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A Simple Proof And Some Applications Of An Integral Representation For The Catalan Numbers*

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Abstract

A simple proof for an integral representation of the Catalan numbers that makes use of an ordinary generating function approach and relies on essentially nothing beyond elementary integral calculus is presented. Application of the integral representation given is then made in the evaluation of a family of infinite sums consisting of products between the central binomial coefficient and the bicentral binomial coefficient.

1 Introduction

A representation of special numbers obtained from various counting sequences using an integral is one of a number of important tools available in their analysis. Integral representations for a number of classical counting numbers are known with the greatest number of representations found for the Catalan numbers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

Recall the nth Catalan number C_n is defined by the recurrence relation $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$ for $n \in \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ with $C_0 = 1$. In the case of the Catalan numbers a great many integral representations are known. A summary of these are to be found in [12]. A canonical example of an integral representation for the Catalan numbers is given in the following theorem.

Theorem 1 (Penson and Sixdeniers [1, p. 2, Eq. (10)]) *For $n \in \mathbb{Z}_{\geqslant 0}$ the Catalan numbers C_n can be represented by the integral*

$$C_n = \frac{1}{2\pi} \int_0^4 x^n \sqrt{\frac{4-x}{x}}\, dx. \tag{1}$$

The integral representation for the Catalan numbers given in (1) was first established by Penson and Sixdeniers using the Mellin transform [1]. A few years later it was proved by Dana-Picard using a recurrence relation combined with a telescoping process [2].

A wealth of alternative integral representations for the Catalan numbers stem from (1) on applying various substitutions. One such alternative representation is obtained on enforcing a substitution of $x \mapsto \frac{1}{x+\frac{1}{4}}$. Doing so yields

$$C_n = \frac{4^{n+2}}{\pi} \int_0^\infty \frac{\sqrt{x}}{(4x+1)^{n+2}}\, dx, \tag{2}$$

and is the form of the integral representation for the Catalan numbers to be used in section 3. The result was recently established using a particularly interesting application of Cauchy's integral formula [10]. As an alternative to this advanced technique it can be proved using an ordinary generating function approach to be introduced in the next section.

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2 Proof Using an Ordinary Generating Function Approach

We give a new proof for the integral representation of the Catalan numbers given by (1) using an ordinary generating function approach. Recalling the well-known ordinary generating function for the Catalan numbers of [13, p. 4, Eq. (1.3)]

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}, \quad |x| < \frac{1}{4}, \tag{3}$$

the proof to be given relies on essentially nothing beyond elementary integral calculus.

Proof. Let

$$\Lambda_n = \int_0^4 x^n \sqrt{\frac{4-x}{x}}\, dx.$$

Enforcing a substitution of $x \mapsto 4x$ gives

$$\Lambda_n = 4\int_0^1 (4x)^n \sqrt{\frac{1-x}{x}}\, dx.$$

Now consider the following ordinary generating function given by

$$G(t) = \sum_{n=0}^{\infty} \Lambda_n t^n = 4\sum_{n=0}^{\infty} \int_0^1 (4tx)^n \sqrt{\frac{1-x}{x}}\, dx. \tag{4}$$

After interchanging the summation and the integration, which is permissible due to Fubini's theorem [14, Thm. 2.25, p. 55], for all $|t| < \frac{1}{4}$ (4) becomes

$$\begin{aligned} G(t) &= 4\int_0^1 \sqrt{\frac{1-x}{x}} \sum_{n=0}^{\infty} (4tx)^n\, dx = 4\int_0^1 \sqrt{\frac{1-x}{x}} \frac{dx}{1-4tx} \\ &= \frac{2}{t}\left[\sqrt{1-4t}\arctan\left(\sqrt{\frac{1-x}{x}}\frac{1}{\sqrt{1-4t}}\right) - \arctan\left(\sqrt{\frac{1-x}{x}}\right)\right]_0^1 \\ &= \frac{\pi\left(1-\sqrt{1-4t}\right)}{t} = 2\pi\sum_{n=0}^{\infty} C_n t^n, \end{aligned} \tag{5}$$

where in the second line the integration performed is elementary while in the third line we have made use of the ordinary generating function for the Catalan numbers given in (3). Equating equal coefficients for t^n in (5) gives $\Lambda_n = 2\pi C_n$ from which the desired integral representation for the Catalan numbers then follows and completes the proof. ∎

3 Some Applications to Infinite Sums

One application for the integral representations found for the Catalan numbers is in their use in the evaluation of a family of infinite sums that contain the product between the central binomial coefficient $\binom{2n}{n}$ and the bicentral binomial coefficient $\binom{4n}{2n}$. We give six examples and indicate how others in a certain family of infinite sums can be found. Another example not given here that employs an integral representation for the Catalan numbers in the evaluation of a similar looking infinite sum can be found in [15].

Proposition 1

$$\sum_{n=0}^{\infty} \binom{4n}{2n}\binom{2n}{n}\frac{1}{64^n(n+1)} = \frac{8\sqrt{2}}{3\pi}. \tag{6}$$

Proof. Denote the value for the infinite sum to be found by S. From the well-known closed-form expression for the Catalan numbers in terms of the central binomial coefficient, namely [13, p. 4, Eq. (1.6)]

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

we may write

$$S = \sum_{n=0}^{\infty}\binom{4n}{2n}\frac{C_n}{64^n} = \frac{16}{\pi}\sum_{n=0}^{\infty}\binom{4n}{2n}\frac{1}{16^n}\int_0^{\infty}\frac{\sqrt{t}}{(4t+1)^{n+2}}\,dt,$$

after the integral representation for the Catalan numbers given in (2) has been used. Due to the positivity of all terms involved, the summation and the integration may be interchanged. Doing so yields

$$S = \frac{16}{\pi}\int_0^{\infty}\frac{\sqrt{t}}{(4t+1)^2}\sum_{n=0}^{\infty}\binom{4n}{2n}\frac{1}{(4\sqrt{4t+1})^{2n}}\,dt. \tag{7}$$

For the infinite sum appearing in (7), applying the classical result for absolutely convergent series of

$$\sum_{n=0}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=0}^{\infty} a_n + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n a_n, \tag{8}$$

we find

$$\sum_{n=0}^{\infty}\binom{4n}{2n}\frac{1}{(4\sqrt{4t+1})^{2n}} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{(4\sqrt{4t+1})^{n}} + \frac{1}{2}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n}{(4\sqrt{4t+1})^{n}}. \tag{9}$$

Recalling the ordinary generating function for the central binomial coefficients of

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}, \quad |x| < \frac{1}{4}, \tag{10}$$

setting, in turn, $x = \frac{1}{4\sqrt{4t+1}}$ and $x = \frac{-1}{4\sqrt{4t+1}}$ in (10) with the results valid for all $t > 0$, (9) reduces to

$$\sum_{n=0}^{\infty}\binom{4n}{2n}\frac{1}{(4\sqrt{4t+1})^{2n}} = \frac{\sqrt[4]{4t+1}}{2}\left[\frac{1}{\sqrt{\sqrt{4t+1}-1}} + \frac{1}{\sqrt{\sqrt{4t+1}+1}}\right] = \frac{\sqrt{2}\sqrt[4]{4t+1}}{4\sqrt{t}}\sqrt{\sqrt{4t+1}+2\sqrt{t}}.$$

Note here the term appearing in square brackets has been algebraically rearranged by finding it square before taking its square root to give the final expression on the right. Thus (7) becomes

$$S = \frac{4\sqrt{2}}{\pi}\int_0^{\infty}\frac{\sqrt{\sqrt{4t+1}+2\sqrt{t}}}{(4t+1)^{\frac{7}{4}}}\,dt.$$

Substituting $t = \frac{1}{4}\tan^2\theta$ produces

$$S = \frac{2\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}\sin\theta\sqrt{1+\sin\theta}\,d\theta = \frac{2\sqrt{2}}{\pi}\cdot\frac{4}{3} = \frac{8\sqrt{2}}{3\pi},$$

where the last of these integrals is elementary and can be found, for example, by substituting $t = \tan\frac{\theta}{2}$, and completes the proof. ■

Proposition 2

$$\sum_{n=0}^{\infty}\binom{4n}{2n}\binom{2n}{n}\frac{1}{64^n(2n+1)(n+1)} = \frac{4}{\pi}\log\left(3+2\sqrt{2}\right) - \frac{8\sqrt{2}}{3\pi}. \tag{11}$$

Proof. The proof proceeds in a similar fashion to that given for Proposition 1. Denoting the sum to be found by S, in terms of the Catalan numbers which is then expressed in terms of the integral representation given by (2) one has

$$S = \sum_{n=0}^{\infty} \binom{4n}{2n} \frac{C_n}{64^n(2n+1)} = \frac{16}{\pi} \int_0^{\infty} \frac{\sqrt{t}}{(4t+1)^2} \sum_{n=0}^{\infty} \binom{4n}{2n} \frac{1}{16^n(2n+1)(4t+1)^n}\, dt.$$

Here an interchange between the summation and the integration has been made and is permissible due to the positivity of all terms involved. Applying the result in (8) to the infinite sum that appears one has

$$\sum_{n=0}^{\infty} \binom{4n}{2n} \frac{1}{(4\sqrt{4t+1})^{2n}(2n+1)} = \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{(4\sqrt{4t+1})^n(n+1)} + \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n}{(4\sqrt{4t+1})^n(n+1)}. \quad (12)$$

If now in the ordinary generating function for the central binomial coefficients given by (10) one replaces x with t and integrates with respect to t from 0 to x one obtains

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1} = \frac{1-\sqrt{1-4x}}{2x}, \quad |x| < \frac{1}{4}. \quad (13)$$

Applying this result, in turn, with x replaced with $\frac{1}{4\sqrt{4t+1}}$ and $\frac{-1}{4\sqrt{4t+1}}$ in (13), the sums in (12) can be found and leads to

$$S = \frac{16\sqrt{2}}{\pi} \int_0^{\infty} \frac{\sqrt{t}}{(4t+1)^{\frac{7}{4}}} \sqrt{\sqrt{4t+1} - 2\sqrt{t}}\, dt,$$

or

$$S = \frac{4\sqrt{2}}{\pi} \int_0^{\frac{\pi}{2}} \sin\theta \tan\theta \sqrt{1-\sin\theta}\, d\theta,$$

after the same substitution of $t = \frac{1}{4}\tan^2\theta$ as made in Proposition 1 has been used. The final integral is elementary and can be found on substituting, for example, $t = \tan\frac{\theta}{2}$. The result is

$$S = \frac{4\sqrt{2}}{\pi}\left(\frac{1}{\sqrt{2}} \log\left(3+2\sqrt{2}\right) - \frac{2}{3}\right) = \frac{4}{\pi}\log\left(3+2\sqrt{2}\right) - \frac{8\sqrt{2}}{3\pi},$$

and completes the proof. ∎

Proposition 3

$$\sum_{n=0}^{\infty} \binom{4n}{2n} \binom{2n}{n} \frac{1}{64^n(2n+1)} = \frac{4}{\pi} \log\left(1+\sqrt{2}\right).$$

Proof. From the partial fraction decomposition of

$$\frac{1}{(n+1)(2n+1)} = \frac{2}{2n+1} - \frac{1}{n+1},$$

combining this result with the values for the sums found in (6) and (11), the result immediately follows and completes the proof. ∎

Remark 1 *An alternative evaluation for the sum found in Proposition 3 is given by Campbell, D'Aurizio, and Sondow in [16, p. 99].*

Proposition 4

$$\sum_{n=0}^{\infty} \binom{4n}{2n} \binom{2n}{n} \frac{2n+1}{64^n(n+1)(n+2)} = \frac{176\sqrt{2}}{105\pi}. \quad (14)$$

Proof. We begin by observing that

$$C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1} = \frac{2(2n+1)}{(n+1)(n+2)}\binom{2n}{n}.$$

Denoting the sum to be found by S, in terms of the Catalan number C_{n+1} it becomes

$$S = \frac{1}{2}\sum_{n=0}^{\infty}\binom{4n}{2n}\frac{C_{n+1}}{64^n} = \frac{32}{\pi}\int_0^{\infty}\frac{\sqrt{t}}{(4t+1)^3}\sum_{n=0}^{\infty}\binom{4n}{2n}\frac{1}{16^n(4t+1)^n}\,dt. \tag{15}$$

Here the integral representation for the Catalan numbers given by (2) with n replaced with $n+1$ has been used while the interchange made between the summation and the integration is permissible due to the positivity of all terms involved. Next applying result (8) to the infinite sum appearing in (15) followed by the result for the ordinary generating function for the central binomial coefficients given in (10) one arrives at

$$S = \frac{8\sqrt{2}}{\pi}\int_0^{\infty}\frac{\sqrt{\sqrt{4t+1}+2\sqrt{t}}}{(4t+1)^{\frac{11}{4}}}\,dt,$$

or

$$S = \frac{4\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}\sin\theta\cos^2\theta\sqrt{1+\sin\theta}\,d\theta,$$

after the same substitution of $t = \frac{1}{4}\tan^2\theta$ as made in Proposition 1 has been used. The remaining integral can be again found by elementary means. The result is

$$S = \frac{4\sqrt{2}}{\pi}\cdot\frac{44}{105} = \frac{176\sqrt{2}}{105\pi},$$

and completes the proof. ■

Proposition 5

$$\sum_{n=0}^{\infty}\binom{4n}{2n}\binom{2n}{n}\frac{1}{64^n(n+2)} = \frac{152\sqrt{2}}{105\pi}. \tag{16}$$

Proof. From the partial fraction decomposition of

$$\frac{2n+1}{(n+1)(n+2)} = \frac{3}{n+2} - \frac{1}{n+1},$$

combining this result with the values for the sums found in (6) and (14), the result immediately follows and completes the proof. ■

Proposition 6

$$\sum_{n=0}^{\infty}\binom{4n}{2n}\binom{2n}{n}\frac{1}{64^n(n+1)(n+2)} = \frac{128\sqrt{2}}{105\pi}.$$

Proof. From the partial decomposition of

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2},$$

the result is an immediate consequence of the difference between (6) and (16) and completes the proof. ■

Generalising the methods used in obtaining Propositions 4 and 5 it is possible to find in closed form the value for any sum of the form

$$\sum_{n=0}^{\infty}\binom{4n}{2n}\binom{2n}{n}\frac{1}{64^n(n+k+1)}, \tag{17}$$

where $k \in \mathbb{Z}_{\geqslant 0}$. That such sums could be found in closed form was first observed by Campbell, D'Aurizio, and Sondow in [16]. These are built up from the base case of $k = 0$ given by (6). By finding the value for the sum

$$\sum_{n=0}^{\infty} \binom{4n}{2n} \frac{C_{n+k}}{64^n},$$

using the result for the integral representation of the Catalan numbers given in (2) with n replaced with $n + k$, one finds

$$\sum_{n=0}^{\infty} \binom{4n}{2n} \frac{C_{n+k}}{64^n} = \frac{2^{2k+1}\sqrt{2}}{\pi} \int_0^{\frac{\pi}{2}} \sin\theta \cos^{2k}\theta \sqrt{1+\sin\theta}\, d\theta = \frac{2^{2k+1}\sqrt{2}}{\pi} I_k. \tag{18}$$

As

$$C_{n+k} = \frac{1}{n+k+1}\binom{2n+2k}{n+k} = \frac{2^k(2n+2k-1)(2n+2k-3)\cdots(2n+1)}{(n+k+1)(n+k)\cdots(n+1)}\binom{2n}{n},$$

the result for (17) then follows from a partial fraction decomposition of the rational function term appearing in front of the central binomial coefficient term in C_{n+k}. For example, the first four sums after the base sum found this way are:

$$\begin{aligned}
k = 1 &: \sum_{n=0}^{\infty} \binom{4n}{2n}\binom{2n}{n} \frac{1}{64^k(n+2)} = \frac{152\sqrt{2}}{105\pi}; \\
k = 2 &: \sum_{n=0}^{\infty} \binom{4n}{2n}\binom{2n}{n} \frac{1}{64^k(n+3)} = \frac{10\,568\sqrt{2}}{10\,395\pi}; \\
k = 3 &: \sum_{n=0}^{\infty} \binom{4n}{2n}\binom{2n}{n} \frac{1}{64^k(n+4)} = \frac{178\,328\sqrt{2}}{225\,225\pi}; \\
k = 4 &: \sum_{n=0}^{\infty} \binom{4n}{2n}\binom{2n}{n} \frac{1}{64^k(n+5)} = \frac{47\,453\,768\sqrt{2}}{72\,747\,675\pi}.
\end{aligned}$$

As the above four examples show, for low orders of k values for the integral I_k appearing in (18) can be readily found, these being a positive rational number for each $k \in \mathbb{Z}_{\geqslant 0}$. It would however be nice if a general expression for the integral I_k in terms of k could be found. To this end it is possible to express I_k in closed form as the sum of two hypergeometric functions [17, 18]. Here it can be shown that

$$\begin{aligned}
I_k &= \frac{2}{3\sqrt{2}}\, {}_2F_1\left(-2k, \frac{3}{2}; \frac{5}{2}; 1\right) + \frac{1}{2k+1}\, {}_2F_1\left(-\frac{1}{2}, 1; 2k+2; \frac{1}{2}\right) \\
&= \frac{4^{2k+1}(2k+2)!(2k)!\sqrt{2}}{(4k+4)!} + \frac{1}{2k+1}\, {}_2F_1\left(-\frac{1}{2}, 1; 2k+2; \frac{1}{2}\right).
\end{aligned}$$

In the second line the first of the hypergeometric functions has been simplified using the Chu–Vandermonde identity [19, Entry 15.4.24, p. 387]. Further simplification does not appear possible. Alternatively, as a simple positive rational number we conjecture that

$$I_k = 2\sum_{n=0}^{k+1} \sum_{i=n}^{\lfloor \frac{n+k+1}{2} \rfloor} \frac{(-1)^n}{2n+2k+1} \binom{n+k+1}{2i}\binom{i}{n},$$

and appears the best we can do.

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