(z)| \geq \left(s + \sum_{j=1}^{n-s} \frac{K}{K + |z_j|}\right) \max_{|z|=1} |G(z)|, \tag{16}$$

or equivalently

$$K \max_{|z|=1} |P'(Kz)| \geq \left(s + \sum_{j=1}^{n-s} \frac{K}{K + |z_j|}\right) \max_{|z|=1} |P(Kz)|.$$

Using Lemma 4 and the fact that $K^{n-1} \max_{|z|=1} |P'(z)| \geq |P'(Kz)|$, we get

$$K^n \max_{|z|=1} |P'(z)| \geq \left(s + \sum_{j=1}^{n-s} \frac{K}{K + |z_j|}\right) \times \left(\frac{2K^n}{1 + K^{n-s}} + \frac{K^n(K-1)(|a_{n-s}|K^{n-s} - |a_0|)}{(1 + K^{n-s})(|a_{n-s}|K^{n-s} + |a_0|K)}\right) \max_{|z|=1} |P(z)|,$$

which is equivalent to

$$\max_{|z|=1} |P'(z)| \geq \left(\frac{2}{1 + K^{n-s}} + \frac{(K-1)(|a_{n-s}|K^{n-s} - |a_0|)}{(1 + K^{n-s})(|a_{n-s}|K^{n-s} + |a_0|K)}\right) \times \left(s + \sum_{j=1}^{n-s} \frac{K}{K + |z_j|}\right) \max_{|z|=1} |P(z)|.$$

This completes the proof of Theorem 1. ∎

Proof of Theorem 2. Since $P(z)$ has all its zeros in $|z| \leq K,\ K \geq 1$, the polynomial $G(z) = P(Kz)$ has all its zeros in $|z| \leq 1$. Therefore for $|\alpha|/K \geq 1$, it can be easily seen that

$$\max_{|z|=1} |D_{\alpha/K} G(z)| \geq \frac{(|\alpha| - K)}{K} \max_{|z|=1} |G'(z)|,$$

or

$$\max_{|z|=K} |D_\alpha P(z)| \geq \frac{(|\alpha| - K)}{K} \max_{|z|=1} |G'(z)|.$$

Using inequality (16), we have

$$\max_{|z|=K} |D_\alpha P(z)| \geq \frac{(|\alpha| - K)}{K} \left(s + \sum_{j=1}^{n-s} \frac{K}{K + |z_j|}\right) \max_{|z|=1} |G(z)|,$$

which is equivalent to

$$\max_{|z|=K}|D_\alpha P(z)| \geq \frac{(|\alpha|-K)}{K}\left(s+\sum_{j=1}^{n-s}\frac{K}{K+|z_j|}\right)\max_{|z|=K}|P(z)|.$$

Now applying Lemma 4 in the right hand side of above inequality, we get

$$\begin{aligned}\max_{|z|=K}|D_\alpha P(z)| \quad &\geq \quad \frac{(|\alpha|-K)}{K}\left(s+\sum_{j=1}^{n-s}\frac{K}{K+|z_j|}\right)\\ &\times\left(\frac{2K^n}{1+K^{n-s}}+\frac{K^n(K-1)(|a_{n-s}|K^{n-s}-|a_0|)}{(1+K^{n-s})(|a_{n-s}|K^{n-s}+|a_0|K)}\right)\max_{|z|=1}|P(z)|. \qquad (17)\end{aligned}$$

Since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, it follows that

$$\max_{|z|=K}|D_\alpha P(z)| \leq K^{n-1}\max_{|z|=1}|D_\alpha P(z)|.$$

Using this observation in (17), we obtain

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \quad &\geq \quad \left(s(|\alpha|-K)+\sum_{j=1}^{n-s}\frac{K(|\alpha|-K)}{K+|z_j|}\right)\\ &\times\left(\frac{2}{1+K^{n-s}}+\frac{(K-1)(|a_{n-s}|K^{n-s}-|a_0|)}{(1+K^{n-s})(|a_{n-s}|K^{n-s}+|a_0|K)}\right)\max_{|z|=1}|P(z)|.\end{aligned}$$

Which is the desired inequality and completes the proof of Theorem 2. ■

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References

[1] A. Aziz, Integral mean estimates for polynomials with restricted zeros, J. Approx. Theory, 55(1988), 232–239.

[2] A. Aziz and Q. G. Mohammad, Growth of polynomials with zeros outside a circle, Proc. Am. Math. Soc., 81(1981), 549–553.

[3] A. Aziz and N. A. Rather, A refinement of a theorem of Paul Turan concerning polynomials, Math. inequal. Appl., 1(1998), 231–238.

[4] A. Aziz and N. A. Rather, Inequalities for the polar derivative of a polynomial with restricted zeros, Math. Balk., 17(2003), 15–28.

[5] S. Bernstein, Lecons Sur Les Proprietes Extremales Et La Meilleure Approximation Des Fonctions Analytiques D'une Variable Reelle, Gauthier-Villars, Paris (1926).

[6] N. K. Govil, On the derivative of a polynomial, Proc. Am. Math. Soc., 41(1973), 543–546.

[7] N. K. Govil and P. Kumar, On sharpening of an inequality of Turan, Appl. Anal. Discrete Math., 13(2019), 711–720.

[8] P. Kumar, On the inequalities concerning polynomials, Complex Anal. Oper. Theory, 14(2020), 1–11.

[9] P. D. Lax, Proof of a conjecture due to Erdös on the derivative of a polynomial, Bull. Am. Math. Soc., 50(1944), 509–513.

[10] G. V. Milovanovic, D. S. Mitrinovic and Th. M. Rassias, Topics in Polynomials, Extremal Properties, Inequalities, Zeros, World Scientific Publishing co., Singapore 1994.

[11] R. Osserman, A sharp Schwarz inequality on the boundary, Proc. Am. Math. Soc., 128 (2000), 3513–3517.

[12] N. A. Rather, A. Iqbal and I. Dar, Inequalities concerning the polar derivatives of polynomials with restricted zeros, Non linear Funct. Anal. Appl., 24(2019), 813–826.

[13] N. A. Rather, I. Dar, A. Iqbal, Inequalities for the derivative of polynomials with restricted zeros, Korean J. Math., 28(2020), 931–942.

[14] P. Turan, Űber die Ableitung von Polynomen, Compositio Math., 7(1939), 89–95.