Applied Mathematics E-Notes, 22(2022), 660-667 © ISSN 1607-2510
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On The Eneström-Kakeya Theorem*

Nisar Ahmad Rather[†], Mohmmad Shafi Wani[‡], Ishfaq Dar[§]

Received 11 November 2021

Abstract

In this paper, by relaxing the hypothesis of the famous Eneström-Kakeya Theorem, we establish a zero bound in the theory of distribution of the zeros of polynomials. Our results not only generalizes the Eneström-Kakeya Theorem but are also applicable to the larger class of polynomials. We also show by examples that established results give better bounds than Cauchy bound.

1 Introduction and Main Results

Polynomials perpetrate and much that is beautiful in mathematics is related to polynomials. Virtually every branch of mathematics, from Algebraic number theory and Algebraic Geometry to Applied Analysis, Fourier Analysis and Computer Sciences, has its corpus of theory arising from the study of polynomials. The subject is now much too large to attempt an encyclopaedic coverage.

The most amusing problem of the algebra is to find the zeros of a polynomial. But as the degree of a polynomial shoot up, it is very difficult to find the zeros of a polynomial. This makes identification of regions containing zeros of a polynomial a significant problem. In 1829, A. L Cauchy [2] gave a very simple expression for the zero-bound in terms of the coefficients of a polynomial. In fact he proved that all the zeros of the polynomial

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, a_n \neq 0$$

lie in the disk

$$|z| \leq 1 + M, \quad where \quad M = \max_{1\leq k\leq n} \left|\frac{a_{n-k}}{a_n}\right|.$$

The remarkable property of this result is its simplicity of computations. In literature [5], there exists several results concerning the bounds for zeros of polynomials. The following elegant result on the location of zeros of a polynomial with restricted coefficients is known as Eneström-Kakeya Theorem [5, 9] which states that:

Theorem 1 *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with real coefficients satisfying $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0$. Then all the zeros of $p(z)$ lie in $|z| \leq 1$.*

Joyal, Labelle and Rahman [4] extended Theorem 1 to polynomials whose coefficients are monotonic but are not necessarily non-negative and proved that:

Theorem 2 *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with real coefficients satisfying $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$. Then all the zeros of $p(z)$ lie in*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

*Mathematics Subject Classifications: 30C10, 30C15, 30E10.

†Department of Mathematics, University of Kashmir, Srinagar-190006, India
‡Department of Mathematics, University of Kashmir, Srinagar-190006, India
§Department of Mathematics, University of Kashmir, Srinagar-190006, India

Aziz and Zargar [1] also relaxed the hypothesis of Theorem 2 in several ways and among other things they proved the following result:

Theorem 3 *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with real coefficients such that for some $k \geq 1$, $ka_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$. Then all the zeros of $p(z)$ lie in*

$$|z + k - 1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|}.$$

Shah and Liman [10] extended Theorem 3 to the polynomials with complex coefficients by proving that:

Theorem 4 *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, 2, \ldots, n$ and $k \geq 1$,*

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|.$$

Then all the zeros of $p(z)$ lie in

$$|z + k - 1| \leq \frac{1}{|a_n|} \left\{ (k|a_n| - |a_0|)(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \right\}.$$

Rather et al. [6] relaxed the hypothesis of Theorem 3 and they proved the following result:

Theorem 5 *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with real coefficients such that for some $k_j \geq 1$, $a_{n-j+1} > 0$, $j = 1, 2, \ldots, r$ where $1 \leq r \leq n$,*

$$k_1 a_n \geq k_2 a_{n-1} \geq k_3 a_{n-2} \geq \cdots \geq k_r a_{n-r+1} \geq a_{n-r} \geq \cdots \geq a_1 \geq a_0.$$

Then all the zeros of $P(z)$ lie in

$$\left| z + k_1 - 1 - (k_2 - 1)\frac{a_{n-1}}{a_n} \right| \leq \frac{1}{|a_n|} \left\{ k_1 a_n - (k_2 - 1)|a_{n-1}| + 2\sum_{j=2}^{r} (k_j - 1)|a_{n-j+1}| - a_0 + |a_0| \right\}.$$

Recently Rather et al ([8], [7]), extended Theorem 5 to the polynomials with complex coefficients and proved following two results:

Theorem 6 *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, 2, \ldots, n$ and $k_j \geq 1$, $a_{n-j} \neq 0$, $j = 0, 1, \ldots, r$ where $1 \leq r \leq n-1$,*

$$k_0|a_n| \geq k_1|a_{n-1}| \geq k_2|a_{n-2}| \geq \cdots \geq k_r|a_{n-r}| \geq |a_{n-r-1}| \geq \cdots \geq |a_1| \geq |a_0|.$$

Then all the zeros of $P(z)$ lie in

$$\begin{aligned} \left| z + k_0 - 1 - (k_1 - 1)\frac{a_{n-1}}{a_n} \right| \leq \; & \frac{1}{|a_n|} \Bigg\{ (k_0|a_n| - |a_0|)(\cos\alpha + \sin\alpha) \\ & + 2\sin\alpha \left(\sum_{j=1}^{r} k_j |a_{n-j}| + \sum_{j=r+1}^{n} |a_{n-j}| \right) \\ & - (k_1 - 1)|a_{n-1}| + 2\sum_{j=1}^{r} (k_j - 1)|a_{n-j}| + |a_0| \Bigg\}. \end{aligned}$$

Theorem 7 *Let* $P(z)=\sum_{j=0}^{n} a_j z^j,\ a_j=\alpha_j+i\gamma_j$ *be a polynomial of degree* n *with complex coefficients such that for some* $k_j \geq 1, \alpha_{n-j+1}>0,\ j=1,2,\ldots,r$ *where* $1\leq r\leq n$,

$$k_1\alpha_n \geq k_2\alpha_{n-1} \geq k_3\alpha_{n-2} \geq \cdots \geq k_r\alpha_{n-r+1} \geq \alpha_{n-r} \geq \cdots \geq \alpha_1 \geq \alpha_0.$$

Then all the zeros of $P(z)$ *lie in*

$$\left|z+(k_1-1)\frac{\alpha_n}{a_n}-(k_2-1)\frac{\alpha_{n-1}}{a_n}\right| \leq \frac{1}{|a_n|}\Bigg[k_1\alpha_n-(k_2-1)|\alpha_{n-1}|+2\Bigg(\sum_{j=2}^{r}(k_j-1)|\alpha_{n-j+1}| + \sum_{j=0}^{n-1}|\gamma_j|\Bigg)-\alpha_0+|\alpha_0|+|\gamma_n|\Bigg].$$

Although Theorems 5–7 are applicable to the larger class of polynomials as compared to all other Eneström-Kakeya type results, but are not applicable to the polynomials with one coefficient equals zero. For instance, if we consider the polynomial $P(z)=5z^4+4z^3+0z^2+2z+1$, then one can note that all Eneström-Kakeya type results including Theorems 5–7 are not applicable to this polynomial. So it is interesting to look for the results applicable to such class of polynomials. Motivated by this, here we establish the following results applicable to such class of polynomials. We begin, by proving the following result.

Theorem 8 *If* $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ *is a polynomial of degree* n *with complex coefficients such that for some real* β, $|\arg(k_j+a_{n-j})-\beta| \leq \alpha \leq \frac{\pi}{2}$, $j=0,1,2,\ldots,n$ *and for some numbers* k_j, $j=0,1,2,\ldots,r$ *where* $1\leq r\leq n-1$,

$$|k_0+a_n| \geq |k_1+a_{n-1}| \geq |k_2+a_{n-2}| \geq \cdots \geq |k_r+a_{n-r}| \geq |a_{n-r-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ *lie in*

$$\left|z+\frac{k_0-k_1}{a_n}\right| \leq \frac{1}{|a_n|}\Bigg\{(|k_0+a_n|-|a_0|)(cos\alpha+sin\alpha)+2\sin\alpha\left(\sum_{j=1}^{r}(|k_j+a_{n-j}|)+\sum_{j=r+1}^{n}|a_{n-j}|\right) + \sum_{j=1}^{r-1}|k_j-k_{j+1}|+|k_r|+|a_0|\Bigg\}.$$

Remark 1 *If we take transformation* $k_j=(\lambda_j-1)a_{n-j},\ a_{n-j}\neq 0, \lambda_j\geq 1,\ j=0,1,2,\ldots r$, *then Theorem 8 reduces to improved version of Theorem 6.*

Remark 2 *For* $\alpha=\beta=0$, *Theorem 8 reduces to the following result.*

Corollary 1 *Let* $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ *be a polynomial of degree* n *with real coefficients such that for some* $k_j\geq 0,\ j=0,1,2,\ldots,r$ *where* $1\leq r\leq n-1$,

$$k_0+a_n \geq k_1+a_{n-1} \geq k_2+a_{n-2} \geq \cdots \geq k_r+a_{n-r} \geq a_{n-r-1} \geq \cdots \geq a_1 \geq a_0 \geq 0.$$

Then all the zeros of $P(z)$ *lie in*

$$\left|z+\frac{k_0-k_1}{a_n}\right| \leq 1+\frac{1}{a_n}\left(k_0+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+k_r\right).$$

Theorem 9 *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *where* $a_j=\alpha_j+i\gamma_j$ *be a polynomial of degree* n *with complex coefficients such that for some* $k_j\geq 0,\ j=0,1,2,\ldots,r$ *where* $1\leq r\leq n-1$,

$$k_0+\alpha_n \geq k_1+\alpha_{n-1} \geq k_2+\alpha_{n-2} \geq \cdots \geq k_r+\alpha_{n-r} \geq \alpha_{n-r-1} \geq \cdots \geq \alpha_1 \geq \alpha_0.$$

Then all the zeros of $P(z)$ *lie in*

$$\left|z+\frac{k_0-k_1}{a_n}\right| \leq \frac{1}{|a_n|}\left\{\alpha_n+|\alpha_0|-\alpha_0+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+\sum_{j=0}^{n-1}|\gamma_{j+1}-\gamma_j|+|\gamma_0|+k_r+k_0\right\}$$

If all the coefficients of $P(z)$ are real then the above theorem reduces to the following result:

Corollary 2 *Let* $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ *be a polynomial of degree* n *with real coefficients such that for some* $k_j\geq 0$, $j=0,1,2,\ldots,r$ *where* $1\leq r\leq n-1$,

$$k_0+a_n\geq k_1+a_{n-1}\geq k_2+a_{n-2}\geq\cdots\geq k_r+a_{n-r}\geq a_{n-r-1}\geq\cdots\geq a_1\geq a_0.$$

Then all the zeros of $P(z)$ *lie in*

$$\left|z+\frac{k_0-k_1}{a_n}\right|\leq\frac{1}{a_n}\left(a_n+k_0+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+k_r+|a_0|-a_0\right).$$

Remark 3 *In Corollary 2, if we take* $k_j=0$, $j=0,1,2,\ldots,r$ *we get theorem 2.*

Remark 4 *Corollary 2 reduces to the Corollary 1 If we take* $a_0\geq 0$. *Further for* $k_j=0$, $j=0,1,2,\ldots,r$ *Corollary 1 reduces to the famous Eneström-Kakeya Theorem [5, 9].*

Remark 5 *It is worth mentioning here that if we use same transformation as in Remark 1, then Theorem 9 reduces to Theorem 7 and Corollary 2 reduces to Theorem 5.*

2 Computations and Analysis

In this section, we present some examples of polynomials to show that our results give better information about the location of the zeros than Cauchy's Theorem.

Example 1 *Let* $P(z)=3z^4+2.8z^3+2.6z^2+3z+1$. *By taking* $\beta=\alpha=0$ *and* $r=2$ *with* $k_0=0$, $k_1=0.2$, $k_2=0.4$ *in Theorem 8, it follows that all the zeros of* $P(z)$ *lie in the disc* $|z-\frac{1}{15}|\leq 1.2$. *Whereas, if we use Cauchy's Theorem, it follows that all the zeros of* $P(z)$ *lie in the disc* $|z|\leq 2$.

Example 2 *Let* $P(z)=4z^5+3.2z^4+3z^3+2.8z^2+3z+1$. *By taking* $r=3$ *with* $k_0=0$, $k_1=0$, $k_2=0$, $k_3=0.2$ *in Corollary 1, it follows that all the zeros of* $P(z)$ *lie in the disc* $|z|\leq 1.1$. *Whereas, if we use Cauchy's Theorem, it follows that all the zeros of* $P(z)$ *lie in the disc* $|z|\leq 1.75$.

Example 3 *Let* $P(z)=10z^3+10z^2+1$. *By taking* $r=2$ *with* $k_0=0$, $k_1=0$, $k_2=1$ *in Corollary 1, it follows that all the zeros of* $P(z)$ *lie in the disc* $|z|\leq 1.2$. *Whereas, if we use Cauchy's Theorem, it follows that all the zeros of* $P(z)$ *lie in the disc* $|z|\leq 2$.

From the above examples, it is evident that our results give better bound than the bound obtained by using Cauchy's Theorem.

3 Lemma

For the proof of Theorem 8, we need the following lemma which is due to Govil and Rahman [3].

Lemma 1 *If for some numbers* k_j *and* k_{j+1}, $|k_j+a_{n-j}|\geq|k_{j+1}+a_{n-j-1}|$ *and* $|\arg(k_j+a_{n-j})-\beta|\leq\alpha\leq\frac{\pi}{2}$, *for some real* β, *then*

$$|(k_j+a_{n-j})-(k_{j+1}+a_{n-j-1})|\leq(|k_j+a_{n-j}|-|k_{j+1}+a_{n-j-1}|)\cos\alpha+(|k_j+a_{n-j}|+|k_{j+1}+a_{n-j-1}|)\sin\alpha.$$

4 Proofs of the Theorems

Proof of Theorem 8. Consider the polynomial,

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{n-r} - a_{n-r-1})z^{n-r} + \cdots + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + [(k_0 + a_n) - (k_1 + a_{n-1})]z^n - (k_0 - k_1)z^n + [(k_1 + a_{n-1}) - (k_2 + a_{n-2})]z^{n-1} \\
&\quad -(k_1 - k_2)z^{n-1} + \cdots + [(k_{r-2} + a_{n-r+2}) - (k_{r-1} + a_{n-r+1})]z^{n-r+2} \\
&\quad -(k_{r-2} - k_{r-1})z^{n-r+2} + [(k_{r-1} + a_{n-r+1}) - (k_r + a_{n-r})]z^{n-r+1} - (k_{r-1} - k_r)z^{n-r+1} \\
&\quad +[(k_r + a_{n-r}) - a_{n-r-1}]z^{n-r} - k_r z^{n-r} + (a_{n-r-1} - a_{n-r-2})z^{n-r-1} + \cdots + (a_1 - a_0)z + a_0.
\end{aligned}$$

This implies,

$$\begin{aligned}
|F(z)| &= |-a_n z^{n+1} - (k_0 - k_1)z^n + [(k_0 + a_n) - (k_1 + a_{n-1})]z^n + [(k_1 + a_{n-1}) - (k_2 + a_{n-2})]z^{n-1} \\
&\quad -(k_1 - k_2)z^{n-1} + \cdots + [(k_{r-2} + a_{n-r+2}) - (k_{r-1} + a_{n-r+1})]z^{n-r+2} - (k_{r-2} - k_{r-1})z^{n-r+2} \\
&\quad +[(k_{r-1} + a_{n-r+1}) - (k_r + a_{n-r})]z^{n-r+1} - (k_{r-1} - k_r)z^{n-r+1} + [(k_r + a_{n-r}) - a_{n-r-1}]z^{n-r} \\
&\quad -k_r z^{n-r} + (a_{n-r-1} - a_{n-r-2})z^{n-r-1} + \cdots + (a_1 - a_0)z + a_0|.
\end{aligned}$$

That is,

$$\begin{aligned}
|F(z)| &\geq |z|^n \Bigg\{ |a_n z + (k_0 - k_1)| - \Bigg(|(k_0 + a_n) - (k_1 + a_{n-1})| + \frac{|(k_1 + a_{n-1}) - (k_2 + a_{n-2})|}{|z|} + \frac{|k_1 - k_2|}{|z|} \\
&\quad + \cdots + \frac{|(k_{r-2} + a_{n-r+2}) - (k_{r-1} + a_{n-r+1})|}{|z|^{r-2}} + \frac{|k_{r-2} - k_{r-1}|}{|z|^{r-2}} + \frac{|(k_{r-1} + a_{n-r+1}) - (k_r + a_{n-r})|}{|z|^{r-1}} \\
&\quad + \frac{|k_{r-1} - k_r|}{|z|^{r-1}} + \frac{|(k_r + a_{n-r}) - (a_{n-r-1})|}{|z|^r} + \frac{|k_r|}{|z|^r} + \frac{|a_{n-r-1} - a_{n-r-2}|}{|z|^{r+1}} + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Bigg) \Bigg\}.
\end{aligned}$$

Let $|z| > 1$ so that $\frac{1}{|z|} < 1$. Then we have

$$\begin{aligned}
|F(z)| &> |z|^n \Bigg\{ |a_n z + (k_0 - k_1)| - \Bigg(|(k_0 + a_n) - (k_1 + a_{n-1})| + |(k_1 + a_{n-1}) - (k_2 + a_{n-2})| + |k_1 - k_2| \\
&\quad + \cdots + |(k_{r-2} + a_{n-r+2}) - (k_{r-1} + a_{n-r+1})| + |k_{r-2} - k_{r-1}| + |(k_{r-1} + a_{n-r+1}) - (k_r + a_{n-r})| \\
&\quad + |k_{r-1} - k_r| + |(k_r + a_{n-r}) - (a_{n-r-1})| + |k_r| + |a_{n-r-1} - a_{n-r-2}| + \cdots + |a_1 - a_0| + |a_0| \Bigg) \Bigg\}.
\end{aligned}$$

Applying Lemma 1, we have for $|z| > 1$,

$$\begin{aligned}
|F(z)| > |a_n||z|^n \Bigg\{ &\Big|z + \frac{(k_0 - k_1)}{a_n}\Big| - \frac{1}{|a_n|} \Bigg[\Bigg(|k_0 + a_n| - |k_1 + a_{n-1}| + |k_1 + a_{n-1}| - |k_2 + a_{n-2}| + \ldots \\
&+ |k_{r-1} + a_{n-r+1}| - |k_r + a_{n-r}| + |k_r + a_{n-r}| - |a_{n-r-1}| + |a_{n-r-1}| - |a_{n-r-2}| + \ldots \\
&- |a_1| + |a_1| - |a_0| \Bigg) \cos\alpha + \Bigg(|k_0 + a_n| + |k_1 + a_{n-1}| + |k_1 + a_{n-1}| + |k_2 + a_{n-2}| + \ldots \\
&+ |k_{r-1} + a_{n-r+1}| + |k_r + a_{n-r}| + |k_r + a_{n-r}| + |a_{n-r-1}| + |a_{n-r-1}| + |a_{n-r-2}| + \ldots \\
&+ |a_1| + |a_1| + |a_0| \Bigg) \sin\alpha + \sum_{j=1}^{r-1} |k_j - k_{j+1}| + |k_r| + |a_0| \Bigg] \Bigg\},
\end{aligned}$$

which gives

$$\begin{aligned}|F(z)| &> |a_n||z|^n\Bigg\{\left|z+\frac{(k_0-k_1)}{a_n}\right|-\frac{1}{|a_n|}\Bigg[(|k_0+a_n|-|a_0|)(\cos\alpha+\sin\alpha)\\ &+2\sin\alpha\Bigg(\sum_{j=1}^{r}|k_j+a_{n-j}|+\sum_{j=r+1}^{n}|a_{n-j}|\Bigg)+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+|k_r|+|a_0|\Bigg]\Bigg\}\\ &> 0\end{aligned}$$

if

$$\begin{aligned}\left|z+\frac{k_0-k_1}{a_n}\right| &> \frac{1}{|a_n|}\Bigg\{(|k_0+a_n|-|a_0|)(\cos\alpha+\sin\alpha)+2\sin\alpha\left(\sum_{j=1}^{r}(|k_j+a_{n-j}|)+\sum_{j=r+1}^{n}|a_{n-j}|\right)\\ &+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+|k_r|+|a_0|\Bigg\}.\end{aligned}$$

This shows that those zeros of $F(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned}\left|z+\frac{k_0-k_1}{a_n}\right| &\leq \frac{1}{|a_n|}\Bigg\{(|k_0+a_n|-|a_0|)(\cos\alpha+\sin\alpha)+2\sin\alpha\left(\sum_{j=1}^{r}(|k_j+a_{n-j}|)+\sum_{j=r+1}^{n}|a_{n-j}|\right)\\ &+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+|k_r|+|a_0|\Bigg\}.\end{aligned}$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already lie in this region. Hence it follows that all the zeros of $F(z)$ and therefore of $P(z)$ lie in

$$\begin{aligned}\left|z+\frac{k_0-k_1}{a_n}\right| &\leq \frac{1}{|a_n|}\Bigg\{(|k_0+a_n|-|a_0|)(\cos\alpha+\sin\alpha)+2\sin\alpha\left(\sum_{j=1}^{r}(|k_j+a_{n-j}|)+\sum_{j=r+1}^{n}|a_{n-j}|\right)\\ &+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+|k_r|+|a_0|\Bigg\}.\end{aligned}$$

This completes the proof of Theorem 8. ■

Proof of Theorem 9. Consider the polynomial

$$\begin{aligned}F(z) &= (1-z)P(z)\\ &= -a_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+\cdots+(\alpha_{n-r}-\alpha_{n-r-1})z^{n-r}+\dots\\ &\quad+(\alpha_1-\alpha_0)z+\alpha_0+i\big[(\gamma_n-\gamma_{n-1})z^n+(\gamma_{n-1}-\gamma_{n-2})z^{n-1}+\cdots+(\gamma_1-\gamma_0)z+\gamma_0\big]\\ &= -a_nz^{n+1}+[(k_0+\alpha_n)-(k_1+\alpha_{n-1})-(k_0-k_1)]z^n+[(k_1+\alpha_{n-1})-(k_2+\alpha_{n-2})-(k_1-k_2)]z^{n-1}\\ &\quad+\cdots+[(k_{r-2}+\alpha_{n-r+2})-(k_{r-1}+\alpha_{n-r+1})-(k_{r-2}-k_{r-1})]z^{n-r+2}\\ &\quad+[(k_{r-1}+\alpha_{n-r+1})-(k_r+\alpha_{n-r})-(k_{r-1}-k_r)]z^{n-r+1}+[k_r+\alpha_{n-r}-\alpha_{n-r-1}-k_r]z^{n-r}\dots\\ &\quad+(\alpha_1-\alpha_0)z+\alpha_0+i\big[(\gamma_n-\gamma_{n-1})z^n+(\gamma_{n-1}-\gamma_{n-2})z^{n-1}+\cdots+(\gamma_1-\gamma_0)z+\gamma_0\big],\end{aligned}$$

which implies,

$$\begin{aligned}|F(z)| &= \big|-a_nz^{n+1}-(k_0-k_1)z^n+[(k_0+\alpha_n)-(k_1+\alpha_{n-1})]z^n+[(k_1\alpha_{n-1})-(k_2+\alpha_{n-2})]z^{n-1}\\ &\quad-(k_1-k_2)z^{n-1}+\cdots+[(k_{r-2}+\alpha_{n-r+2})-(k_{r-1}+\alpha_{n-r+1})]z^{n-r+2}-(k_{r-2}-k_{r-1})]z^{n-r+2}\\ &\quad+[(k_{r-1}+\alpha_{n-r+1})-(k_r+\alpha_{n-r})]z^{n-r+1}-(k_{r-1}-k_r)z^{n-r+1}+[k_r+\alpha_{n-r}-\alpha_{n-r-1}]z^{n-r}\\ &\quad-k_rz^{n-r}\cdots+(\alpha_1-\alpha_0)z+\alpha_0\\ &\quad+i\big[(\gamma_n-\gamma_{n-1})z^n+(\gamma_{n-1}-\gamma_{n-2})z^{n-1}+\cdots+(\gamma_1-\gamma_0)z+\gamma_0\big]\big|,\end{aligned}$$

that is,

$$\begin{aligned}|F(z)| \geq\ & |z|^n\big[|za_n+k_0-k_1|-\big(|(k_0+\alpha_n)-(k_1+\alpha_{n-1})|+|(k_1+\alpha_{n-1})-(k_2+\alpha_{n-2})|/|z|\\ &+|k_1-k_2|/|z|+\cdots+|(k_{r-2}+\alpha_{n-r+2})-(k_{r-1}+\alpha_{n-r+1})|/|z|^{r-2}\\ &+|k_{r-2}-k_{r-1}|/|z|^{r-2}+|(k_{r-1}+\alpha_{n-r+1})-(k_r+\alpha_{n-r})|/|z|^{r-1}+|k_{r-1}-k_r|/|z|^{r-1}\\ &+|k_r+\alpha_{n-r}-\alpha_{n-r-1}|/|z|^r+|k_r|/|z|^r\\ &+|\alpha_{n-r-1}-\alpha_{n-r-2}|/|z|^{r+1}+\cdots+|\alpha_1-\alpha_0|/|z|^{n-1}+|\alpha_0|/|z|^n\\ &+|\gamma_n-\gamma_{n-1}|+|\gamma_{n-1}-\gamma_{n-2}|/|z|+\cdots+|\gamma_1-\gamma_0|/|z|^{n-1}+|\gamma_0|/|z|^n\big)\big].\end{aligned}$$

By using hypothesis, we have for $|z|>1$,

$$\begin{aligned}|F(z)| >\ & |a_n||z|^n\big[|z+\frac{k_0-k_1}{a_n}|-\frac{1}{|a_n|}\big((k_0+\alpha_n)-(k_1+\alpha_{n-1})+(k_1+\alpha_{n-1})-(k_2+\alpha_{n-2})\\ &+|k_1-k_2|+\cdots+(k_{r-2}+\alpha_{n-r+2})-(k_{r-1}+\alpha_{n-r+1})\\ &+|k_{r-2}-k_{r-1}|+(k_{r-1}+\alpha_{n-r+1})-(k_r+\alpha_{n-r})+|k_{r-1}-k_r|\\ &+k_r+\alpha_{n-r}-\alpha_{n-r-1}+k_r\\ &+\alpha_{n-r-1}-\alpha_{n-r-2}+\cdots+\alpha_1-\alpha_0+|\alpha_0|\\ &+|\gamma_n-\gamma_{n-1}|+|\gamma_{n-1}-\gamma_{n-2}|+\cdots+|\gamma_1-\gamma_0|+|\gamma_0|\big)\big].\end{aligned}$$

implies,

$$\begin{aligned}|F(z)| >\ & |a_n||z|^n\Bigg[|z+\frac{k_0-k_1}{a_n}|-\frac{1}{|a_n|}\Bigg\{\alpha_n+|\alpha_0|-\alpha_0+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+\sum_{j=0}^{n-1}|\gamma_{j+1}-\gamma_j|\\ &+|\gamma_0|+k_r+k_0\Bigg\}\Bigg]\\ >\ & 0\end{aligned}$$

if

$$\big|z+\frac{k_0-k_1}{a_n}\big|>\frac{1}{|a_n|}\Bigg\{\alpha_n+|\alpha_0|-\alpha_0+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+\sum_{j=0}^{n-1}|\gamma_{j+1}-\gamma_j|+|\gamma_0|+k_r+k_0\Bigg\}.$$

This shows that those zeros of $F(z)$ whose modulus is greater than 1 lie in

$$\big|z+\frac{k_0-k_1}{a_n}\big|\leq\frac{1}{|a_n|}\Bigg\{\alpha_n+|\alpha_0|-\alpha_0+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+\sum_{j=0}^{n-1}|\gamma_{j+1}-\gamma_j|+|\gamma_0|+k_r+k_0\Bigg\}.$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already lie in this region. Hence it follows that all the zeros of $F(z)$ and therefore of $P(z)$ lie in

$$\big|z+\frac{k_0-k_1}{a_n}\big|\leq\frac{1}{|a_n|}\Bigg\{\alpha_n+|\alpha_0|-\alpha_0+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+\sum_{j=0}^{n-1}|\gamma_{j+1}-\gamma_j|+|\gamma_0|+k_r+k_0\Bigg\}.$$

This completes the proof of Theorem 9.
■

Acknowledgement. The authors are highly grateful to the referees for their valuable suggestions and comments.

References

[1] A. Aziz and B. A. Zargar, Some extensions of Eneström-Kakeya theorem, Glas. Mat. Ser. III, 31(1996), 239–244.

[2] A. L. Cauchy, Exercises de mathématique, in Oeuvres, 9(1829), 122.

[3] N. K. Govil and Q. I. Rahman, On the Eneström-Kakeya theorem II, Tohoku Math. J., 20(1968), 126–136.

[4] A. Joyal, G. Labelle and Q. I. Rahman, On the location of zeros of polynomials, Canadian Math. Bull., 10(1967), 55–63.

[5] M. Marden, The Geometry of the Zeros of a Polynomial in a Complex Variable, Mathematical Surveys, No. 3. American Mathematical Society, New York, N. Y., 1949. ix+183 pp.

[6] N. A. Rather, I. Dar and A. Iqbal, Generalizations of Eneström-Kakeya theorem and its extensions to analytic functions, J. Class. Anal., 16(2020), 37–44.

[7] N. A. Rather, I. Dar and A. Iqbal, On the regions containing all the zeros of polynomials and related analytic functions, Vestn. St.-Peterbg. Univ. Mat. Mekh. Astron., 8(2021), 331–337.

[8] N. A. Rather, I. Dar and Mohammad Shafi, On the zero bounds of polynomials and related analytic functions, Appl. Math. E-Notes, 21(2021), 525–532.

[9] Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, London Mathematical Society Monographs. New Series, 26. The Clarendon Press, Oxford University Press, Oxford, 2002.

[10] W. M. Shah and A. Liman, On Eneström-Kakeya theorem and related analytic functions, Proc. Indian Acad. Sci., 3(2007), 359–370.