Applied Mathematics E-Notes, 22(2022), 731-740 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

A Generalization Of Refined Young's Inequality *

Mohamed Amine Ighachane[†], Mohamed Akkouchi[‡]

Received 15 February 2021

Abstract

In this paper, we prove that for positive real numbers a, b, τ and μ with $0 \le \mu \le \frac{1}{2} \le \tau \le 1$ and for a positive integer m, we have

$$\frac{\mu^m}{\tau}\Big(a^m \nabla_\tau b^m - (a\sharp_\tau b)^m\Big) + r_m\Big(b^{\frac{m}{2}} - (a\sharp_\tau b)^{\frac{m}{2}}\Big)^2 \le \Big(a\nabla_\mu b\Big)^m - \Big(a\sharp_\mu b\Big)^m$$

and

$$\frac{(1-\tau)^m}{1-\mu}\Big(a^m \nabla_\mu b^m - (a\sharp_\mu b)^m\Big) + r'_m\Big(a^{\frac{m}{2}} - (a\sharp_\mu b)^{\frac{m}{2}}\Big)^2 \le \Big(a\nabla_\tau b\Big)^m - \Big(a\sharp_\tau b\Big)^m,$$

where $a\nabla_\mu b = \mu a + (1-\mu)b$ and $a\sharp_\mu b = a^\mu b^{1-\mu}$ are, respectively, the weighted arithmetic and geometric means, and r_m and r'_m are two positive numbers. Our results extend some fresh results obtained by Kittaneh, Manasrah, Zhao and Wu. As applications we give some Young's type inequalities for operators and matrices.

1 Introduction

We start by reviewing some important facts concerning the classical Young's inequality and its refinements. The famous Young inequality (for scalars) says that for $a, b > 0$ and $0 \le \mu \le 1$, we have

$$a^\mu b^{1-\mu} \le \mu a + (1-\mu)b.$$

Even though this inequality looks very simple, it is of great interest in operator theory. Refining this inequality has taken the attention of many researchers in the field, where adding a positive term to the left side is possible.

Throughout, we denote $\mu a + (1-\mu)b$ and $a^\mu b^{1-\mu}$, respectively by $a\nabla_\mu b$ and $a\sharp_\mu b$. When $\mu = \frac{1}{2}$, the arithmetic and geometric means can be rewritten by simplification as $a\nabla b$, $a\sharp b$, and the Young inequality reduces to the arithmetic and geometric means inequality

$$\sqrt{ab} \le \frac{a+b}{2}.$$

The first refinement of Young inequality is the squared version proved in [8] as follows

$$(a^\mu b^{1-\mu})^2 + r_0^2(a-b)^2 \le (\mu a + (1-\mu)b)^2, \tag{1}$$

where $r_0 = \min\{\mu, 1-\mu\}$.

Later, Kittaneh and Manasrah [14], obtained the other interesting refinement of Young's inequality

$$a^\mu b^{1-\mu} + r_0(\sqrt{a} - \sqrt{b})^2 \le \mu a + (1-\mu)b, \tag{2}$$

where $r_0 = \min\{\mu, 1-\mu\}$.

Zhao and Wu [16], obtained another refinement of inequality (2) as follows:

*Mathematics Subject Classifications: 47A63, 15A60.

†Department of Mathematics, Faculty of Sciences-Semlalia, University Cadi Ayyad, Av. Prince My. Abdellah, BP: 2390, Marrakesh (40.000-Marrakech), Morocco (Maroc)

‡Department of Mathematics, Faculty of Sciences-Semlalia, University Cadi Ayyad, Av. Prince My. Abdellah, BP: 2390, Marrakesh (40.000-Marrakech), Morocco (Maroc)

1. If $0 < \mu \leq \frac{1}{2}$, then
$$a^{\mu}b^{1-\mu} + \mu(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{b})^2 \leq \mu a + (1-\mu)b. \tag{3}$$
2. If $\frac{1}{2} < \mu \leq 1$, then
$$a^{\mu}b^{1-\mu} + (1-\mu)(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{a})^2 \leq \mu a + (1-\mu)b, \tag{4}$$
where $r_0 = \min\{\mu, 1-\mu\}$, and $r_1 = \min\{2r_0, 1-2r_0\}$.

Manasrah and Kittaneh [1] gave generalized refinements of (1) and (2) as follows

Theorem 1 **([1])** *Let $a, b > 0$ and m be a positive integer, then we have*
$$\Big(a^{\mu}b^{1-\mu}\Big)^m + r_0^m\Big(a^{\frac{m}{2}} - b^{\frac{m}{2}}\Big)^2 \leq \Big(\mu a + (1-\mu)b\Big)^m, \tag{5}$$
where $r_0 = \min\{\mu, 1-\mu\}$.

The interested reader is referred to [2, 6, 9, 10, 11, 12, 14] as a sample of recent progress in this direction.

Let $B(\mathcal{H})$ denote the $\mathbb{C}^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ is called positive, denoted as $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. The set of all positive operators is denoted by $B(\mathcal{H})^+$. The set of all invertible operators in $B(\mathcal{H})^+$, is denoted by $B(\mathcal{H})^{++}$, and $\mathbf{M}_n(\mathbb{C})$ denotes the space of $n \times n$ complex matrices. The singular values of a matrix $A \in \mathbf{M}_n(\mathbb{C})$ are the eigenvalues of the positive semi-definite matrix $|A| = (A^*A)^{1/2}$, denoted by $s_i(A)$ for $i = 1, 2, 3, \ldots, n$. A norm $|||.|||$ on $\mathbf{M}_n(\mathbb{C})$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbf{M}_n(\mathbb{C})$ and all unitary matrices $U, V \in \mathbf{M}_n(\mathbb{C})$. The trace norm is given by $||A||_1 = tr|A| = \sum_{k=1}^n s_k(A)$, where tr is the usual trace. This norm is unitarily invariant.

Let $A, B \in B(\mathcal{H})^{++}$ and $\mu \in [0, 1]$. The $\mu-$weighted operators geometric mean of A and B, denoted by $A\sharp_{\mu}B$, is defined as
$$A\sharp_{\mu}B := A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{\mu}A^{1/2},$$
and the μ-weighted operators arithmetic mean of A and B is defined as
$$A\nabla_{\mu}B := (1-\mu)A + \mu B.$$
If $\mu = \frac{1}{2}$, these operators can be rewritten by simplification as $A\nabla B$ and $A\sharp B$.

This paper is organised as follows. In Section 2, we establish in Theorem 3 a new generalized refinements of Young's inequality. Section 3, is devoted to certain applications of Theorem 3, to deduce (see Theorem 4) Young's type inequalities for operators. Section 4, is devoted to certain applications of the main result of the second section therein obtain a new refinement of Young's type inequalities for determinants, trace and norms of positive definite matrices.

2 A Generalization of Refined Young's Inequality

Before giving the main result, we need the following theorem concerning the arithmetic-geometric mean inequality.

Theorem 2 *Let n be a positive integer. For $k = 1, 2, \ldots, n$, let $x_k > 0$ and let $\mu_k \geq 0$ satisfy $\sum_{k=1}^n \mu_k = 1$. Then, we have*
$$\prod_{k=1}^n x_k^{\mu_k} \leq \sum_{k=1}^n \mu_k x_k. \tag{6}$$

We need also the following two lemmas.

Lemma 1 *Let m be a positive integer and let μ be a positive number, such that $0 \leq \mu \leq 1$. Then we have*

$$\sum_{k=1}^{m} \binom{m}{k} k\mu^k (1-\mu)^{m-k} = m\mu$$

and

$$\sum_{k=0}^{m-1} \binom{m}{k} (m-k)\mu^k (1-\mu)^{m-k} = m(1-\mu)$$

where $\binom{m}{k}$ is the binomial coefficient.

Proof. For any non-negative real numbers x_1 and x_2, we have

$$(x_1 + x_2)^m = \sum_{k=0}^{m} \binom{m}{k} x_1^k x_2^{m-k}, \tag{7}$$

by derivation of (7) with respect x_1 and x_2 respectively we find that

$$m(x_1 + x_2)^{m-1} = \sum_{k=1}^{m} \binom{m}{k} k x_1^{k-1} x_2^{m-k}, \tag{8}$$

and

$$m(x_1 + x_2)^{m-1} = \sum_{k=0}^{m-1} \binom{m}{k} (m-k) x_1^k x_2^{m-k-1}. \tag{9}$$

By multiplying (8) and (9) by x_1 and x_2 respectively, we get

$$m x_1 (x_1 + x_2)^{m-1} = \sum_{k=1}^{m} \binom{m}{k} k x_1^k x_2^{m-k}, \tag{10}$$

and

$$m x_2 (x_1 + x_2)^{m-1} = \sum_{k=0}^{m-1} \binom{m}{k} (m-k) x_1^k x_2^{m-k}. \tag{11}$$

By setting $x_1 = \mu$ and $x_2 = 1-\mu$ in (10) and (11), respectively, we deduce the result. This completes the proof. ∎

Lemma 2 *Let μ and τ be a two positive numbers such that $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$, and m be a positive number. Then we have*

$$(1-\mu)^m - (1-\tau)\frac{\mu^m}{\tau} \geq 0 \quad \textit{and} \;\; \tau^m - (1-\tau)^m \frac{\mu}{1-\mu} \geq 0.$$

Proof. Under the condition $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$, we have $\frac{1-\tau}{\tau} \leq 1$ and $\frac{\mu}{1-\mu} \leq 1$. So,

$$(1-\mu)^m - (1-\tau)\frac{\mu^m}{\tau} \geq (1-\mu)^m - \mu^m \geq 0$$

and

$$\tau^m - (1-\tau)^m \frac{\mu}{1-\mu} \geq \tau^m - (1-\tau)^m \geq 0.$$

This completes the proof. ∎

Throughout the rest of this paper, $r_m = \min\{\frac{\mu^m}{\tau}, (1-\mu)^m - (1-\tau)\frac{\mu^m}{\tau}\}$ and $r'_m = \min\{\frac{(1-\tau)^m}{1-\mu}, \tau^m - (1-\tau)^m \frac{\mu}{1-\mu}\}$.

The main result to be proved in this paper is the following theorem.

Theorem 3 *Let $a, b > 0$ and $0 \le \mu \le \frac{1}{2} \le \tau \le 1$. Then for all positive integer m,*

$$\frac{\mu^m}{\tau}\Big(a^m \nabla_\tau b^m - (a\sharp_\tau b)^m\Big) + r_m\Big(b^{\frac{m}{2}} - (a\sharp_\tau b)^{\frac{m}{2}}\Big)^2 \le (a\nabla_\mu b)^m - \Big(a\sharp_\mu b\Big)^m \tag{12}$$

and

$$\frac{(1-\tau)^m}{1-\mu}\Big(a^m \nabla_\mu b^m - (a\sharp_\mu b)^m\Big) + r_m^{'}\Big(a^{\frac{m}{2}} - (a\sharp_\mu b)^{\frac{m}{2}}\Big)^2 \le (a\nabla_\tau b)^m - \Big(a\sharp_\tau b\Big)^m. \tag{13}$$

Proof. Suppose that $0 \le \mu \le \frac{1}{2} \le \tau \le 1$. We claim that

$$\Big(a\sharp_\mu b\Big)^m + \frac{\mu^m}{\tau}\Big(a^m \nabla_\tau b^m - (a\sharp_\tau b)^m\Big) + r_m\Big(b^{\frac{m}{2}} - (a\sharp_\tau b)^{\frac{m}{2}}\Big)^2 \le (a\nabla_\mu b)^m. \tag{14}$$

We have, the following identities

$$\begin{aligned}
& (a\nabla_\mu b)^m - \frac{\mu^m}{\tau}\Big(a^m \nabla_\tau b^m - (a\sharp_\tau b)^m\Big) - r_m\Big(b^{\frac{m}{2}} - (a\sharp_\tau b)^{\frac{m}{2}}\Big)^2 \\
= & \sum_{k=0}^{m} \binom{m}{k} \mu^k (1-\mu)^{m-k} a^k b^{m-k} - \frac{\mu^m}{\tau}\Big(\tau a^m + (1-\tau) b^m - (a\sharp_\tau b)^m\Big) \\
& - r_m\Big(b^{\frac{m}{2}} - (a\sharp_\tau b)^{\frac{m}{2}}\Big)^2 \\
= & \sum_{k=0}^{m} \binom{m}{k} \mu^k (1-\mu)^{m-k} a^k b^{m-k} - \mu^m a^m - \mu^m \frac{1-\tau}{\tau} b^m + \frac{\mu^m}{\tau}(a\sharp_\tau b)^m \\
& - r_m\Big((a\sharp_\tau b)^m + b^m - 2(a\sharp_\tau b)^{\frac{m}{2}} b^{\frac{m}{2}}\Big) \\
= & \sum_{k=1}^{m-1} \binom{m}{k} \mu^k (1-\mu)^{m-k} a^k b^{m-k} + \Big((1-\mu)^m - \mu^m \frac{1-\tau}{\tau} - r_m\Big) b^m \\
& + \Big(\frac{\mu^m}{\tau} - r_m\Big)(a\sharp_\tau b)^m + 2 r_m (a\sharp_\tau b)^{\frac{m}{2}} b^{\frac{m}{2}} \\
= & \sum_{k=0}^{m+1} \mu_k x_k,
\end{aligned}$$

where x_k is given by

$$x_0 := b^m, \quad \textit{with} \quad \mu_0 := (1-\mu)^m - \mu^m \frac{1-\tau}{\tau} - r_m,$$

and for $1 \le k \le m-1$,

$$x_k := a^k b^{m-k}, \quad \textit{with} \quad \mu_k := \binom{m}{k} \mu^k (1-\mu)^{m-k},$$

$$x_m := (a\sharp_\tau b)^m, \quad \textit{with} \quad \mu_m := \frac{\mu^m}{\tau} - r_m$$

and

$$x_{m+1} := (a\sharp_\tau b)^{\frac{m}{2}} b^{\frac{m}{2}}, \quad \textit{with} \quad \mu_{m+1} := 2 r_m.$$

By using Lemma 2, we have

1. $x_k > 0$ for all $k \in \{0, 1, \ldots, m, m+1\}$,

2. $\mu_k \ge 0$ for all $k \in \{0, 1, \ldots, m, m+1\}$, with $\sum_{k=0}^{m+1} \mu_k = 1$.

Hence by Theorem 2, we get

$$\begin{aligned}&(a\nabla_\mu b)^m - \frac{\mu^m}{\tau}\Big(a^m\nabla_\tau b^m - (a\sharp_\tau b)^m\Big) - r_m\Big(b^{\frac{m}{2}} - (a\sharp_\tau b)^{\frac{m}{2}}\Big)^2\\ \geq\ & \prod_{k=0}^{m+1} x_k^{\mu_k} = a^{\mu(m)} b^{\tau(m)},\end{aligned}$$

where

$$\begin{aligned}\mu(m) &= \sum_{k=1}^{m-1}\binom{m}{k}(m-k)\mu^k(1-\mu)^{m-k} + m\tau\Big(\frac{\mu^m}{\tau} - r_m\Big)\\ &\quad + \frac{m\tau}{2}2r_m\\ &= \sum_{k=1}^{m}\binom{m}{k}k\mu^k(1-\mu)^{m-k}\\ &= m\mu, \quad \text{(by Lemma 1)}\end{aligned}$$

and

$$\begin{aligned}\tau(m) &= \sum_{k=1}^{m-1}\binom{m}{k}(m-k)\mu^k(1-\mu)^{m-k}\\ &\quad + m\Big((1-\mu)^m - \mu^m\frac{1-\tau}{\tau} - r_m\Big)\\ &\quad + m(1-\tau)\Big(\frac{\mu^m}{\tau} - r_m\Big) + (m - \frac{m\tau}{2})2r_m\\ &= \sum_{k=0}^{m-1}\binom{m}{k}(m-k)\mu^k(1-\mu)^{m-k}\\ &= m(1-\mu) \quad \text{(by Lemma 1)}.\end{aligned}$$

If $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$, then we have $0 \leq 1-\tau \leq \frac{1}{2} \leq 1-\mu \leq 1$, So by changing two elements a, b and two weights μ, τ by $1-\tau$, $1-\mu$ respectively in inequality (14), the desired inequality is obtained. ■

Remark 1 *If we set $m = 1$ in Theorem 3 and $\tau = \frac{1}{2}$ in inequality (12) and $\mu = \frac{1}{2}$ in inequality (13) then we recapture the inequalities (3) and (4).*

3 A Refined Operator Version of Young's Inequalities

In this section, we are concerned by the investigation of operator version of Young's type inequalities for operators.

Before stating and proving our result, we need to recall the following lemma.

Lemma 3 ([15], p. 3) *Let $T \in B(\mathcal{H})$ be self-adjoint. If f and g are both continuous functions with $f(t) \geq g(t)$ for $t \in Sp(T)$ (where the sign $Sp(T)$ denotes the spectrum of operator T), then $f(T) \geq g(T)$.*

The main result of this section reads as follows.

Theorem 4 *Let $A, B \in \mathcal{B}(H)^{++}$ and $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$. Then for all positive integer m, we have*

$$\begin{aligned}&\frac{\mu^m}{\tau}\Big(A\nabla_\tau(A\sharp_m B) - A\sharp_{m\tau}B\Big) + r_m\Big(A + A\sharp_{m\tau}B - 2A\sharp_{\frac{m\tau}{2}}B\Big)\\ \leq\ & A\sharp_m(A\nabla_\mu B) - A\sharp_{m\mu}B\end{aligned}$$

and

$$\begin{aligned}&\frac{(1-\mu)^m}{1-\tau}\Big(A\nabla_\mu(A\sharp_m B)-A\sharp_{m\mu}B\Big)+r'_m\Big(A\sharp_m B+A\sharp_{m\tau}B-2A\sharp_{\frac{m}{2}+\frac{m\tau}{2}}B\Big)\\ \leq\ & A\sharp_m(A\nabla_\tau B)-A\sharp_{m\tau}B.\end{aligned}\tag{15}$$

Proof. Let $b=1$ in inequality (12), then we get

$$\begin{aligned}\frac{\mu^m}{\tau}\Big((\tau a^m+(1-\tau))-a^{m\tau}\Big)\quad &+\quad r_m\Big(1+a^{m\tau}-2a^{\frac{m\tau}{2}}\Big)\\ &\leq\quad (\mu a+(1-\mu))^m-a^{m\mu}.\end{aligned}\tag{16}$$

The operator $C=A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ has a positive spectrum, then by Lemma 3 and inequality (16) we get

$$\begin{aligned}&\frac{\mu^m}{\tau}\Big((\tau C^m+(1-\tau)I)-C^{m\tau}\Big)+r_m\Big(I+C^{m\tau}-2C^{\frac{m\tau}{2}}\Big)^2\\ \leq\ & (\mu C+(1-\mu)I)^m-C^{m\mu}.\end{aligned}\tag{17}$$

Finally, multiplying inequality (17) by $A^{\frac{1}{2}}$ on the left and right hand sides, we get

$$\begin{aligned}&\frac{\mu^m}{\tau}\Big(A\nabla_\tau(A\sharp_m B)-A\sharp_{m\tau}B\Big)+r_m\Big(A+A\sharp_{m\tau}B-2A\sharp_{\frac{m\tau}{2}}B\Big)\\ \leq\ & A\sharp_m(A\nabla_\mu B)-A\sharp_{m\mu}B.\end{aligned}$$

Using the same technique in inequality (13), we get inequality (15). This completes the proof. ■

4 Refinements of Young's Type Inequalities for Matrices

In this section, we give some refined Young type inequalities for traces, determinants, and norms of positive definite matrices. A matrix Young's inequality due to Ando [4] asserts that

$$s_j(A^\mu B^{1-\mu})\leq s_j(\mu A+(1-\mu)B),$$

the above singular value inequality entails the following unitarily invariant norm inequality

$$|||A^\mu B^{1-\mu}|||\leq|||\mu A+(1-\mu)B|||.$$

4.1 Refinements of Young's Type Inequality for Determinants

A determinant version of Young's inequalities is also known [7, p. 467]: For positive semi-definite matrices A, B and $0\leq\mu\leq1$,

$$\det(A^\mu B^{1-\mu})\leq det(\mu A+(1-\mu)B).\tag{18}$$

To prove the main result of this section, we need the following lemma (see, e.g., [7, p. 482,]) is the Minkowski inequality for determinants.

Lemma 4 *Let $A,B\in\mathbf{M}_n(\mathbb{C})$ be positive definite matrices. Then we have*

$$\det{(A+B)^{\frac{1}{n}}}\geq\det(A)^{\frac{1}{n}}+det(B)^{\frac{1}{n}}.\tag{19}$$

The main result to be proved in this subsection is the following theorem.

Theorem 5 *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices and $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$. Then for all positive integer m, we have*

$$\begin{aligned} & \frac{\mu^m}{\tau}\Big(\det(A)^m \nabla_\tau \det(B)^m - (\det(A)\sharp_\tau \det(B))^m\Big) \\ & + r_m\Big(\det(B)^{\frac{m}{2}} - (\det(A)\sharp_\tau \det(B))^{\frac{m}{2}}\Big)^2 \\ \leq & \Big(\det(\mu A + (1-\mu)B)\Big)^m - \Big(\det(A^\mu B^{1-\mu})\Big)^m, \end{aligned} \tag{20}$$

and

$$\begin{aligned} & \frac{(1-\mu)^m}{1-\tau}\Big(\det(A)^m \nabla_\mu \det(B)^m - (\det(A)\sharp_\mu \det(B))^m\Big) \\ & + r'_m\Big(\det(A)^{\frac{m}{2}} - (\det(A)\sharp_\mu \det(B))^{\frac{m}{2}}\Big)^2 \\ \leq & \Big(\det(\tau A + (1-\tau)B)\Big)^m - \Big(\det(A^\tau B^{1-\tau})\Big)^m. \end{aligned} \tag{21}$$

Proof. We have

$$\begin{aligned} \det\Big(\mu A + (1-\mu)B\Big)^m &= \Big[\det\Big(\mu A + (1-\mu)B\Big)^{\frac{1}{n}}\Big]^{nm} \\ &\geq \Big[\det(\mu A)^{\frac{1}{n}} + \det((1-\mu)B)^{\frac{1}{n}}\Big]^{nm} \text{ (by Lemma 4)} \\ &= \Big[\mu \det(A)^{\frac{1}{n}} + (1-\mu)\det(B)^{\frac{1}{n}}\Big]^{nm} \\ &\geq \frac{\mu^{nm}}{\tau}\Big(\det(A)^m \nabla_\tau \det(B)^m - (\det(A)\sharp_\tau \det(B))^m\Big) \\ &\quad + r_{nm}\Big(\det(B)^{\frac{m}{2}} - (\det(A)\sharp_\tau \det(B))^{\frac{m}{2}}\Big)^2 \\ &\quad + \Big(\det(A)^\mu \det(B)^{1-\mu}\Big)^m \\ &= \frac{\mu^{nm}}{\tau}\Big(\det(A)^m \nabla_\tau \det(B)^m - (\det(A)\sharp_\tau \det(B))^m\Big) \\ &\quad + r_{nm}\Big(\det(B)^{\frac{m}{2}} - (\det(A)\sharp_\tau \det(B))^{\frac{m}{2}}\Big)^2 \\ &\quad + \Big(\det(A^\mu B^{1-\mu})\Big)^m. \end{aligned}$$

Using the same technique in inequality (13), we get inequality (21). This completes the proof. ■

4.2 Refinements of Young's Type Inequality for Norms

The main result to be proved in this subsection, concerns the norms of positive semi-definite matrices. Before giving our result, we need the following lemma.

Lemma 5 *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive semi-definite matrices. Then we have*

$$|||A^\mu X B^{1-\mu}||| \leq |||AX|||^\mu |||XB|||^{1-\mu}. \tag{22}$$

In particular

$$tr|A^\mu B^{1-\mu}| \leq (trA)^\mu (trB)^{1-\mu}. \tag{23}$$

Theorem 6 *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive semi-definite matrices and $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$. Then for all positive integer m, we have*

$$\begin{aligned} & \frac{\mu^m}{\tau}\Big(|||AX|||^m \nabla_\tau |||XB|||^m - (|||AX|||\sharp_\tau |||XB|||)^m\Big) \\ & +r_m\Big(|||XB|||^{\frac{m}{2}} - (|||AX|||\sharp_\tau |||XB|||)^{\frac{m}{2}}\Big)^2 \\ \leq\ & \Big(\mu|||AX||| + (1-\mu)|||XB|||\Big)^m - \Big(|||A^\mu XB^{1-\mu}|||\Big)^m, \end{aligned} \tag{24}$$

and

$$\begin{aligned} & \frac{(1-\mu)^m}{1-\tau}\Big(|||AX|||^m \nabla_\mu |||XB|||^m - (|||AX|||\sharp_\mu |||XB|||)^m\Big) \\ & +r'_m\Big(|||AX|||^{\frac{m}{2}} - (|||AX|||\sharp_\mu |||XB|||)^{\frac{m}{2}}\Big)^2 \\ \leq\ & \Big(\tau|||AX||| + (1-\tau)|||XB|||\Big)^m - \Big(|||A^\tau XB^{1-\tau}|||\Big)^m. \end{aligned} \tag{25}$$

Proof. *By using Theorem 3 and Lemma 5 we have*

$$\begin{aligned} & \Big(|||A^\mu XB^{1-\mu}|||\Big)^m + \frac{\mu^m}{\tau}\Big(|||AX|||^m \nabla_\tau |||XB|||^m - (|||AX|||\sharp_\tau |||XB|||)^m\Big) \\ & +r_m\Big(|||XB|||^{\frac{m}{2}} - (|||AX|||\sharp_\tau |||XB|||)^{\frac{m}{2}}\Big)^2 \\ \leq\ & \Big(|||AX|||^\mu |||XB|||^{1-\mu}\Big)^m \\ & +\frac{\mu^m}{\tau}\Big(|||AX|||^m \nabla_\tau |||XB|||^m - (|||AX|||\sharp_\tau |||XB|||)^m\Big) \\ & +r_m\Big(|||XB|||^{\frac{m}{2}} - (|||AX|||\sharp_\tau |||XB|||)^{\frac{m}{2}}\Big)^2 \\ \leq\ & \Big(\mu|||AX||| + (1-\mu)|||XB|||\Big)^m. \end{aligned}$$

Using the same technique in inequality (13), we get inequality (25). This completes the proof. ■

4.3 Refinements of Young's Type Inequality for Traces

The main result to be proved in this section, concerns the traces of positive semi-definite matrices which can be reads as follows:

Theorem 7 *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive semi-definite matrices and $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$. Then for all positive integer m, we have*

$$\begin{aligned} & \frac{\mu^m}{\tau}\Big(tr(A)^m \nabla_\tau tr(B)^m - (tr(A)\sharp_\tau tr(B))^m\Big) \\ & +r_m\Big(tr(B)^{\frac{m}{2}} - (tr(A)\sharp_\tau tr(B))^{\frac{m}{2}}\Big)^2 \\ \leq\ & \Big(tr(\mu A + (1-\mu)B)\Big)^m - \Big(tr(A^\mu B^{1-\mu})\Big)^m, \end{aligned} \tag{26}$$

and

$$\begin{aligned} & \frac{(1-\mu)^m}{1-\tau}\Big(tr(A)^m \nabla_\mu tr(B)^m - (tr(A)\sharp_\mu tr(B))^m\Big) \\ & +r'_m\Big(tr(A)^{\frac{m}{2}} - (tr(A)\sharp_\mu tr(B))^{\frac{m}{2}}\Big)^2 \\ \leq\ & \Big(tr(\tau A + (1-\tau)B)\Big)^m - \Big(tr(A^\tau B^{1-\tau})\Big)^m. \end{aligned} \tag{27}$$

Proof. *By using Theorem 3 and Lemma 5 we have*

$$\begin{aligned}
&\Big(tr(A^{\mu}B^{1-\mu})\Big)^m + \frac{\mu^m}{\tau}\Big(tr(A)^m \nabla_\tau tr(B)^m - (tr(A)\sharp_\tau tr(B))^m\Big) \\
&+r_m\Big(tr(B)^{\frac{m}{2}} - (tr(A)\sharp_\tau tr(B))^{\frac{m}{2}}\Big)^2 \\
\leq\ &\Big(tr(A)^{\mu} tr(B)^{1-\mu}\Big)^m + \frac{\mu^m}{\tau}\Big(tr(A)^m \nabla_\tau tr(B)^m - (tr(A)\sharp_\tau tr(B))^m\Big) \\
&+r_m\Big(tr(B)^{\frac{m}{2}} - (tr(A)\sharp_\tau tr(B))^{\frac{m}{2}}\Big)^2 \\
\leq\ &\Big(tr(\mu A + (1-\mu)B)\Big)^m.
\end{aligned}$$

Using the same technique in inequality (13), we get inequality (27). This completes the proof. ∎

Acknowledgment. The authors would like to express their deep thanks to the anonymous referees for their helpful comments and suggestions on the initial version of the manuscript which lead to the improvement of this paper.

References

[1] Y. Al-Manasrah and F. Kittaneh, A generalization of two refined Young inequalities, Positivity, 19(2015), 757–768.

[2] Y. Al-Manasrah and F. Kittaneh, Further generalization refinements and reverses of the Young and Heinz inequalities, Results, Math., 19(2016), 757–768.

[3] H. Alzer, C. M. Fonseca and A. Kovačes, Young-type inequalities and their matrix analogues, Linear Multilinear Algebra, 63(2015), 622–635.

[4] T. Ando, Matrix Young inequality, Oper. Theory Adv. Appl., 75(1995), 33–38.

[5] A. Burqan and M. Khandaqji, Reverses of Young type inequalities, J. Math. Inequal., 9(2015), 113–120.

[6] D. Choi, A generalization of Young-type inequalities, Math Inequalities Appl., 21(2018), 99–106.

[7] R. A. Horn and C. R, Johnson, Matrix Analysis, Cambridge Univ. Press, New-York, (1985).

[8] O. Hirzallah and F. Kittaneh, Matrix Young inequalities for the Hilbert-Schmidt norm, Linear Algebra Appl., 308(2000), 77–84.

[9] M. A. Ighachane and M. Akkouchi, A new generalization of two refined Young inequalities and applications, Moroccan J. Pure Appl. Anal., 6(2020), 155–167.

[10] M. A. Ighachane and M. Akkouchi, Further generalized refinement of Young's inequalities for τ-measurable operators, Moroccan J. Pure Appl. Anal., 7(2021), 214–226.

[11] M. A. Ighachane, M. Akkouchi and E. H. Benabdi, A new generalized refinement of the weighted arithmetic-geometric mean inequality, Math. Ineq. Appl., 23(2020), 1079–1085.

[12] M. A. Ighachane and M. Akkouchi, Further refinement of Young's type inequality for positive linear maps, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 115(2021), 19 pp.

[13] F. Kittaneh, Norm inequalities for fractionl powers of positive operators, Lett. Math. Phys. 27 (1993), 279-285.

[14] F. Kittaneh and Y. Al- Manasrah, Improved Young and Heinz inequalities for matrices, J. Math. Anal. Appl., 36(2010), 292–269.

[15] J. Pećarić, T. Furuta, T. Mićić Hot and T. Seo, Mond-Pečarić Method in Operator Inequalities, Inequalities for bounded selfadjoint operators on a Hilbert space. Monographs in Inequalities, 1. ELEMENT, Zagreb, 2005.

[16] J. Zhao and J. Wu, Operator inequalities involving improved Young and its reverse inequalities, J. Math. Anal. Appl., 421(2015), 1779–1789.