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On A Family Of Kannan Type Selfmaps Of b-Metric Spaces*

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Abstract

Let $(X, d; s)$ be a b-metric space with parameter $s \geq 1$. Let T be a selfmap of X satisfying the following condition:

$$d(Tx, Ty) \leq \frac{1}{s+\beta}\{d(x, Tx) + d(y, Ty)\}, \quad \text{for all } x, y \in X, \qquad (K(\tfrac{1}{s+\beta}))$$

where $\beta \geq 0$ is another parameter. When $s = 1 = \beta$, we recapture the usual condition $(K(\frac{1}{2}))$ due to Kannan. We prove here, in the case $s + \beta > 2$ with $\beta > 0$, that the selfmap T is a Picard operator. The case $s + \beta = 2$ needs more conditions. Indeed, we provide an example of a continuous selfmap T of a complete b-metric space X satisfying $(K(\frac{1}{2}))$ but without being a Picard operator. So, in this case, a natural question arises: what are the mildest conditions ensuring the Picard property for T ? The purpose of this paper is to give answers to this question.

1 Introduction

Concerning the metric spaces, the following fixed point result was proved in 1968 by Kannan [19].

Theorem 1 ([19]) *Let (M, d) be a complete metric space and T be a selfmapping of M satisfying*

$$d(Tx, Ty) \leq \lambda \{d(x, Tx) + d(y, Ty)\}, \quad \forall x, y \in M, \qquad (K(\lambda))$$

where $\lambda \in [0, \frac{1}{2})$. Then T is a Picard operator.

Contrary to the case of the Banach contractive condition (see [5]), the condition $(K(\lambda))$ of Theorem 1 does not force T to be continuous. In 1975, V. Subrahmanyam (see [30]) has proved that Theorem 1 gives a characterization of the completeness of metric spaces.

We point out that the value $\frac{1}{2}$ can not be taken in Theorem 1. Indeed, there exist continuous selfmaps of compact metric spaces without having fixed points. This case was studied by R. Kannan in [20] and [21]. Next, we recall the following theorem.

Theorem 2 ([20, 21]) *Let (M, d) be a compact metric space and let T be a continuous mapping of M into itself such that*

(i) $d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$ *for all $x, y \in M$.*

(ii) For every closed subset F of M which contains more than one element and is mapped into itself by T there exists $x \in F$ such that

$$d(x, Tx) < \sup_{y \in F} d(y, Ty).$$

Then T has a unique fixed point u in M. If in addition to the hypothesis (i) and (ii), we have

(iii) $d(Tx, u) < d(x, u)$ *if $x \neq u$, where u is the unique fixed point of T.*

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Then T is a Picard operator.

In Theorem 2, we see that a continuous selfmap T of a compact metric space satisfying $K(\frac{1}{2})$ is far from being a Picard operator.

In 2017, J. Górnicki [18] has made some new contributions to selfmappings of a metric space (M, d) satisfying the condition:

$$d(Tx, Ty) < k\{d(x, Tx) + d(y, Ty)\}, \quad \text{for all } x \neq y \in M, \qquad (K - G)$$

where $k \in [0, 1]$.

Also, in [18] one can find a new proof to the following (known) result.

Theorem 3 ([18]) *Let (M, d) be a compact metric space and $T : M \to M$ be a continuous mapping satisfying*

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\} \quad \textit{for all } x \neq y \in M. \qquad (K - S)$$

Then T is a Picard operator.

We observe that if T satisfies the condition (K-S), then T satisfies the condition $(K(\frac{1}{2}))$. We notice that other results refining and completing results of Theorem 2 are established in [33]. For other related works in metric spaces, the reader is invited to see [22], [16], [8] and others.

Contrary to the case of metric spaces, in a b-metric space $(X, d; s)$, one can conceive many types of Kannan mappings. In this paper, we are interested by the following family of contractive conditions:

$$d(Tx, Ty) \leq \frac{1}{s + \beta}\{d(x, Tx) + d(y, Ty)\}, \quad \text{for all } x, y \in X, \qquad (K(\tfrac{1}{s+\beta}))$$

where T is a selfmap of the b-metric space $(X, d; s)$ and $\beta > 0$ is a parameter. The set of such selfmaps will be denoted by $\mathcal{K}(\frac{1}{s+\beta}))$.

We deal with two cases: (i) the case where $s + \beta > 2$ and (ii) the case where $s + \beta = 2$ which is a critical case. Indeed, we prove (through an example) that there exist selfmaps satisfying $(K(\frac{1}{2}))$ in a complete b-metric space but having no fixed point. So a natural question arises: what kind of (mild) condition has to be added to garantee fixed point ? The aim of this paper is to study this problem.

This paper is organized as follows: In Section 1, we present and introduction. In Section 2, we provide a recall of some notations and results needed in the paper and make some definitions. In Section 3, (see Theorem 4), we prove that if X is complete and if T satisfies the condition $K(\frac{1}{s+\beta})$ with $s + \beta > 2$, then T is a Picard operator.

If $s + \beta = 2$, then the condition $K(\frac{1}{s+\beta})$ coincides with the condition $K(\frac{1}{2})$. If T satisfies this condition then we show by Example 1 that the Picard property is not automatically ensured. However, when in addition, $1 \leq s < 2$, we prove (see Theorem 5) that if T satisfies the condition $K(\frac{1}{2})$, then the following conditions are equivalent:

(A) T is a Picard operator.

(C) T is asymptotically regular on X.

(D) T is a Cauchy operator (see Definition 2).

In the fourth section, we establish the same results for an orbitally continuous selfmap T for a complete b-metric space $(X, d; s)$ with any arbitrary $s \geq 1$ (see Theorem 6).

We have introduced the following new condition

(S) For any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(T^i x, T^j x) < \epsilon + \delta \text{ implies } d(T^{i+1}x, T^{j+1}x) < \epsilon \text{ for any } i, j \in \mathbb{N} \cup \{0\}.$$

We point out that the condition (S) is a variant of the condition (D), given by T. Suzuki, in the paper [32].

In Theorem 7, we prove that if $(X, d; s)$ is a complete b-metric space with parameter $s \geq 1$ and if the selfmap T satisfies the condition (S), then T is a Picard operator. Thus (S) is sufficient condition. We do not know if it is also necessary.

In the fifth section we suppose that $(X, d; s)$ is a compact b-metric space with parameter $s \geq 1$ and that the selfmap T satisfies the condition $K(\frac{1}{2})$. Then we prove (see Theorem 8) that the conditions (A), (C) and (D) are still equivalent and that are equivalent with the following condition:

(B) For all $x, u \in X$ satisfying $\lim_{k\to\infty} T^{n_k}x = u$, then we have $\lim_{k\to\infty} T(T^{n_k}x) = u$.

We provide Example 2 to support Theorem 8.

2 Definitions and Preliminaries

In all this paper, we let $(X, d; s)$ be a b-metric space with parameter $s \geq 1$. Next, we recall the definition of this concept, see for more informations the references: [4], [12], [13] and the book [23] by W. Kirk and N. Shahzad.

Definition 1 *Let X be a non-empty set and let $d : X \times X \to [0, +\infty)$ be a function. Then d is said to be a b-metric on the set X, if the following conditions are satisfied:*

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) there exists a real number $s \geq 1$ such that: $d(x, y) \leq s\,[d(x, u) + d(u, y)]$, for all $x, y \in X$.

The triplet $(X, d; s)$ is said to be a b-metric space with parameter s. The inequality (iii) is called the s-triangle inequality.

Let $T : X \to X$ be a selfmapping. We set $\mathrm{Fix}\,(T) := \{x \in X : Tx = x\}$. ($\mathrm{Fix}\,(T)$ is the fixed point set of T). For all $n \in \mathbb{N}$, we set $T^{n+1} := T \circ T^n$, $T^0 = I_X$ (the identity map of X) and $T^1 := T$. The orbit of T at $x \in X$ is defined as $O_T(x) = \{x, Tx, T^2x, T^3x, \dots\}$.

We extend some notions from [29] due to I. A. Rus to the context of b-metric spaces.

Definition 2 *Let $(X, d; s)$ be a b-metric space with parameter $s \geq 1$ and let $T : X \to X$ be a selfmapping.*

(a) We say that T is weakly Picard operator (WPO) if the sequence $(T^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of T.

(b) (b) If the operator T is WPO and $\mathrm{Fix}\,(T) = \{x\}$ (for some $x \in X$), then T is said to be a Picard operator (PO).

(c) We say that T is Cauchy operator (CO) if the sequence $(T^n(x))_{n\in\mathbb{N}}$ is Cauchy, for all $x \in X$.

Some concepts used in Definition 2 above are precised below.

Definition 3 *Let $(X, d; s)$ be a b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then, the following are defined as follows:*

(i) The sequence $\{x_n\}$ is said to be a Cauchy sequence, if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that: for all $n \geq N$ and all $p \in \mathbb{N}$, we have $d(x_n, x_{n+p}) < \varepsilon$.

(ii) *The sequence* $\{x_n\}$ *is said to be convergent to* x*, if for any* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that: for all* $n \geq N$*, we have* $d(x_n, x) < \varepsilon$*. In this case, we write*

$$\lim_{n\to\infty} x_n = x \quad or \quad x_n \to x \quad as \quad n \to \infty.$$

(iii) $(X, d; s)$ *is said to be a complete* b*-metric space if every Cauchy sequence in* X *converges to some* $x \in X$*.*

(iv) $(X, d; s)$ *is said to be (sequentially) compact if from each sequence* $\{x_n\}$ *of* X *one can extract a subsequence* $\{x_{\varphi(n)}\}$ *which is convergent in* X*.*

Notice that we observe that every converging sequence in a b-metric space has a unique limit; and every converging sequence in a b-metric space is a Cauchy sequence. In general, The converse is not true.

We extend to b-metric spaces the definition of orbitally continuous maps wich was introduced by Ćirić in [9] for metric spaces.

Definition 4 *Let* $(X, d; s)$ *be a* b*-metric space. A selfmapping* T *of* X *is said to be* T*-orbitally continuous on* X*, if for all* $x, u \in X$ *and every strictly increasing sequence* $(n_k)_{k\in\mathbb{N}}$ *of positive integers such that* $\lim_{k\to\infty} T^{n_k}x = u$*, then we have* $\lim_{k\to\infty} T(T^{n_k}x) = Tu$*.*

In connection with the definition above, we have the following lemma.

Lemma 1 *Let* $(X, d; s)$ *be a* b*-metric space with parameter* $s \geq 1$ *and let* T *be a selfmapping of* X*. If* T *is a weakly Picard operator on* X*, then* T *is orbitally continuous on* X*.*

Proof. Let $x, u \in X$ and let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $\lim_{k\to\infty} T^{n_k}x = u$. Since T is supposed to be a weakly Picard operator, there exists a fixed point (say) $z \in X$ such that $z = \lim_{n\to+\infty} T^n x$. Since $(T^{n_k}x)_k$ is a subsequence of $(T^n x)_n$, we have $z = \lim_{k\to+\infty} T^{n_k}x$. By the uniqueness of the limit, we have $u = z$. Moreover, the sequence $(T(T^{n_k}x))_k = (T^{n_k+1}x)_k$ is also a subsequence of $(T^n x)_n$, and we have

$$\lim_{k\to+\infty} T(T^{n_k}x) = z = Tz = Tu.$$

Thus we have proved that T is orbitally continuous on X. This ends the proof. ∎

In 1966, the concept of asymptotically regular mappings in metric spaces was introduced by Browder and Petryshyn (see [7]). Next we extend this concept to the b-metric spaces.

Definition 5 *Let* $(X, d; s)$ *be a* b*-metric space with parameter* $s \geq 1$ *and let* T *be a selfmapping of* X*. Then,* T *is said to be asymptotically regular at a point* x *in* X*, if* $\lim_{n\to\infty} d(T^n x, T^n Tx) = 0$*.* T *is said to be asymptotically regular on* X *if it is asymptotically regular at any point* x *in* X*.*

We end this section by recalling the following important lemma.

Lemma 2 *Let* $(X, d; s)$ *be a* b*-metric space with parameter* $s \geq 1$*. Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of elements from* X *having the property that there exists* $\gamma \in [0, 1)$ *such that*

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}), \quad \forall n \in \mathbb{N}.$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ *is Cauchy.*

Lemma 2 was established in 2017 by R. Miculescu and A. Mihail in their paper [24]. In the same year, it was also proved and by T. Suzuki in his paper [31] by a different method.

In [14], T. M. Došenović, M. V. Pavlović and S. N. Radenović made a nice discussion concerning Lemma 2 and showed that various known fixed point results in b-metric spaces can be shortened by using this lemma.

3 Results in Complete b-Metric Spaces Without Continuity

3.1 Study of the Case Where $s+\beta>2$

Next, we state our first main result in a complete b-metric space for the case where $s+\beta>2$ with $\beta>0$.

Theorem 4 *Let $(X,d;s)$ be a complete b-metric space with parameter $s\geq 1$. Let $\beta>0$ be such that $s+\beta>2$. Let $T:X\to X$ be a mapping satisfying*

$$d(Tx,Ty)\leq \frac{1}{s+\beta}\{d(x,Tx)+d(y,Ty)\}, \quad \textit{for all } x,y\in X. \qquad (K(\tfrac{1}{s+\beta}))$$

Then T is a Picard operator.

Proof. Let $x_0\in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n=T^nx_0$ for each $n\in\mathbb{N}$. We set $\tau_n=d(x_n,x_{n+1})$ for each $n\in\mathbb{N}\cup\{0\}$. Then we have

$$\begin{aligned}
\tau_n &= d(T^nx_0,T^{n+1}x_0)\\
&= d(T(T^{n-1}x_0),T(T^nx_0))\\
&\leq \frac{1}{s+\beta}\{d(T^{n-1}x_0,T^nx_0)+d(T^nx_0,T^{n+1}x_0)\}\\
&= \frac{1}{s+\beta}(\tau_{n-1}+\tau_n).
\end{aligned}$$

Then

$$\tau_n\leq \frac{1}{s+\beta-1}\tau_{n-1}:=q\tau_{n-1}.$$

We have $s+\beta-1>2-1=1$, therefore, we have $0<q<1$. By Lemma 2, (x_n) is a Cauchy sequence. By the completeness of X, the sequence (x_n) converges to some (unique) point (say) $z=z_{x_0}$ in X, which may depend on x_0.

Let us show that this point z is a fixed point of T. Indeed, we have

$$\begin{aligned}
d(z,Tz) &\leq sd(z,T^{n+1}x_0)+sd(T^{n+1}x_0,Tz)\\
&\leq sd(z,T^{n+1}x_0)+\frac{s}{s+\beta}\{d(T^nx_0,T^{n+1}x_0)+d(z,Tz)\}.
\end{aligned}$$

Then

$$\frac{\beta}{s+\beta}d(z,Tz)\leq \Big(sd(z,T^{n+1}x_0)+\frac{s}{s+\beta}d(T^nx_0,T^{n+1}x_0)\Big)\longrightarrow 0 \text{ as n}\to\infty.$$

Since $\beta>0$, this implies that $z=Tz$, i.e., z is a fixed point of T.

Next, we prove the uniqueness of z. We argue by contradiction, let z^* be another (different) fixed point of T, then

$$d(z,z^*)=d(Tz,Tz^*)\leq \frac{1}{s+\beta}\{d(z,Tz)+d(z^*,Tz^*)\}.$$

Then

$$d(z,z^*)\leq 0,$$

which leads to a contradiction. Hence, our assumption was wrong. Therefore, z must be the unique fixed point T.

Let $y_0\in X$ be arbitrary but fixed and consider the iterated sequence (y_n) where $y_n=T^ny_0$ for each $n\in\mathbb{N}$. We denote $\sigma_n=d(y_n,y_{n+1})$ for each $n\in\mathbb{N}\cup\{0\}$. By assumption we know that $\lim_{n\to\infty}\tau_n=0=\lim_{n\to\infty}\sigma_n$.

Then for all integer $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(z_{x_0}, z_{y_0}) &\leq sd(z_{x_0}, x_n) + s^2 d(x_n, y_n) + s^2 d(y_n, z_{y_0}) \\
&= sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + d(T^n x_0, T^n y_0) \\
&= sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + s^2 d(T(T^{n-1}x_0), T(T^{n-1}y_0) \\
&\leq sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + \frac{s^2}{s+\beta}\{d(T^{n-1}x_0, T^n x_0) + d(T^{n-1}y_0, T^n y_0)\} \\
&= \Big[sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + \frac{s^2}{s+\beta}(\tau_{n-1} + \sigma_{n-1})\Big] \longrightarrow 0 \text{ as n} \to \infty.
\end{aligned}$$

Then

$$d(z_{x_0}, z_{y_0}) = 0.$$

This says that the fixed point does not depend on the initial value x_0 for any arbitrary point, so for every $x \in X$, the iterated sequence $(T^n x)$ converges to the unique fixed point of T, i.e., T is a Picard operator. This ends the proof. ■

3.2 Study of the Case Where $s + \beta = 2$.

This subsection deals with the case where $s + \beta = 2$ with $\beta = 2 - s > 0$ in a complete b-metric space $(X, d; s)$(with $1 \leq s < 2$). So in this case, we are concerned with selfmaps T of X satisfying the following contractive condition:

$$d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \quad \text{for all } x, y \in X. \qquad (K(\tfrac{1}{2}))$$

Before giving our second main result, we provide an example of a continuous selfmap T satisfying $(K(\frac{1}{2}))$ in a complete b-metric space (with parameter $s \geq 2$) but T has no fixed point. This will explain why we need to take $\beta = 2 - s > 0$.

Example 1 *Let us choose $X = \mathbb{N}$ and we define $d : X \times X \to \mathbb{R}$ by*

$$d(x, y) = \begin{cases} 1 + (\frac{1}{x} - \frac{1}{y})^2, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

Then it is eaasy to see that d is a b-metric on X with parameter $s = 2$. Observe that every Cauchy sequence in X is eventually constant and hence $(X, d; 2)$ is a complete b-metric space.

We define a function $T : X \to X$ by setting

$$Tx = 3x,$$

for all $x \in X$. It is easy to verify that T is continuous and a fixed point free map.

Now, we show that T satisfies the condition $(K(\frac{1}{2}))$. Indeed, for all $x, y \in X$ with $x \neq y$, we have

$$\begin{aligned}
d(Tx, Ty) &= 1 + (\frac{1}{3x} - \frac{1}{3y})^2 \\
&= 1 + \frac{1}{9x^2} + \frac{1}{9y^2} - \frac{2}{9xy} \leq 1 + \frac{1}{9x^2} + \frac{1}{9y^2};
\end{aligned}$$

whereas,

$$\begin{aligned}
\frac{1}{2}\{d(x, Tx) + d(y, Ty)\} &= \frac{1}{2}\big\{1 + (\frac{1}{x} - \frac{1}{3x})^2 + 1 + (\frac{1}{y} - \frac{1}{3y})^2\big\} \\
&= 1 + \frac{2}{9x^2} + \frac{2}{9y^2} \geq 1 + \frac{1}{9x^2} + \frac{1}{9y^2}.
\end{aligned}$$

Therefore, we deduce that

$$d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\},$$

for all x, y in X. Thus, T is a continuous selfmap of the complete b-metric space X but T is fixed point free. Therefore, T is not a Picard operator.

The example above shows that for a selfmap T satisfying $(K(\frac{1}{2}))$ in a complete b-metric space $(X, d; s)$ some supplementary conditions are required to ensure the Picard property for T. In fact, we investigate the mildest supplementary conditions upon a selfmap $T \in \mathcal{K}(\frac{1}{2})$ to be a Picard operator in a complete b-metric space.

According to the example above, one needs also to suppose that $s \in [1, 2)$. Next we provide our second main result.

Theorem 5 *Let $(X, d; s)$ be a complete b-metric space with parameter such that $1 \leq s < 2$. Let $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \quad \text{for all } x, y \in X. \tag{$K(\frac{1}{2})$}$$

Then the following assertions are equivalent:

(1) T is a Picard operator.

(2) T is a Cauchy operator.

(3) T is asymptotically regular on X.

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are clear.

Let us prove that (3) $\Longrightarrow$ (1). To this end, let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. As before, we set $\tau_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By assumption, we know that $\lim_{n\to\infty} \tau_n = 0$.

Now, we show that the sequence $(x_n = T^n(x_0))$ is a Cauchy sequence. To this end, we observe that for all integers $n, m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\ &= d(T(T^{n-1} x_0), T(T^{m-1} x_0) \\ &\leq \frac{1}{2}\{d(T^{n-1} x_0, T^n x_0) + d(T^{m-1} x_0, T^m x_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \tau_{m-1}). \end{aligned}$$

Then

$$\lim_{n,\, m\to\infty} d(x_n, x_m) = 0.$$

This says that (x_n) is a Cauchy sequence. Since $(X, d; s)$ is complete, this sequence converges to some (unique) point (say) $z = z_{x_0}$ in X, which may depend on x_0. Let us show that this point z is a fixed point of T. Indeed, we have

$$\begin{aligned} d(z, Tz) &\leq sd(z, T^{n+1} x_0) + sd(T^{n+1} x_0, Tz) \\ &\leq sd(z, T^{n+1} x_0) + \frac{s}{2}\{d(T^n x_0, T^{n+1} x_0) + d(z, Tz)\}. \end{aligned}$$

Then

$$(1 - \frac{s}{2})d(z, Tz) \leq \Big(sd(z, T^{n+1} x_0) + \frac{s}{2}d(T^n x_0, T^{n+1} x_0)\Big) \longrightarrow 0 \text{ as n} \to \infty.$$

Since $2 > s$, this implies that $z = Tz$, i.e., z is a fixed point of T.

Next, we prove the uniqueness of z. We argue by contradiction, let z^* be another (different) fixed point of T, then

$$d(z, z^*) = d(Tz, Tz^*) \leq \frac{1}{2}\{d(z, Tz) + d(z^*, Tz^*)\}.$$

Then

$$d(z, z^*) \leq 0,$$

which leads to a contradiction. Hence, our assumption was wrong. Therefore, z must be the unique fixed point T.

Let $y_0 \in X$ be arbitrary but fixed and consider the iterated sequence (y_n) where $y_n = T^n y_0$ for each $n \in \mathbb{N}$. We denote $\sigma_n = d(y_n, y_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By assumption we know that $\lim_{n\to\infty} \tau_n = 0 = \lim_{n\to\infty} \sigma_n$. Then for all integer $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(z_{x_0}, z_{y_0}) &\leq sd(z_{x_0}, x_n) + s^2 d(x_n, y_n) + s^2 d(y_n, z_{y_0}) \\
&= sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + d(T^n x_0, T^n y_0) \\
&= sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + s^2 d(T(T^{n-1} x_0), T(T^{n-1} y_0)) \\
&\leq sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + \frac{s^2}{2}\{d(T^{n-1} x_0, T^n x_0) + d(T^{n-1} y_0, T^n y_0)\} \\
&= \Big[sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + \frac{s^2}{2}(\tau_{n-1} + \sigma_{n-1})\Big] \longrightarrow 0 \text{ as n} \to \infty.
\end{aligned}$$

Then

$$d(z_{x_0}, z_{y_0}) = 0.$$

This says that the fixed point does not depend on the initial value x_0 for any arbitrary point, so for every $x \in X$, the iterated sequence $(T^n x)$ converges to the unique fixed point of T, i.e., T is a Picard operator. This ends the proof. ∎

4 Results in Complete b-Metric Spaces for Orbitally Continuous Maps

It is worthy to notice that the in all the results of Section three, we did not required continuity for the selmappings. The aim of this section is to establish a general case for continuous selfmappings of complete b-metric spaces. Our result in this direction reads as follows.

Theorem 6 *Let $(X, d; s)$ be a complete b-metric space with parameter $sd \geq 1$. Let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \quad \text{for all } x, y \in X. \tag{$K(\frac{1}{2})$}$$

We suppose that T is orbitally continuous. Then the following assertions are equivalent:

(1) T is a Picard operator.

(2) T is a Cauchy operator.

(3) T is asymptotically regular on X.

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are evident.

We need to prove the implication (3) $\Longrightarrow$ (1). To this end, let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. As before, we set $\tau_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By assumption, we know that $\lim_{n\to\infty} \tau_n = 0$.

Now, we show that the sequence $(x_n = T^n(x_0))$ is a Cauchy sequence. To this end, we observe that for all integers $n, m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\ &= d(T(T^{n-1}x_0), T(T^{m-1}x_0) \\ &\leq \frac{1}{2}\{d(T^{n-1}x_0, T^n x_0) + d(T^{m-1}x_0, T^m x_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \tau_{m-1}). \end{aligned}$$

Then

$$\lim_{n,\, m \to \infty} d(x_n, x_m) = 0.$$

This says that (x_n) is a Cauchy sequence. Since $(X, d; s)$ is complete, this sequence converges to some (unique) point (say) $z = z_{x_0}$ in X, which may depend on x_0. Let us show that this point z is a fixed point of T. Indeed, since T is orbitally continuous on X, we deduce that the sequence $(x_{n+1} = T(x_n))_n$ converges to Tz. But $(x_{n+1})_n$ converges also to z. By uniqueness of the limit, we conclude that $Tz = z$. Thus z is a fixed point of T.

To prove uniqueness of this fixed point. We argue by contradiction, let z^* be another (different) fixed point of T, then

$$d(z, z^*) = d(Tz, Tz^*) \leq \frac{1}{2}\{d(z, Tz) + d(z^*, Tz^*)\}.$$

Then

$$d(z, z^*) \leq 0,$$

which leads to a contradiction. Hence, our assumption was wrong. Therefore, z must be the unique fixed point T. Let $y_0 \in X$ be arbitrary but fixed and consider the iterated sequence (y_n) where $y_n = T^n y_0$ for each $n \in \mathbb{N}$. We denote $\sigma_n = d(y_n, y_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By assumption we know that $\lim_{n\to\infty} \tau_n = 0 = \lim_{n\to\infty} \sigma_n$. Then for all integer $n \in \mathbb{N}$, we have

$$\begin{aligned} d(z_{x_0}, z_{y_0}) &\leq sd(z_{x_0}, x_n) + s^2 d(x_n, y_n) + s^2 d(y_n, z_{y_0}) \\ &= sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + d(T^n x_0, T^n y_0) \\ &= sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + s^2 d(T(T^{n-1}x_0), T(T^{n-1}y_0) \\ &\leq sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + \frac{s^2}{2}\{d(T^{n-1}x_0, T^n x_0) + d(T^{n-1}y_0, T^n y_0)\} \\ &= \Big[sd(z_{x_0}, x_n) + s^2 d(y_n, z_{y_0}) + \frac{s^2}{2}(\tau_{n-1} + \sigma_{n-1})\Big] \longrightarrow 0 \text{ as n} \to \infty. \end{aligned}$$

Then

$$d(z_{x_0}, z_{y_0}) = 0.$$

This says that the fixed point does not depend on the initial value x_0 for any arbitrary point, so for every $x \in X$, the iterated sequence $(T^n x)$ converges to the unique fixed point of T, i.e., T is a Picard operator. This ends the proof. ■

We end this section by proving a result using a new condition.

Theorem 7 *Let $(X, d; s)$ be a complete b-metric space with parameter $s \geq 1$ and $T : X \to X$ a selfmap satisfying the following conditions:*

$(K(\frac{1}{2}))$: $d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$ *for all $x, y \in X$,*

(S) *For any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d(T^i x, T^j x) < \epsilon + \delta \textit{ implies } d(T^{i+1}x, T^{j+1}x) < \epsilon \textit{ for any } i, j \in \mathbb{N} \cup \{0\}.$$

- *Then T is a Picard operator.*

Proof. We start by observing that the property (S) implies that T is orbitally continuous on X. Thus according to Theorem 6, it is sufficient to show that T is asymptotically regular on X.

So, let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. As before, we denote $\tau_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. We know, by using the condition $(K(\frac{1}{2}))$, that the sequence (τ_n) is nonincreasing. Since (τ_n) is bounded below, it converges to a nonnegative number (say) τ. We show that $\tau = 0$. To get a contradiction, we suppose that $\tau > 0$. Then there exists an integer n_δ such that $d(x_n, x_{n+1}) = \tau_n < \tau + \delta$ for all integer $n \geq n_\delta$.

Let n be any integer satisfying $n \geq n_\delta$. By the the condition (E), we infer that $d(Tx_n, Tx_{n+1}) = \tau_{n+1} < \tau$. This implies that $0 < \tau \leq \tau_{n+1} < \tau$, which is absurd. Thus we have showed that $\lim_{n\to\infty} \tau_n = 0$. This ends the proof. ■

We point out that Theorem 7 is the analog of Theorem 2.7 of [17]. We point out that the condition (S) is a variant of the condition named (D) in the paper [32] of T. Suzuki. It seems that our condition is stronger than the condition (D) introduced in [32] and considered in Theorem 2.7 of the paper [17].

One can say that the first papers dealing with fixed point theory in b-metric spaces are the articles; [4], [6], [12] and [13]. Nowadays, the theory of fixed point in b-metric spaces is well developed. For other aspects of this theory, we invite the reader to consult the papers: [1], [2], [26], [28], [24], [25], [3], [27], [11] and others.

5 Results in Compact b-Metric Spaces

Now, we state a result in the case where the b-metric space $(Xd; s)$ is compact. In this result, we do not need T to be orbitally continuous.

Theorem 8 *Let $(X, d; s)$ be a compact b-metric space with a parameter $s \geq 1$. Let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \quad \text{for all } x, y \in X. \tag{$K(\frac{1}{2})$}$$

Then the following assertions are equivalent:

(A) T is a Picard operator.

(B) For all $x, u \in X$ and every strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $\lim_{k\to\infty} T^{n_k}x = u$, then we have $\lim_{k\to\infty} T(T^{n_k}x) = u$.

(C) T is asymptotically regular on X.

(D) T is a Cauchy operator.

Proof. (A) $\Longrightarrow$ (B) Since T is a Picard operator, there exists a unique fixed point $z \in X$ such that $z = \lim_{n\to+\infty} T^n x$ for all $x \in X$. Let $u \in X$ and let $(T^{n_k}x)_k$ be a subsequence of $(T^n x)_n$ such that $u = \lim_{k\to+\infty} T^{n_k}x$. Then we infer that $u = z$. Moreover, the sequence $(T(T^{n_k}x))_k = (T^{n_k+1}x)_k$ is a subsequence of $(T^n x)_n$, therefore we have

$$\lim_{k\to+\infty} T(T^{n_k}x) = z = u.$$

Thus we have proved that (B) holds true.

(B) $\Longrightarrow$ (C) Let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. We set $\tau_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Then we have

$$\begin{aligned} \tau_n &= d(T^n x_0, T^{n+1}x_0) \\ &= d(T(T^{n-1}x_0), T(T^n x_0)) \\ &\leq \frac{1}{2}\{d(T^{n-1}x_0, T^n x_0) + d(T^n x_0, T^{n+1}x_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \tau_n). \end{aligned}$$

Then

$$\tau_n \leq \tau_{n-1}.$$

This shows that (τ_n) is a non-increasing sequence of nonegative real numbers, so it must be a convergent sequence. Let us denote $\tau := \lim_{n\to+\infty} \tau_n$. We must show that $\tau = 0$.

Since X is compact, the sequence (x_n) has a convergent subsequence, say (x_{n_k}) which converges to some $z \in X$. By to the condition (B), we infer that $\lim_{k\to+\infty} T(T^{n_k}x) = z$. Accorging to the s-triangle inequality, we have

$$d(x_{n_k}, x_{n_k+1}) \leq sd(x_{n_k}, z) + sd(z, x_{n_k+1}).$$

Then

$$\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = 0.$$

Hence, the subsequence $(\tau_{n_k})_k$ converges to zero. Since the whole sequence (τ_n) converges to τ, we must have $\tau = 0$. So we have proved that T is asymptotically regular on X.

(C) $\Longrightarrow$ (D) Let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. As before, we denote $\tau_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By the assumption (C), we know that $\lim_{n\to\infty} \tau_n = 0$.

Now, we show that the sequence $(x_n = T^n(x_0))$ is a Cauchy sequence. To this end, we observe that for all integers $n, m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\ &= d(T(T^{n-1}x_0), T(T^{m-1}x_0)) \\ &\leq \frac{1}{2}\{d(T^{n-1}x_0, T^n x_0) + d(T^{m-1}x_0, T^m x_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \tau_{m-1}). \end{aligned}$$

Then

$$\lim_{n,\, m\to\infty} d(x_n, x_m) = 0.$$

This says that (x_n) is a Cauchy sequence. hence, we have proved that (C) implies (D).

(D) $\Longrightarrow$ (A) Suppose that T is a Cauchy operator and let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. By assumption, this sequence is a Cauchy sequence. Since X is compact, the sequence (x_n) has a convergent subsequence, say (x_{n_k}) which converges to some $z \in X$. This implies that the whole sequence (x_n) converges to the point $z = z_{x_0}$ which may depend on x_0. Also, we have

$$\begin{aligned} d(z, Tz) &\leq d(z, T^{n+1}x_0) + d(T^{n+1}x_0, Tz) \\ &< d(z, T^{n+1}x_0) + \frac{1}{2}\{d(T^n x_0, T^{n+1}x_0) + d(z, Tz)\}. \end{aligned}$$

Then

$$\frac{1}{2}d(z, Tz) < d(z, T^{n+1}x_0) + \frac{1}{2}d(T^n x_0, T^{n+1}x_0) \to 0 \text{ as n} \to \infty.$$

This implies that $z = Tz$, i.e., z is a fixed point of T.

Next, we prove the uniqueness of z. We argue by contradiction, let z^* be another (different) fixed point of T, then

$$d(z, z^*) = d(Tz, Tz^*) < \frac{1}{2}\{d(z, Tz) + d(z^*, Tz^*)\}.$$

Then

$$d(z, z^*) < 0,$$

which leads to a contradiction. Hence, our assumption was wrong. Therefore, z must be the unique fixed point T.

Let $y_0 \in X$ be arbitrary but fixed and consider the iterated sequence (y_n) where $y_n = T^n y_0$ for each $n \in \mathbb{N}$. We denote $\sigma_n = d(y_n, y_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By assumption we know that $\lim_{n\to\infty} \tau_n = 0 = \lim_{n\to\infty} \sigma_n$. Then for all integer $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, y_n) &= d(T^n x_0, T^n y_0) \\ &= d(T(T^{n-1}x_0), T(T^{n-1}y_0) \\ &\leq \frac{1}{2}\{d(T^{n-1}x_0, T^n x_0) + d(T^{n-1}y_0, T^n y_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \sigma_{n-1}). \end{aligned}$$

Then

$$d(z_{x_0}, z_{y_0}) = \lim_{n\to\infty} d(x_n, y_m) = 0.$$

This says that the fixed point does not depend on the initial value x_0 for any arbitrary point, so for every $x \in X$, the iterated sequence $(T^n x)$ converges to the unique fixed point of T, i.e., T is a Picard operator. This ends the proof. ∎

To support our Theorem 8, we provide the following example.

Example 2 *Let $X = [0, 2]$ endowed with the b-metric d given by $d(x, y) := (x - y)^2$. Then X is a compact b-metric space with parameter $s = 2$. We consider the selfmap $T : X \to X$ given by $T(x) = \frac{x}{\sqrt{6}}$ for all $x \in X$. For all x, y in X, we have*

$$\begin{aligned} d(Tx, Ty) &= (\frac{x}{\sqrt{6}} - \frac{y}{\sqrt{6}})^2 \\ &= \frac{1}{6}(x^2 + y^2 - 2xy) \leq \frac{1}{6}(x^2 + y^2); \end{aligned}$$

whereas,

$$\begin{aligned} \frac{1}{2}\{d(x, Tx) + d(y, Ty)\} &= \frac{1}{2}\{(x - \frac{x}{\sqrt{6}})^2 + (y - \frac{y}{\sqrt{6}})^2\} \\ &= \frac{(\sqrt{6}-1)^2}{2}\{\frac{x^2}{6} + \frac{x^2}{6}\} \\ &\geq \frac{x^2}{6} + \frac{x^2}{6}, \end{aligned}$$

because $2 \leq (\sqrt{6} - 1)^2$. Therefore, we deduce that

$$d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \quad \forall x, y \in X.$$

Thus, T satisfies the condition $(K(\frac{1}{2}))$ on the compact b-metric space X. The selfmap T is asymptotically regular on X. Indeed, for each $x_0 \in X$, we have $T^n(x_0) = (\frac{x_0}{\sqrt{6}})^n$ which converges to zero in X. Hence, the condition (C) of Theorem 8 is satisfied, therefore T is a Picard operator with $\mathrm{Fix}\,(T) = \{0\}$.

6 Concluding Remarks

Let $(X, d; s)$ be a b-metric space with parameter $s \geq 1$. Let T be a selfmap of X. We consider the following contractive conditions:

(Ed): $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$ with $x \neq y$.

(K-S): $d(Tx,Ty) < \frac{1}{2}\{d(x,Tx)+d(y,Ty)\}$, for all $x,y \in X$ with $x \neq y$.

$(K(\frac{1}{2}))$**:** $d(Tx,Ty) \leq \frac{1}{2}\{d(x,Tx)+d(y,Ty)\}$, for all $x,y \in X$.

$K(\frac{1}{s+\beta})$**:** $d(Tx,Ty) \leq \frac{1}{s+\beta}\{d(x,Tx)+d(y,Ty)\}$, for all $x,y \in X$, where $\beta > 0$.

It is well known that each one of all first three conditions is not strong enough to ensure fixed points for T. The purpose of this paper is to investigate the necessary and sufficient conditions for selfmaps T satisfying the condition $K(\frac{1}{s+\beta})$, with $\beta > 0$ under which T becomes a Picard operator. The main results of this paper are described as follows.

In Theorem 4, we have proved that if X is complete and if T satisfies the condition $K(\frac{1}{s+\beta})$ with $s+\beta > 2$, then T is a Picard operator.

If $s+\beta = 2$, then the condition $K(\frac{1}{s+\beta})$ coincides with the condition $K(\frac{1}{2})$. Let T satisfy this condition then we show by Example 1 that, in the case where $s \geq 2$, the Picard property is not automatically ensured even for continuous selfmaps. However, when $1 \leq s < 2$, we have proved in Theorem 5 that if T satisfies the condition $K(\frac{1}{2})$, then the following conditions are equivalent:

(A) T is a Picard operator.

(C) T is asymptotically regular on X.

(D) T is a Cauchy operator (see Definition 2).

In Theorem 6, we have established the same results for any orbitally continuous selfmap T of a complete b-metric space $(X,d;s)$ with any arbitrary $s \geq 1$.

We have introduced the following new condition

(S) For any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(T^i x, T^j x) < \epsilon + \delta \text{ implies } d(T^{i+1}x, T^{j+1}x) < \epsilon \text{ for any } i,j \in \mathbb{N} \cup \{0\}.$$

In Theorem 7, we have shown that if $(X,d;s)$ is a complete b-metric space with parameter $s \geq 1$ and if the selfmap T satisfies the condition (S), then T is a Picard operator. Thus (S) is sufficient condition. We do not know if it is also necessary.

Suppose that $(X,d;s)$ is a compact b-metric space with parameter $s \geq 1$ and that the selfmap T satisfies the condition $K(\frac{1}{2})$. Then in Theorem 8, we have established that the conditions (A), (C) and (D) are still equivalent and that are equivalent with the following condition:

(B) For all $x,u \in X$ and every strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $\lim_{k\to\infty} T^{n_k}x = u$, then we have $\lim_{k\to\infty} T(T^{n_k}x) = u$.

We have furnished Example 2 to support Theorem 8.

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