

# Epi Convergence Of Double Function Sequences\*

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## Abstract

In this paper, we introduce and study the concept of epi convergence for double function sequences and establish the equivalence between epi convergence of a sequence of functions and the Kuratowski convergence of their epigraphs.

## 1 Introduction

The concept of epi convergence is important because it provides a convenient concept of convergence to approximate minimization problems in the field of mathematical optimization. Recently, studies on double sequences have gained importance. In this study, we will generalize the concept of epi convergence, which is known for single function sequences, to the double function sequences.

Double sequences may arise as solutions of partial difference equations. Convergence properties of these solutions are as important as many others such as boundedness, positivity, oscillatory behaviors, periodicity (see the articles [7], [12] and [13]. For the detailed information on the subject and the studies on convergence methods and their applications (see, for examples, [6, 8, 10, 14, 15, 16, 17, 18, 19, 22, 23]).

Now, we recall the basic definitions and concepts. Let's start by giving the definition of convergence of a double sequence. The notion of convergence for double sequences was presented by Pringsheim [21].

A double sequence  $x = (x_{nm})_{n,m \in \mathbb{N}}$  of real numbers is said to be convergent to  $L \in \mathbb{R}$  in Pringsheim's sense if for any  $\varepsilon > 0$ , there exists  $N_\varepsilon \in \mathbb{N}$  such that  $|x_{nm} - L| < \varepsilon$ , whenever  $n, m > N_\varepsilon$ . In this case we write

$$P - \lim_{n,m \rightarrow \infty} x_{nm} = L.$$

A double sequence  $x = (x_{nm})$  is bounded if and only if there exist a positive integer  $M$  such that  $|x_{nm}| < M$  for all  $n$  and  $m$ .

A double subsequence of a double sequence is defined as follows (see, for example, [20]):

A double subsequence  $y = (y_{nm})$  is a double subsequence of  $x = (x_{nm})$  provided that there exist increasing index sequences  $(n_j)$  and  $(m_j)$  such that  $y$  is formed by

$$\begin{array}{cccc} x_{n_1 m_1} & x_{n_2 m_2} & x_{n_5 m_5} & x_{n_{10} m_{10}} \\ x_{n_4 m_4} & x_{n_3 m_3} & x_{n_6 m_6} & - \\ x_{n_9 m_9} & x_{n_8 m_8} & x_{n_7 m_7} & - \\ - & - & - & - \end{array}$$

A double sequence  $x = (x_{nm})$  is said to converge regularly if it converges in Pringsheim's sense and, in addition, the following finite limits exist:

$$\lim_{n \rightarrow \infty} x_{nm} = a_m \quad (m = 1, 2, 3, \dots),$$

$$\lim_{m \rightarrow \infty} x_{nm} = a_n \quad (n = 1, 2, 3, \dots).$$

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Note that the main drawback of the Pringsheim's convergence is that a convergent double sequence fails in general to be bounded. The notion of regular convergence introduced by Hardy [9] lacks this disadvantage. In addition to the Pringsheim's convergence, the regular convergence requires the convergence of rows and columns of a double sequence.

Let  $x = (x_{nm})$  be a double subsequence of real numbers and, for each  $k$ , let  $\alpha_k = \sup_k \{x_{mn} : m, n \geq k\}$ . The Pringsheim limit superior of  $x = (x_{nm})$  is defined as follows:

- (i) if  $\alpha_k = +\infty$  for each  $k$ , then  $P - \limsup x_{nm} := +\infty$ ;
- (ii) if  $\alpha_k < +\infty$  for some  $k$ , then  $P - \limsup x_{nm} := \inf_k \{\alpha_k\}$ .

Similarly, for each  $k$ , let  $\beta_k = \inf_k \{x_{mn} : m, n \geq k\}$ . Then the Pringsheim limit inferior of  $x = (x_{nm})$  is defined as follows:

- (iii) if  $\beta_k = -\infty$  for each  $k$ , then  $P - \liminf x_{nm} := -\infty$ ;
- (iv) if  $\beta_k > -\infty$  for some  $k$ , then  $P - \liminf x_{nm} := \sup_k \{\beta_k\}$ .

**Example 1** Let the double sequence  $(x_{nm})$  defined by

$$x_{nm} = \begin{cases} n, & \text{if } m = 0, \\ -m, & \text{if } n = 0, \\ (-1)^m, & \text{if } m = n > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have  $P - \liminf x_{nm} = -1$  and  $P - \limsup x_{nm} = 1$ . This sequence is neither bounded above nor bounded below; however, the Pringsheim limit superior and inferior are both finite numbers.

A double sequence of functions  $(f_{nm})$  is said to be pointwise convergent to  $f$  on a set  $S \subset \mathbb{R}$  if for each point  $x$  in  $S$  and for each  $\epsilon > 0$ , there exists a positive integer  $N = N(x, \epsilon)$  such that

$$|f_{nm}(x) - f(x)| < \epsilon, \quad \text{for all } n, m > N.$$

In this case, we write  $P - \lim f_{nm}(x) = f(x)$  or  $f_{nm} \rightarrow f$  on  $S$ .

An extended real-valued function  $f : X \rightarrow [-\infty, \infty]$  on a metrizable space  $X$  is called lower semicontinuous provided its epigraph

$$epif \equiv \{(x, \alpha) : x \in X, \alpha \in \mathbb{R} \text{ and } \alpha \geq f(x)\}$$

is a closed subset of  $X \times \mathbb{R}$ .

A fundamental convergence concept for sequences of lower semicontinuous functions in optimization theory, decision theory, homogenization problems and variational analysis is the notion of epi convergence. During the last three decades the concept of epi convergence was introduced and then it was used in various investigations in optimization and related areas. There are many papers dealing with the epi convergence of single sequences of functions. In this paper we will introduce and study the concept of epi convergence for double sequences of functions.

## 2 Epi Convergence of Double Sequence of Functions

In order to give the definition of epi convergence of a double sequence of functions, we will first give the following definition.

**Definition 1** Let  $(X, d)$  be a metric space. For every  $x \in X$  let us denote the system of the neighbourhood of  $x$  by  $U(x)$ . To any double sequence  $(f_{nm})$  of functions from  $X$  into  $[-\infty, \infty]$ , we have associated two limit functions:

(a) The epi limit inferior of the sequence  $(f_{nm})$ , denoted by  $li_e f_{nm}$ , is defined by

$$(li_e f_{nm})(x) = \sup_{V \in \mathcal{U}(x)} P - \liminf_{n,m} \inf_{u \in V} f_{nm}(u).$$

(b) The epi limit superior of the sequence  $(f_{nm})$ , denoted by  $ls_e f_{nm}$ , is defined

$$(ls_e f_{nm})(x) = \sup_{V \in \mathcal{U}(x)} P - \limsup_{n,m} \inf_{u \in V} f_{nm}(u).$$

**Definition 2** Let  $(X, d)$  be a metric space and  $(f_{nm})$  be a double sequence of functions from  $X$  into  $[-\infty, \infty]$ . This sequence  $(f_{nm})$  is said to be epi convergent at  $x$  if the following equality holds:

$$(li_e f_{nm})(x) = (ls_e f_{nm})(x).$$

This common value is then denoted  $\lim_e f_{nm}(x)$ :

$$\lim_e f_{nm}(x) = li_e f_{nm}(x) = ls_e f_{nm}(x).$$

For lower semicontinuous functions equivalent definition can be given as following:

**Definition 3** Let  $(f_{nm})$  be a double sequence of lower semicontinuous function on a metric space  $(X, d)$ . We say that  $(f_{nm})$  is epi convergent to  $f$ , in sense of Pringsheim, and we write  $f = \lim_e f_{nm}$ , provided at each  $x \in X$ , the following two conditions both hold:

(c) whenever  $(x_{nm})$  is  $P$ -convergent to  $x$ , we have

$$f(x) \leq P - \liminf_{nm} f_{nm}(x_{nm}),$$

$$f(x) \leq \liminf_n f_{nm}(x_{nm})$$

and

$$f(x) \leq \liminf_m f_{nm}(x_{nm});$$

(d) there exists a sequence  $(x_{nm})$  is  $P$ -convergent to  $x$  such that

$$f(x) = P - \lim_{nm} f_{nm}(x_{nm}).$$

### Some Examples

**Example 2** Let  $f_{mn} : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f_{nm}(x) = \max\{\frac{x}{nm}, -1\}$  and  $f(x) \equiv 0$ . Then the double function sequence  $(f_{nm})$  is epi convergent to the function  $f$ .

**Example 3** Let  $f_{mn} : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f_{2n,2m}(x) = \begin{cases} \frac{1}{nm}, & \text{if } x \text{ is rational,} \\ 1 - \frac{1}{nm}, & \text{otherwise,} \end{cases}$$

and

$$f_{2n+1,2m+1}(x) = 1 - 2f_{2n,2m} = \begin{cases} 1 - \frac{1}{nm}, & \text{if } x \text{ is rational,} \\ \frac{1}{nm}, & \text{otherwise.} \end{cases}$$

Then we have the subsequences  $(f_{2n,2m}(x))$  converging to

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational,} \\ 1, & \text{otherwise,} \end{cases}$$

and  $(f_{2n+1,2m+1}(x))$  converging to

$$h(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{otherwise.} \end{cases}$$

Since  $g(x) \neq h(x) \quad \forall x \in \mathbb{R}$ , the sequence  $(f_{kl}(x))$  does not convergence pointwise. On the other hand, this sequence is easily shown to be epi convergent with the epi-limit  $f(x) \equiv 0$ , by choosing for any arbitrary  $x$  and a sequence  $(y_{nm})$  according to

$$y_{2n,2m} \text{ rational such that } |y_{2n,2m} - x| < \frac{1}{nm}$$

and

$$y_{2n+1,2m+1} \text{ irrational such that } |y_{2n+1,2m+1} - x| < \frac{1}{nm}.$$

Then  $y_{kl} \rightarrow x$  and  $f_{kl}(y_{kl}) = \frac{1}{kl/2} \rightarrow 0$  thus  $\limsup_{kl} f_{kl} < f(x) = 0$  whereas  $0 = f(x) \leq P\text{-}\liminf_{kl} f_{kl}(x_{kl})$ ,  $0 = f(x) \leq \liminf_k f_{kl}(x_{kl})$  and  $0 = f(x) \leq \liminf_l f_{kl}(x_{kl})$  are trivially satisfied for any sequence  $(x_{kl})$  converging to  $x$ .

**Example 4** Take  $f_{nm} = g_{nm}$ , where

$$g_{nm}(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \text{ or } x = \frac{k}{j+l}, k \in \mathbb{Z}, j \in \{1, 2, \dots, n\}, l \in \{1, 2, \dots, m\}, \\ -1, & \text{otherwise.} \end{cases}$$

Then we have  $f_{nm} \rightarrow 0$  pointwise, but  $\lim_e f_{nm} = -1$ . If we take  $f_{nm} = (-1)^{n+m}g_{nm}$ , then we have  $f_{nm} \rightarrow 0$  pointwise, but  $\lim_e f_{nm}$  does not exists at any point.

**Example 5** Let  $f_{mn} : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f_{mn}(x) = mne^{(m+n)x}$ . Then we have

$$P\text{-}\lim f_{mn} = \begin{cases} 0, & \text{if } x \leq 0, \\ +\infty, & \text{if } x > 0, \end{cases}$$

and

$$\lim_e f_{mn} = \begin{cases} 0, & \text{if } x < 0, \\ -\frac{1}{e}, & \text{if } x = 0, \\ +\infty, & \text{if } x > 0. \end{cases}$$

**Example 6** Let  $f_{mn} : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f_{mn}(x) = \begin{cases} 0, & \text{if } x \leq -\frac{1}{m+n}, \\ \frac{(m+n)x+1}{3}, & \text{if } -\frac{1}{m+n} \leq x \leq \frac{1}{m+n}, \\ 1, & \text{if } x \geq \frac{1}{m+n}. \end{cases}$$

Then we have

$$P\text{-}\lim f_{mn} = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{3}, & \text{if } x = 0, \\ 1, & \text{if } x > 0 \end{cases}$$

and

$$\lim_e f_{mn} = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 0. \end{cases}$$

The lower semicontinuity property of functionals has often played an important role in proving the existence of the extremals of the functionals.

**Theorem 1** *Let  $(f_{nm})$  be a sequence of lower semicontinuous functional on a metric space  $(X, d)$ . If  $(f_{nm})$  is epi convergent to  $f$  then the functional  $f$  is lower-semicontinuous (l.s.c.).*

**Proof.** Assume that at some  $x$  the limit  $f$  is not l.s.c. Then there exists an  $\epsilon > 0$  such that in any neighbourhood  $U(x, \frac{1}{nm}) = \{w : d(w, x) < \frac{1}{nm}\}$  we can find an element  $w_{nm} \in U(x, \frac{1}{nm})$  satisfying  $f(w_{nm}) < f(x) - \epsilon$ . Since  $(f_{nm})$  is epi convergent to  $f$ , we have for all double sequence  $(w_{nm})$  there exists subsequence  $(\xi_{v_{nm}})$  such that

$$d(\xi_{v_{nm}}, w_{nm}) < \frac{1}{nm}$$

and

$$f_{v_{nm}}(\xi_{v_{nm}}) < f(w_{nm}) + \frac{1}{nm}$$

where  $(v_{nm})$  can be chosen as to be strictly increasing in the Pringsheim sense. Hence we have a double sequence  $(\xi_{v_{nm}})$  which is P-convergent to  $x$  with

$$f_{v_{nm}}(\xi_{v_{nm}}) < f(w_{nm}) + \frac{1}{nm} < f(x) - \epsilon + \frac{1}{nm}$$

yielding

$$P - \liminf_{nm} f_{v_{nm}}(\xi_{v_{nm}}) \leq f(x) - \epsilon$$

in contradiction to for all  $(\xi_{v_{nm}})$ , P-convergent to  $x$ ,

$$f(x) \leq P - \liminf_{nm} f_{v_{nm}}(\xi_{v_{nm}})$$

according to the assumption of epi-convergence. ■

**Definition 4** *We say that a double function sequence  $f_{nm} : X \rightarrow [0, +\infty]$  is equicoercive if there exists a compact set  $K$  (independent of  $n$  and  $m$ ) such that*

$$\inf\{f_{nm}(x) : x \in X\} = \inf\{f_{nm}(x) : x \in K\}.$$

**Theorem 2** *Let  $(f_{mn})$  be an equicoercive double sequence of functional which is epi converges in the Pringsheim sense to the functional  $f$ . Assume that  $(x_{nm})$  is a double sequence of minimizers for  $(f_{nm})$ , respectively and  $P - \lim_{nm} x_{nm} = L$  then  $L$  is a minimizer of  $f$  and*

$$f(L) = \min_{x \in X} f(x) = P - \lim_{nm} f_{nm}(x_{nm}) = P - \lim_{nm} \min_{x \in X} f_{nm}(x).$$

**Proof.** From the (c) part of the definition of epi convergence in the sense of Pringsheim we are granted the following

$$f(L) \leq P - \liminf_{nm} f_{nm}(x_{nm}).$$

Now, for any  $L_0 \in X$ , we know that there exists a double sequence  $\{y_{nm}\}$  such that  $P - \lim_{nm} y_{nm} = L_0$  and

$$P - \lim_{nm} f_{nm}(y_{nm}) = f(L_0).$$

Using the fact that  $x_{nm}$  is a minimizer of  $f_{nm}$ , we obtain the following

$$f(L_0) = P - \lim_{nm} f_{nm}(y_{nm}) \geq P - \limsup_{nm} f_{nm}(x_{nm}) \geq P - \liminf_{nm} f_{nm}(x_{nm}) \geq f(L)$$

and therefore  $L$  is a minimizer of  $f$ . Eventually, letting  $L_0 = L$  leads to the fact that all inequalities are in fact equalities and we have the result. ■

### 3 Kuratowski Convergence of Double Sequences of Sets

The study of Kuratowski convergence of epigraphs originated with Wijsman [24, 25], and has its roots in convex analysis. This mode of convergence is now usually called epiconvergence in the literature (see, for example, [1]-[5],[11]). In this section, this concept will be generalized to double set sequences.

**Definition 5** Let  $(A_{nm})$  be double sequence of subsets of  $X$ . The lower limit of the sequence  $(A_{nm})$  is defined by

$$LiA_{nm} = cl \left( \bigcup_{(n,m) \in \mathbb{N} \times \mathbb{N}} \bigcap_{\{(k,l): k \geq n, l \geq m\}} clA_{kl} \right)$$

where  $cl$  denotes the closure operation.

The upper limit of the double sequence  $(A_{nm})$  is defined by

$$LsA_{nm} = \bigcap_{(n,m) \in \mathbb{N} \times \mathbb{N}} cl \left( \bigcup_{\{(k,l): k \geq n, l \geq m\}} clA_{kl} \right).$$

The double sequence  $(A_{nm})$  is said to be convergent if the following equality holds:

$$LiA_{nm} = LsA_{nm}.$$

Its limit, denoted  $A = LimA_{nm}$ , is the subset  $A$  of  $X$  equal to this common value. We shall refer to this notion of set convergence under the name Kuratowski convergence.

The following theorem make these limits easy to handle:

**Theorem 3** Let  $(A_{nm})$  be a double sequence of subset of  $X$ . Then

$$LiA_{nm} = \bigcap_{N \in \mathcal{N}_{\infty}^{\#}} cl \left( \bigcup_{(k,l) \in N} A_{kl} \right) \tag{1}$$

and

$$LsA_{nm} = \bigcap_{N \in \mathcal{N}_{\infty}} cl \left( \bigcup_{(k,l) \in N} A_{kl} \right) \tag{2}$$

where

$$\begin{aligned} \mathcal{N}_{\infty} &= \{N \subset \mathbb{N} \times \mathbb{N} : \mathbb{N} \times \mathbb{N} \setminus N \text{ is finite}\} \\ \mathcal{N}_{\infty}^{\#} &= \{N \subset \mathbb{N} \times \mathbb{N} : N \text{ is infinite}\}. \end{aligned}$$

**Proof.** Formula (2) follows directly from the definition of  $LsA_{nm}$ . Let us prove (1).

$$LiA_{nm} = cl \left( \bigcup_{(n,m) \in \mathbb{N} \times \mathbb{N}} \bigcap_{\{(k,l): k \geq n, l \geq m\}} clA_{kl} \right) \subset \bigcap_{N \in \mathcal{N}_{\infty}^{\#}} cl \left( \bigcup_{(k,l) \in N} A_{kl} \right).$$

Since the second member is closed, it is equivalent to prove that

$$\bigcap_{\{(k,l): k \geq n, l \geq m\}} clA_{kl} \subset cl \left( \bigcup_{(k,l) \in N} A_{kl} \right)$$

for all  $(n, m) \in \mathbb{N} \times \mathbb{N}$  and for all  $N \in \mathcal{N}_\infty^\sharp$ . By definition of  $\mathcal{N}_\infty^\sharp$ , since  $N \in \mathcal{N}_\infty^\sharp$  for any  $(n, m) \in \mathbb{N} \times \mathbb{N}$

$$\{(k, l) \in \mathbb{N} \times \mathbb{N} : k \geq n, l \geq m\} \cap N \neq \emptyset.$$

Therefore

$$\bigcap_{\{(k,l):k \geq n, l \geq m\}} clA_{kl} \subset \bigcup_{(k,l) \in N} clA_{kl} \subset cl \left( \bigcup_{(k,l) \in N} clA_{kl} \right) = \bigcup_{(k,l) \in N} clA_{kl}.$$

Let us now prove the opposite inclusion

$$cl \left( \bigcup_{(n,m) \in \mathbb{N} \times \mathbb{N}} \bigcap_{\{(k,l):k \geq n, l \geq m\}} clA_{kl} \right) \supset \bigcap_{N \in \mathcal{N}_\infty^\sharp} cl \left( \bigcup_{(k,l) \in N} A_{kl} \right).$$

It is equivalent to prove that for any sequence of closed sets  $(A_{nm})$

$$R = cl \left( \bigcup_{(n,m) \in \mathbb{N} \times \mathbb{N}} \bigcap_{\{(k,l):k \geq n, l \geq m\}} A_{kl} \right) \supset \bigcap_{N \in \mathcal{N}_\infty^\sharp} cl \left( \bigcup_{(k,l) \in N} A_{kl} \right) = Q.$$

Let us take  $x \notin R$  and prove that  $x \notin Q$ . Since  $x \notin R$ , there exists a neighbourhood  $U$  which contains  $x$  and satisfies; for all  $(n, m) \in \mathbb{N} \times \mathbb{N}$  there exists  $k_n > n, l_m > m$  such that

$$U \cap A_{k_n, l_m} = \emptyset.$$

Let us define  $N = \{(k_n, l_m) : (n, m) \in \mathbb{N} \times \mathbb{N}\}$ .  $N$  belongs to  $\mathcal{N}_\infty^\sharp$  and for all  $(k, l) \in N$

$$U \cap A_{kl} = \emptyset;$$

equivalently

$$U \cap \left( \bigcup_{(k,l) \in N} A_{kl} \right) = \emptyset.$$

So we have found an  $N \in \mathcal{N}_\infty^\sharp$  such that

$$x \notin cl \left( \bigcup_{(k,l) \in N} A_{kl} \right),$$

which is equivalent to say that  $x \notin Q$ . ■

**Theorem 4** Let  $(X, d)$  be a metric space and  $(A_{nm})$  be a double sequence of subsets of  $X$ . Then  $LiA_{nm}$  and  $LsA_{nm}$  are two closed subsets of  $X$  and  $LiA_{nm} \subset LsA_{nm}$ .

**Proof.** Since intersection of closed sets is closed, from (1) and (2) it follows that  $LiA_{nm}$  and  $LsA_{nm}$  are closed. The inclusion  $LiA_{nm} \subset LsA_{nm}$  follows from (1), (2) and the inclusion  $\mathcal{N}_\infty \subset \mathcal{N}_\infty^\sharp$ . ■

**Proposition 1** Let  $(X, d)$  be a metric space. For any double sequence  $(A_{nm})$  of subset of  $X$  the following sequential formulations hold:

$$LiA_{nm} = \{x \in X : \exists(x_{nm}), \forall(n, m) \in \mathbb{N} \times \mathbb{N}, x_{nm} \in A_{nm} \quad P - \lim_{n, m \rightarrow \infty} x_{nm} = x\} \quad (3)$$

and

$$LsA_{nm} = \{x \in X : \exists(n_k, m_l), \exists(x_{kl}), \forall(k, l) \in \mathbb{N} \times \mathbb{N}, x_{kl} \in A_{n_k m_l} \quad P - \lim_{k, l \rightarrow \infty} x_{kl} = x\}. \quad (4)$$

**Proof.** Let us first prove (3); the inclusion

$$LiA_{nm} \supset \{x = P - \lim_{n,m \rightarrow \infty} x_{nm} : x_{nm} \in A_{nm}\}$$

is true in a metric space. Let  $(x_{nm})$  be such that for all  $(n, m) \in \mathbb{N} \times \mathbb{N}$   $x_{nm} \in A_{nm}$  and

$$P - \lim_{n,m \rightarrow \infty} x_{nm} = x;$$

then for any neighbourhood  $U$  of  $x$ , there exists  $N_U \in \mathcal{N}_\infty$  such that

$$\bigcup_{(k,l) \in N_U} \{x_{kl}\} \subset U.$$

Since for any  $N \in \mathcal{N}_\infty^\sharp$ ,  $N \cap N_U \neq \emptyset$ , it follows preceding inclusion that

$$U \cap \left( \bigcup_{(k,l) \in N} A_{kl} \right) \neq \emptyset.$$

This being true for any neighbourhood of  $x$  and any  $N \in \mathcal{N}_\infty^\sharp$ , it follows from (1) that

$$x \in \bigcap_{N \in \mathcal{N}_\infty^\sharp} cl\left( \bigcup_{(k,l) \in N} A_{kl} \right) = LiA_{nm}.$$

Let us prove the opposite inclusion. Let  $x \in LiA_{nm}$  and  $(U_{kl})$  be a countable open neighbourhoods of  $x$  which is decreasing and such that  $\bigcap_{(k,l) \in \mathbb{N} \times \mathbb{N}} U_{kl} = \{x\}$ . By definition of  $LiA_{nm}$ , for every  $(k, l) \in \mathbb{N} \times \mathbb{N}$  there exist integers  $n_k$  and  $m_l$  such that

$$U_{kl} \cap A_{nm} \neq \emptyset$$

for all  $n \geq n_k, m \geq m_l$ . Let us define a sequence  $(x_{nm})$  in the following way: For each  $n$  such that  $n_k \leq n < n_{k+1}$  and for each  $m$  such that  $m_l \leq m < m_{k+1}$ , take  $x_{nm} \in U_{kl} \cap A_{nm}$ , which is nonvoid. The defined sequence  $(x_{nm})$  satisfies for all  $(n, m) \in \mathbb{N} \times \mathbb{N}$   $x_{nm} \in A_{nm}$  and for all  $(k, l) \in \mathbb{N} \times \mathbb{N}$   $x_{nm} \in U_{kl}$  for all  $n \geq n_k$  and  $m \geq m_l$ . This exactly the definition of the convergence of the sequence  $(x_{nm})$  to  $x$ .

The proof of (4) can be obtained in a similar way. ■

From this proposition, we get the sequential definitions of convergence of a double sequence of sets.

**Definition 6** Let  $(A_{nm})$  be a double sequence of closed subsets of  $X$ . We say that  $(A_{nm})$  is Kuratowski convergent to a closed subset  $A$  of  $X$  provided  $A = LiA_{nm} = LsA_{nm}$ , where

$$LiA_{nm} = \{x \in X : \text{there exist a sequence } (a_{nm}) \text{ } P\text{-convergent to } x \text{ with } a_{nm} \in A_{nm} \text{ for all but finitely many pairs of integers } (n, m)\}$$

and

$$LsA_{nm} = \{x \in X : \text{there exists positive integers } n_1 < n_2 < n_3 < \dots, m_1 < m_2 < m_3 < \dots \text{ and } a_{kl} \in A_{n_k m_l} \text{ such that } P - \lim_{k,l \rightarrow \infty} a_{kl} = x\}.$$

When  $A = LiA_{nm} = LsA_{nm}$ , we write  $A = LimA_{nm}$ .



**Example 7** Let  $(A_{nm})$  be the following double sequence of sets:

$$A_{nm} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n+m}\right\}.$$

For any  $n, m \in \mathbb{N}$

$$cl \left( \bigcup_{k \geq n, l \geq m}^{\infty} A_{kl} \right) = \{0\} \cup \left\{ \frac{1}{n+m} : n, m \in \mathbb{N} \right\}$$

and

$$\bigcap_{n, m=1}^{\infty} cl \left( \bigcup_{k \geq n, l \geq m}^{\infty} A_{kl} \right) = \{0\} \cup \left\{ \frac{1}{n+m} : n, m \in \mathbb{N} \right\}.$$

Similarly

$$\bigcap_{k \geq n, l \geq m}^{\infty} A_{kl} = A_{nm}, \quad \bigcup_{n, m=1}^{\infty} \bigcap_{k \geq n, l \geq m}^{\infty} A_{kl} = \left\{ \frac{1}{n+m} : n, m \in \mathbb{N} \right\}$$

and

$$cl \left( \bigcup_{n, m=1}^{\infty} \bigcap_{k \geq n, l \geq m}^{\infty} A_{kl} \right) = \{0\} \cup \left\{ \frac{1}{n+m} : n, m \in \mathbb{N} \right\}$$

and then the double sequence  $(A_{nm})$  is Kuratowski convergent to

$$\{0\} \cup \left\{ \frac{1}{n+m} : n, m \in \mathbb{N} \right\}.$$

**Example 8** Let  $(A_{nm})$  be the following double sequence of sets

$$A_{nm} = \{(x, y) \in \mathbb{R}^2 : y \leq nm x\}.$$

This double sequence of sets is Kuratowski convergent to the set

$$A = \{(x, y) \in \mathbb{R}^2 : 0 \leq x\}$$

**Example 9** Let the double set sequence  $(A_{nm})$  be defined by

$$A_{nm} = \{(x, y) \in \mathbb{R}^2 : |x|^{\frac{n+m}{2}} + |y|^{\frac{n+m}{2}} \leq 1\}.$$

Then this double sequence of sets is Kuratowski convergent to the set

$$A = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}.$$

**Theorem 5** Let  $(X, d)$  be a metric space and  $(f_{nm})$  a double sequence of functions from  $X$  into  $[-\infty, \infty]$ . The limit sets  $Li(epi f_{nm})$  and  $Ls(epi f_{nm})$  are still epigraphs. They are equal to the epigraphs of  $ls_e f_{nm}$  and  $li_e f_{nm}$  respectively, that is,

$$Li(epi f_{nm}) = epi(ls_e f_{nm}) \tag{5}$$

$$Ls(epi f_{nm}) = epi(li_e f_{nm}). \tag{6}$$

**Proof.** Let us first prove (5). By definition  $Li$ ,  $(x, \alpha) \in Li(epi f_{nm})$  if and only if: For all  $V \in U(x)$  and for all  $\epsilon > 0$ , there exists  $(n, m) \in \mathbb{N} \times \mathbb{N}$  such that for all  $k \geq n, l \geq m$

$$(V \times (\alpha - \epsilon, \alpha + \epsilon)) \cap epi f_{kl} \neq \emptyset.$$

This is equivalent to for all  $V \in U(x)$  and for all  $\epsilon > 0$ , there exist  $(n, m) \in \mathbb{N} \times \mathbb{N}$  such that for all  $k \geq n, l \geq m$  and  $x_{kl} \in V$  satisfying

$$\alpha + \epsilon > f_{kl}(x_{kl}).$$

This can be reformulated in the following way:

$$\alpha \geq \sup_{V \in U(x)} \inf_{(n,m) \in \mathbb{N} \times \mathbb{N}} \sup_{\{(k,l): k \geq n, l \geq m\}} \inf_{u \in V} f_{kl}(u)$$

that is

$$\alpha \geq \sup_{V \in U(x)} P - \limsup_{n,m \rightarrow \infty} \inf_{u \in V} f_{nm}(u) = (ls_e f_{nm})(x)$$

which means  $(x, \alpha) \in epi(ls_e f_{nm})$ .

Let us prove (6). By definition  $Ls$ ,  $(x, \alpha) \in Ls(epi f_{nm})$  if and only if: for all  $(n, m) \in \mathbb{N} \times \mathbb{N}$ , for all  $V \in U(x)$  and for all  $\epsilon > 0$ , there exists  $k \geq n, l \geq m$  such that

$$(V \times (\alpha - \epsilon, \alpha + \epsilon)) \cap epi f_{kl} \neq \emptyset.$$

Because the sets are epigraphs, this is equivalent to: for all  $(n, m) \in \mathbb{N} \times \mathbb{N}$ , for all  $V \in U(x)$  and for all  $\epsilon > 0$ , there exists  $x_{km} \in V$  such that

$$\alpha + \epsilon > f_{kl}(x_{kl}).$$

This can be reformulated in the following way:

$$\alpha \geq \sup_{V \in U(x)} \sup_{(n,m) \in \mathbb{N} \times \mathbb{N}} \inf_{\{(k,l): k \geq n, l \geq m\}} \inf_{u \in V} f_{kl}(u)$$

that is,

$$\alpha \geq \sup_{V \in U(x)} P - \liminf_{n,m \rightarrow \infty} \inf_{u \in V} f_{nm}(u) = (li_e f_{nm})(x)$$

which means  $(x, \alpha) \in epi(li_e f_{nm})$ . ■

We are now able to state the main result of this paper and establish the equivalence between the epi convergence of a double sequence of functions and the Kuratowski convergence of double sequences of set of their epigraphs. It is direct consequence of Definition 2 and Theorem 5.

**Theorem 6** *Let  $(X, d)$  be a metric space and  $(f_{nm})$  a double sequence of functions from  $X$  into  $[-\infty, \infty]$ . The sequence  $(f_{nm})$  is epi convergent if and only if the sequence of sets  $(epi f_{nm})$  is convergent in the Kuratowski sense. In that case following equality holds:*

$$epi(\lim_e f_{nm}) = Lim(epi f_{nm}).$$

Theorem 6 allows us to interpret, by consideration of epigraphs, the epi convergence of a double sequence of functions in terms of double set convergence.

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