

Irrigation And Nutrition On The Dynamics Of Huanglongbing*

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Abstract

Huanglongbing disease is a severe disease of the citrus trees and the most destructive disease worldwide, which has proved difficult to control. We study in the present work by formulating a mathematical model for huanglongbing (HLB) transmission between citrus trees and Asian Citrus Psyllid (ACP). The model incorporates a fraction of susceptible citrus trees that undergo irrigation and nutrition with a view to preventing transmission of HLB between citrus trees and ACP population, as irrigation and nutrition have been shown to boost trees' immunity. The basic reproduction number, R_0 , global stability of disease-free and endemic equilibrium points are studied, for understanding and prediction of HLB transmission. We proceed further to carry out sensitivity analysis in order to determine the parameters that R_0 is most sensitive to, positively and negatively. The results indicate that the parameters α (ACP biting rate) and ρ (fraction of susceptible ACP who are sensitive to insecticides) are the most dominant sensitivity indices towards the basic reproduction number. This suggests that constant practice of irrigation and nutrition can effectively lower R_0 . Numerical simulation is performed to confirm the analytical results and to investigate various control strategies. Our result suggests that adoption of constant practice of irrigation and nutrition will lead to reduction in the transmission of HLB in citrus tree population. However the intervention strategy needs to be implemented together with other intervention strategies. The results obtained from this study can help the citrus trees industries and the biologists to adopt better understanding of the modeling strategies to control HLB.

1 Introduction

Huanglongbing (HLB) disease is a disease of citrus trees caused by Candidature Liberibacter species carried by Asian Citrus Psyllid (ACP). It is one of the most devastating diseases in Asian, African and American countries. It is the most severe citrus disease that is currently devastating citrus industries worldwide. Psychologically, HLB acts by disrupting the phloem, the tissue in the inner part of the branches and stem of the tree, resulting in the accumulation of high levels of starch, as well as reducing its ability to uptake nutrients [24]. Practically, all commercial citrus species and cultivars are vulnerable to HLB. The disease has an array of symptoms which can be detected anywhere on the plant, from the roots to the leaves, changing the chemical characteristics sensory attributes of the fruit. Some of HLB symptoms include olive green of leaves, green veins, vein yellowing, mottling, stunted and dieback of twigs [5]. HLB can impact citrus production in many ways. HLB increases the mortality rate of trees. It negatively affects marketable yields per tree. HLB also increases citrus production costs and compromises the influx of nutrients to the fruit as it produces severe adverse effects on it [23].

Citrus trees under any type of stress are less able to resist ACP infestation. According to [22], if stress can be reduced by ensuring irrigation and nutrition, the trees may show less severe disease symptoms, including milder effects on fruit production and yield. While some growers have reported milder disease symptoms, increased fruit yields and improved fruit quality with fertigation, have been reported. A study developed by [3] stated that citrus grooves have not been intensively managed with respect to nutrition and water, such that trees are continually exposed to some level of stress. Attacks of insects and pathogens, either direct or opportunistic, contribute to additional stress that affects fruit yields and quality.

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Mathematical models of infectious diseases have been studied extensively by researchers. See for instance [27, 28, 29, 30, 31, 25, 34, 6, 10, 1, 11, 12, 14, 15, 16, 20, 4, 9, 37, 18, 36, 32, 8, 13] and the references cited therein. Also, mathematical models for plant epidemics have been developed by researchers. [33] developed a mathematical model of huanglongbing with cross protection. A threshold value, R_0 , is established and the existence of a backward bifurcation was presented. Their results suggest that cross protection is a promotion control measure and replanting trees is bad for HLB control. In another development, [17] studied an HLB model by taking into account seasonal fluctuations. Further, they considered switching coefficients and switching control schemes in their model. They specifically studied the effects of switching control schemes on the dynamics of the model. They analysed the global dynamics of the impulsive switching model for huanglongbing. They finally performed numerical simulations which indicate that spring and autumn are the optimum seasons for killing psyllids and winter is the optimum season for removing infected trees. [35] proposed a deterministic dynamical model to explore the transmission dynamics of HLB between citrus tree and Asian citrus psyllid (ACP). They analysed the model using theory of dynamical system and the analytical results show that the disease-free equilibrium is globally asymptotically stable when the basic reproduction number, $R_0 < 1$ and when $R_0 > 1$, the system is uniformly persistent. They further carried out global sensitivity analysis to know the parameters that have great impact on HLB transmission dynamics and the results of the global sensitivity analysis reveal that R_0 is very sensitive to conversion rate, removal rate of symptomatic citrus trees and infectious enhancement coefficient, but not sensitive to the maximum number of citrus trees that can be planted in the grove and the recruitment rate of ACP. They applied optimal control theory to the model which indicates that spraying of insecticides is more effective than other control strategies in reducing the number of ACP in the early phase of transmission of HLB. [26] developed a mathematical model of HLB to investigate the transmission dynamics of the disease between ACP and citrus trees incorporating sensitive and resistant strains of ACP. They derived an expression for the basic reproduction number R_0 of HLB. Their findings reveal that the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$ and if $R_0 > 1$, the system is uniformly persistent. They applied global sensitivity analysis of R_0 , in order to obtain some parameters that have the greatest influence on HLB transmission dynamics. They proceeded further to carry out optimal control analysis accompanied with numerical simulations. The results from sensitivity analysis show that R_0 is very sensitive to environmental carrying capacity of citrus trees k , ACP biting rate b_v , vaccination rate ν_h , transmission probability from ACP to citrus trees α_h and β_h . The optimal control results indicate that the weights in the objective function have little impact on the optimal control strategy. They finally concluded that the intensity of insecticide resistant present in ACP population may slightly reduce the effectiveness of control measures.

Our work differs from the existing work on HLB in the following sense: In [33], the authors developed a mathematical model of HLB incorporating cross protection. They investigated the effect of cross protection in controlling the spread of HLB without considering the impact of irrigation and nutrition on the disease dynamics. Cross protection is a significant protection against a disease due to an immune response elicited against a related organism but irrigation and nutrition is a natural way of boosting citrus trees' immunity against HLB. Irrigation and nutrition have proved to be more efficient than cross protection because it boosts trees' immunity naturally but cross protection (vaccination) may be imperfect and may not even work for citrus trees with compromised immunity. Also in [33], the resistance of ACP (Asian Citrus Psyllid) which may sometimes be limited resources availability, was not considered. This was considered in our work. In [26], they developed HLB model to consider the optimality of intervention strategies i.e., removal of infectious citrus trees and spraying of insecticides, but they did not take into account some important prevention measures (irrigation and nutrition). Irrigation and nutrition have better advantages than removal of symptomatic citrus trees and insecticides spraying because it will not allow HLB to invade citrus trees population in the first place. Our sensitivity analysis results confirm this. That is, R_0 is most sensitive to the fraction of susceptible citrus trees that undergo irrigation and nutrition, in a negative sense. This implies that irrigation and nutrition factors greatly reduce R_0 below one, which indicates that HLB transmission will die out of the citrus tree population if irrigation and nutrition are effectively practised. In summary, the studies mentioned above have considered several factors that contribute to the spread, prevention and containment of HLB in tree population without considering the effect of irrigation and nutrition on the disease dynamics.

Irrigation is the artificial process of applying controlled amounts of water to land to assist in production of crops while nutrition is the process of supplying Nitrogen (N), Phosphorus (P) and Potassium (K) to plants so as to enable them to germinate, grow and fight off diseases and pests. Irrigation and nutrition have been proved to be effective when preventing citrus trees from being attacked and damaged by ACP, as it reduces stress on citrus trees. This is because citrus trees under any type of stress are less able to resist ACP infestation [22]. Motivated by the above, this study seeks to investigate the impact of irrigation and nutrition on the dynamics of HLB in citrus tree population. Based on the consideration of irrigation and nutrition, we formulate a compartmental model to incorporate fraction of susceptible citrus trees that undergo irrigation and nutrition. Basic reproduction number, R_0 , of the formulated model is obtained and global analyses are done for $R_0 < 1$ and $R_0 > 1$. We also introduce sensitivity analysis of R_0 of the model which makes it more realistic and biologically significant.

The rest of the paper is organized in the following way: The description of the model and definitions of variables and parameters are explained in section 2. Section 3 is devoted on the calculation of basic reproduction number and establishment of global stability of disease-free and existence of endemic equilibria. In section 4, sensitivity analysis of R_0 is performed. Analytical results obtained in the section 3 are numerically verified in section 5 with the help of realistic values of the model parameters. Section 6 wraps the modeling work with conclusion.

2 Model Formulation

The total citrus tree population is subdivided into four compartments; S_c, E_c, I_c, Q_c , which is interpreted as (i) susceptible citrus trees, (ii) exposed citrus trees, (iii) infectious citrus trees and (iv) vaccinated citrus trees so that $N_h(t) = S_c(t) + E_c(t) + I_c(t) + Q_c(t)$. The total Asian Citrus Psyllid population denoted as $N_p(t)$, is subdivided into three compartments; (i) susceptible ACP, (ii) exposed ACP and (iii)infectious ACP, so that $N_p(t) = S_p(t) + E_p(t) + I_p(t)$. We assume that emergence and replanting of trees enter citrus tree population at the rate λ . The disease transmission terms for citrus trees and ACP population are given as $\alpha a S_c I_p$ and $\alpha d S_p I_c$ where α is the ACP biting rate, a is the transmission probability from I_p to S_c and d is the transmission probability from I_c to S_p . The homogeneity of the proposed model assumes that all host have identical rates of disease-causing contacts, i.e., average biting rate a . We let θ be a fraction of susceptible citrus trees that undergo irrigation and watering while ρ represents a fraction of susceptible ACP that are sensitive to insecticides. Citrus trees are vaccinated at a rate v . Exposed citrus trees progress to become infectious at a rate ω . Infectious citrus trees are removed at a rate g or die due to huanglongbing infection at a rate δ . There is recruitment rate into susceptible ACP at a rate η . There is progression rate from exposed ACP to infectious ACP at a rate β . Natural death rate occurs in citrus tree population at a rate μ while it occurs in ACP population at a rate κ . The description of variables and parameters used for the model are given in Tables 1 and 2.

Applying the assumptions, description of terms above, Table 1, Table 2 and Figure 1, the transmission dynamics of the disease is formulated below:

Table 1: Description of state variables of model (1)-(7)

Variables	Description
S_c	Susceptible citrus trees
E_c	Exposed citrus trees
I_c	Infectious citrus trees
Q_c	Vaccinated citrus trees
S_p	Susceptible ACP
E_p	Exposed ACP
I_p	Infectious ACP

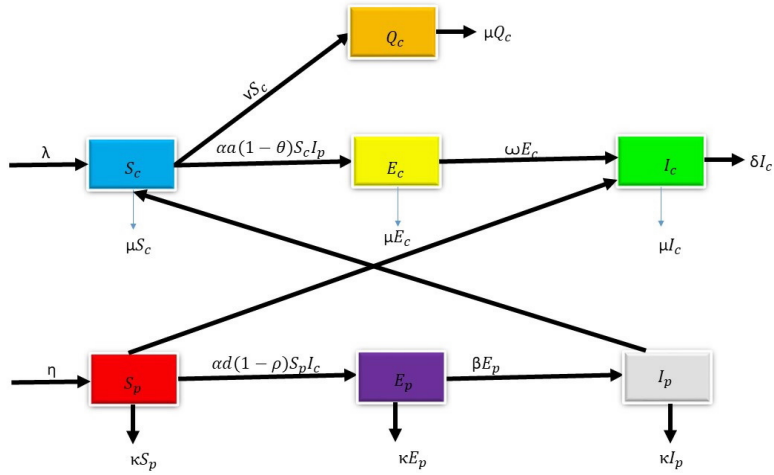


Figure 1: Scheme of Huanglongbing model.

Table 2: Summary of the parameters

Parameter	Meaning	Value (year ⁻¹)	Reference
λ	Emergence and replanting rate of citrus trees	100	Assumed
α	ACP biting rate	10	Assumed
a	Transmission probability from I_p to S_c	0.048	[35]
θ	Fraction of susceptible citrus trees that undergo irrigation and nutrition	0.5	Assumed
μ	Natural mortality rate in citrus trees	0.04	[35]
v	Vaccination rate	2	Assumed
η	Recruitment rate of ACP	5000	Assumed
ω	Disease progression rate of exposed citrus trees	0.17	Assumed
δ	Disease related death rate	0.1	[35]
d	Transmission probability from I_c to S_p	0.00039	[35]
κ	Natural mortality rate of ACP	4.0556	Assumed
β	Progression rate from E_p to I_p	12.1667	Assumed
ρ	Fraction of susceptible ACP that are sensitive to insecticides	0.6	[35]

$$\frac{dS_c}{dt} = \lambda - \alpha a(1 - \theta)S_c I_p - (\mu + v)S_c, \tag{1}$$

$$\frac{dE_c}{dt} = \alpha a(1 - \theta)S_c I_p - (\omega + \mu)E_c, \tag{2}$$

$$\frac{dI_c}{dt} = \omega E_c - (\delta + \mu)I_c, \tag{3}$$

$$\frac{dQ_c}{dt} = vS_c - \mu Q_c, \tag{4}$$

$$\frac{dS_p}{dt} = \eta - \alpha d(1 - \rho)S_p I_c - \kappa S_p, \tag{5}$$

$$\frac{dE_p}{dt} = \alpha d(1 - \rho)S_p I_c - (\beta + \kappa)E_p, \tag{6}$$

$$\frac{dI_p}{dt} = \beta E_p - \kappa I_p. \tag{7}$$

(1)–(3) are independent of the state Q_c and after decoupling the equations for Q_c from the model, we have the remaining equations of the model below

$$\frac{dS_c}{dt} = \lambda - \alpha a(1 - \theta)S_c I_p - (\mu + \nu)S_c, \quad (8)$$

$$\frac{dE_c}{dt} = \alpha a(1 - \theta)S_c I_p - (\omega + \mu)E_c, \quad (9)$$

$$\frac{dI_c}{dt} = \omega E_c - (\delta + \mu)I_c, \quad (10)$$

$$\frac{dS_p}{dt} = \eta - \alpha d(1 - \rho)S_p I_c - \kappa S_p, \quad (11)$$

$$\frac{dE_p}{dt} = \alpha d(1 - \rho)S_p I_c - (\beta + \kappa)E_p, \quad (12)$$

$$\frac{dI_p}{dt} = \beta E_p - \kappa I_p. \quad (13)$$

2.1 Basic Properties of the Model

Theorem 1 *The solutions $S_c(t)$, $E_c(t)$, $I_c(t)$, $S_p(t)$, $E_p(t)$, $I_p(t)$ of the Huanglongbing model with nonnegative initial data $S_c(0)$, $E_c(0)$, $I_c(0)$, $S_p(0)$, $E_p(0)$, $I_p(0)$, remain nonnegative for all time $t > 0$.*

Proof. Equation (8) can be written as

$$\begin{aligned} \frac{dS_c}{dt} + (\alpha a(1 - \theta)I_p + \mu + \nu)S_c &\geq 0, \\ \frac{d}{dt} \left[S_c(t) \exp \left(\int_0^t \alpha a(1 - \theta)I_p(\zeta) + (\mu + \nu)t \right) \right] &\geq 0. \end{aligned} \quad (14)$$

Integrating (14) gives

$$S_c(t) \geq S_c(0) \exp \left(- \int_0^t \alpha a(1 - \theta)I_p(\zeta) + (\mu + \nu)t \right) \geq 0.$$

In a similar manner, it can be shown that other state variables $E_c(t)$, $I_c(t)$, $S_p(t)$, $E_p(t)$, $I_p(t)$ are nonnegative for all $t > 0$. ■

Next, we show that

$$\Gamma = \Gamma_c \times \Gamma_p \subset R_+^3 \times R_+^3,$$

where

$$\Gamma_c = \left\{ (S_c, E_c, I_c) \in R_+^3 : N_c \leq \frac{\lambda}{\nu + \mu} \right\} \quad \text{and} \quad \Gamma_p = \left\{ (S_p, E_p, I_p) \in R_+^3 : N_p \leq \frac{\eta}{\kappa} \right\}$$

is a positive invariant region.

Theorem 2 *The region Γ is positively invariant with respect to the model (8)–(13).*

Proof. The slope of the total citrus tree population is given by $\frac{dN_c}{dt} = \lambda - (\nu + \mu)N_c$ which on solving yields

$$N_c(t) = N_c(0)e^{-(\nu+\mu)t} + \frac{\lambda}{\nu + \mu}(1 - e^{-(\nu+\mu)t}).$$

A similar approach for the ACP population gives

$$N_p(t) = N_p(0)e^{-\kappa t} + \frac{\eta}{\kappa}(1 - e^{-\kappa t}).$$

It follows that $N_c(t) \rightarrow \frac{\lambda}{\nu + \mu}$ and $N_p(t) \rightarrow \frac{\eta}{\kappa}$ as $t \rightarrow \infty$. In particular, $N_c(t) \leq \frac{\lambda}{\nu + \mu}$ if $N_c(0) \leq \frac{\lambda}{\nu + \mu}$ and $N_p(t) \leq \frac{\eta}{\kappa}$ if $N_p(0) \leq \frac{\eta}{\kappa}$. Hence, Γ is positively invariant. Therefore, it is sufficient to study the dynamics of model (8)–(13) in region Γ where the model can be considered as being epidemiologically and mathematically well-posed. ■

3 Model Analysis

The section is devoted to the qualitative analysis of the model (8)–(13). In the sequel, we shall be analyzing the disease-free equilibrium point, endemic equilibrium point and sensitivity of the model parameters.

3.1 Disease-Free Equilibrium

The disease-free equilibrium is the point at which the population is free of the disease. In this study, it refers to the situation where huanglongbing no longer exists in the population. We obtain the disease-free equilibrium point E_0 by setting the disease compartments, (E_c, I_c, E_p, I_p) , to zero and equating the right hand side of (8)–(13) to zero. The resulting system of algebraic equations is then solved and we obtain

$$E_0 = \left(\frac{\lambda}{\nu + \mu}, 0, 0, \frac{\eta}{\kappa}, 0, 0 \right).$$

To analyse the stability of the disease-free equilibrium point, we need to first obtain the basic reproduction number R_0 . The basic reproduction number is the expected number of new cases by primary one in a susceptible population throughout his infectious period. For example $R_0 = 2$ means an infectious individual will spread the infection to two people within the period when he is infectious. The basic reproduction number is obtained by expressing (8)–(13) as the difference between the rate of new infection in each infected compartment F and the rate of transfer between each infected compartment G . Hence, we have

$$\begin{bmatrix} \frac{dE_c}{dt} \\ \frac{dI_c}{dt} \\ \frac{dE_p}{dt} \\ \frac{dI_p}{dt} \end{bmatrix} = F - G = \begin{bmatrix} \alpha a(1 - \theta)S_c I_p \\ 0 \\ \alpha d(1 - \rho)S_p I_c \\ 0 \end{bmatrix} - \begin{bmatrix} r_2 E_c \\ -\omega E_c + r_3 I_c \\ (\kappa + \beta)E_p \\ -\beta E_p + \kappa I_p \end{bmatrix}$$

and

$$S = J_F J_G^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha d(1 - \rho)\eta}{\kappa r_4} & \frac{\alpha d(1 - \rho)\eta\beta}{\kappa r_4} \\ 0 & 0 & 0 & 0 \\ \frac{\alpha a(1 - \theta)\lambda}{r_1 r_2} & \frac{\alpha a(1 - \theta)\lambda\omega}{r_1 r_2 r_3} & 0 & 0 \end{bmatrix}.$$

R_0 is the maximum eigenvalue of matrix S and it's given as

$$R_0 = \sqrt{\frac{a(1 - \theta)(1 - \rho)\lambda\omega d\eta\beta\alpha^2}{r_1 r_2 r_3 \kappa(\kappa + \beta)}},$$

where

$$r_1 = \nu + \mu, \quad r_2 = \omega + \mu, \quad r_3 = \delta + \mu \quad \text{and} \quad r_4 = \kappa + \beta.$$

3.2 Local Stability of Disease-Free Equilibrium Solution

Clearly, the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$. To see this, we obtain the Jacobian matrix of the system (8)–(13) evaluated at E_0 :

$$J_{E_0} = \begin{bmatrix} -r_1 & 0 & 0 & 0 & 0 & \frac{-\alpha a(1-\theta)\lambda}{r_1} \\ 0 & -r_2 & 0 & 0 & 0 & \frac{\alpha a(1-\theta)\lambda}{r_1} \\ 0 & \omega & -r_3 & 0 & 0 & 0 \\ 0 & 0 & \frac{-\alpha d(1-\rho)\eta}{\kappa} & -\kappa & 0 & 0 \\ 0 & 0 & \frac{\alpha d(1-\rho)\eta}{\kappa} & 0 & -(\beta + \kappa) & 0 \\ 0 & 0 & 0 & 0 & \beta & -\kappa \end{bmatrix} \quad (15)$$

Two of the eigenvalues are $-r_1$ and $-\kappa$. The other four are eigenvalues of the matrix

$$\begin{bmatrix} -r_2 & 0 & 0 & \frac{\alpha a(1-\theta)\lambda}{r_1} \\ \omega & -r_3 & 0 & 0 \\ 0 & \frac{\alpha d(1-\rho)\eta}{\kappa} & -(\beta + \kappa) & 0 \\ 0 & 0 & \beta & -\kappa \end{bmatrix} \quad (16)$$

whose characteristic equation is

$$a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \quad (17)$$

where

$$a_0 = r_1 r_2 r_3 \kappa (\beta + \kappa) - \alpha^2 a \lambda \omega \beta d \eta (1 - \theta) (1 - \rho), \quad (18)$$

$$a_1 = r_2 r_3 + 2r_2 r_3 \kappa + r_2 \kappa \beta + r_2 \kappa^2 + r_3 \kappa^2 - r_3 \kappa \beta, \quad (19)$$

$$a_2 = r_2 r_3 + r_2 \beta + r_3 \beta + \kappa \beta + \kappa^2 + 2r_2 \kappa + 2r_3 \kappa, \quad (20)$$

$$a_3 = r_2 + r_3 + \beta + 2\kappa, \quad (21)$$

$$a_4 = 1. \quad (22)$$

Further manipulation of a_0 in terms of the basic reproduction number, R_0 , yields

$$a_0 = r_1 r_2 r_3 \kappa (\beta + \kappa) (1 - R_0^2) \quad (23)$$

One sees from (18)–(22) that all a_i 's are positive. Moreover, if $R_0 < 1$, it follows from (23) that $a_0 > 0$. Therefore the disease-free equilibrium point E_0 is locally asymptotically stable. The following theorem summarizes the above result:

Theorem 3 *The Huanglongbing model is locally asymptotically stable at infection-free equilibrium E_0 if $R_0 < 1$.*

The above theorem implies that a small invasion of infectious citrus trees into a completely susceptible citrus trees population will not lead to an outbreak of the disease.

3.3 Global Stability of Disease-Free Equilibrium Solution

Theorem 4 *The disease-free equilibrium E_0 of the model is globally asymptotically stable in Γ if $R_0 < 1$ and unstable if $R_0 > 1$.*

Proof. Consider the Lyapunov function

$$L = \frac{\omega}{r_2 r_3} E_c + \frac{1}{r_3} I_c + \frac{\kappa R_0}{\alpha d(1-\rho)\eta} E_p + \frac{\kappa(\kappa + \beta)R_0}{\alpha d(1-\rho)\eta\beta} I_p.$$

Its time derivative is

$$\begin{aligned} \dot{L} &= \frac{\omega}{r_2 r_3} (\alpha a(1-\theta)S_c I_p - r_2 E_c) + \frac{1}{r_3} (\omega E_c - r_3 I_c) + \frac{\kappa R_0}{\alpha d(1-\rho)\eta} (\alpha d(1-\rho)S_p I_c - (\beta + \kappa)E_p) \\ &\quad + \frac{\kappa(\kappa + \beta)R_0}{\alpha d(1-\rho)\eta\beta} (\beta E_p - \kappa I_p) \\ &= \frac{\omega \alpha a(1-\theta)S_c}{r_2 r_3} - I_c + \frac{\kappa R_0 S_p I_c}{\eta} - \frac{\kappa^2(\kappa + \beta)R_0 I_p}{\alpha d(1-\rho)\eta\beta} \\ &= \left[\frac{\omega \alpha a(1-\theta)S_c}{r_2 r_3} - \frac{\kappa^2(\kappa + \beta)R_0}{\alpha d(1-\rho)\eta\beta} \right] I_p + \left(\frac{\kappa R_0 S_p}{\eta} - 1 \right) I_c \\ &\leq \left[\sqrt{\frac{\omega a(1-\theta)(\kappa + \beta)\lambda r_1 \kappa^2}{r_2 r_3 d \beta \eta (1-\rho)}} (R_0 - 1) \right] I_p + (R_0 - 1) I_c \\ &\leq \left[\sqrt{\frac{\omega a(1-\theta)(\kappa + \beta)\lambda r_1 \kappa^2}{r_2 r_3 d \beta \eta (1-\rho)}} I_p + I_c \right] (R_0 - 1) \end{aligned}$$

Therefore, $\dot{L} \leq 0$ for $R_0 \leq 1$ and $\dot{L} = 0$ if and only if $R_0 = 1$ or $I_c = 0$ and $I_p = 0$. Consequently, the largest compact invariant set in $\{(S_c, E_c, I_c, Q_c, S_p, E_p, I_p) \in \Gamma : \dot{L} = 0\}$ is the E_0 and by Lyapunov-Lasalle's invariance principle, the disease-free equilibrium point is globally stable in Γ if $R_0 \leq 1$ but globally asymptotically stable in Γ if $R_0 < 1$. Note that Γ is a positive invariant set containing all the solutions of R_+^7 . This completes the proof.

■

Remark 1 *The epidemiological implication of Theorem 4 is that huanglongbing infection can be eradicated irrespective of the initial sizes of the sub-population of the model whenever $R_0 < 1$.*

3.4 Existence of Endemic Equilibrium Point

An infection or a disease is said to be endemic in a population when it persists at a baseline level in a geographic location. The endemic equilibrium point is the baseline value to which the sub-populations finally settle. The endemic equilibrium point is obtained by setting the right side of (8)–(13) to zero and solving the resulting system of algebraic equations. The endemic equilibrium solution is therefore $E_1 = (S_c^*, E_c^*, I_c^*, S_p^*, E_p^*, I_p^*)$, where

$$\begin{aligned} S_c^* &= \frac{\lambda}{G_c^* + r_1}, & E_c^* &= \frac{\lambda G_c^*}{r_2(G_c^* + r_1)}, & I_c^* &= \frac{\omega \lambda G_c^*}{r_2 r_3 (G_c^* + r_1)}, \\ S_p^* &= \frac{\eta}{G_p^* + \kappa}, & E_p^* &= \frac{G_p^* \eta}{(\kappa + G_p^*)(\kappa + \beta)}, & I_p^* &= \frac{\beta \eta G_p^*}{\kappa(G_p^* + \kappa)(\kappa + \beta)}, \end{aligned}$$

where the forces of infection for citrus trees and ACP at equilibrium state are

$$G_c^* = \alpha a(1-\theta)I_p^* \quad \text{and} \quad G_p^* = \alpha d(1-\rho)I_c^*.$$

Substituting I_p^*, G_p^*, I_c^* and G_c^* in $G_c^* = \alpha a(1-\theta)I_p^*$ gives the following linear equation

$$X_1 G_c^* + X_2 = 0,$$

where

$$X_1 = r_1 r_2 (\alpha d (1 - \rho) \lambda \omega r_2 + r_2) \quad \text{and} \quad X_2 = (r_1 r_2)^2 (1 - R_0^2).$$

Therefore,

$$G_c^* = \frac{-X_2}{X_1} \leq 0 \quad \text{if } X_2 \geq 0 \text{ at } R_0 \leq 1,$$

and no endemic equilibrium exists. On the other hand, $G_c^* = \frac{-X_2}{X_1} > 0$ if $X_2 < 0$ at $R_0 > 1$. Hence, an endemic equilibrium exists only at $R_0 > 1$.

The theorem below summarizes the above result.

Theorem 5 *The model (8)–(13) has a unique endemic equilibrium solution whenever $R_0 > 1$, and no endemic equilibrium solution otherwise.*

3.5 Global Stability of the Endemic Equilibrium Solution

The endemic equilibrium solution E_1 is locally asymptotically stable if $R_0 > 1$. This can be shown by linearizing the nonlinear system (8)–(13) around the endemic equilibrium solution. The rest follows by Routh-Hurwitz's theorem [19]. We make use of Goh-Volterra type Lyapunov function [21] and establish the global asymptotic stability of endemic equilibrium in what follows:

Theorem 6 *The unique endemic equilibrium E_1 , is globally asymptotically stable whenever $R_0 > 1$.*

Proof. Given the following equations which are satisfied by the endemic equilibrium point E_1 :

$$\lambda = \alpha a (1 - \theta) S_c^* I_p^* + \mu_h S_c^*, \quad (24)$$

$$\alpha a (1 - \theta) S_c^* I_p^* = r_1 E_c^*, \quad (25)$$

$$\omega E_c^* = r_2 I_c^*, \quad (26)$$

$$\eta = \kappa S_p^* + \alpha d (1 - \rho) S_p^* I_c^*, \quad (27)$$

$$\alpha d (1 - \rho) S_p^* I_c^* = r_4 E_p^*, \quad (28)$$

$$\beta E_p^* = \kappa I_p^*. \quad (29)$$

Consider the following Goh-Volterra Lyapunov function

$$\begin{aligned} V = & \left(S_c - S_c^* - S_c^* \ln \frac{S_c}{S_c^*} \right) + \left(E_c - E_c^* - E_c^* \ln \frac{E_c}{E_c^*} \right) + a \left(I_c - I_c^* - I_c^* \ln \frac{I_c}{I_c^*} \right) \\ & + \left(S_p - S_p^* - S_p^* \ln \frac{S_p}{S_p^*} \right) + \left(E_p - E_p^* - E_p^* \ln \frac{E_p}{E_p^*} \right) + b \left(I_p - I_p^* - I_p^* \ln \frac{I_p}{I_p^*} \right) \end{aligned}$$

where

$$a = \frac{\alpha d S_p^*}{r_3} \quad \text{and} \quad b = \frac{\alpha a (1 - \theta) S_c^*}{\kappa}.$$

The time derivative of the Lyapunov function is obtained as

$$\begin{aligned} \dot{V} = & \left(1 - \frac{S_c^*}{S_c} \right) S'_c + \left(1 - \frac{E_c^*}{E_c} \right) E'_c + a \left(1 - \frac{I_c^*}{I_c} \right) I'_c + \left(1 - \frac{S_p^*}{S_p} \right) S'_p + \left(1 - \frac{E_p^*}{E_p} \right) E'_p + b \left(1 - \frac{I_p^*}{I_p} \right) I'_p \\ = & \left(1 - \frac{S_c^*}{S_c} \right) (\lambda - \alpha a (1 - \theta) S_c I_p - r_1 S_c) + \left(1 - \frac{E_c^*}{E_c} \right) (\alpha a (1 - \theta) S_c I_p - r_2 E_c) \\ & + a \left(1 - \frac{I_c^*}{I_c} \right) (\omega E_c - r_3 I_c) + \left(1 - \frac{S_p^*}{S_p} \right) (\eta - \alpha d S_c I_p - \kappa S_p) + \left(1 - \frac{E_p^*}{E_p} \right) (\alpha d S_p I_c - r_4 E_p) \\ & + b \left(1 - \frac{I_p^*}{I_p} \right) (\beta E_p - \kappa I_p). \end{aligned}$$

Using (24)–(29), we have

$$\begin{aligned} \dot{V} &= \left(1 - \frac{S_c^*}{S_c}\right) (\alpha a(1 - \theta)S_c^*I_p^* + r_1S_c^* - \alpha a(1 - \theta)S_cI_p - r_1S_c) + \left(1 - \frac{E_c^*}{E_c}\right) (\alpha a(1 - \theta)S_cI_p - r_2E_c) \\ &+ a \left(1 - \frac{I_c^*}{I_c}\right) (\omega E_c - r_3I_c) + \left(1 - \frac{S_p^*}{S_p}\right) (\eta - \alpha dS_cI_p - \kappa S_p) + \left(1 - \frac{E_p^*}{E_p}\right) (\alpha dS_pI_c - r_4E_p) \\ &+ b \left(1 - \frac{I_p^*}{I_p}\right) (\beta E_p - \kappa I_p). \end{aligned}$$

Ignoring some terms and further simplification gives

$$\begin{aligned} \dot{V} &= \alpha a(1 - \theta)S_c^*I_p^* + r_1E_c^* + r_4E_p^* + ar_3I_c^* + b\kappa I_p^* - \frac{\alpha a(1 - \theta)(S_c^*)^2I_p^*}{S_c} - \frac{\alpha a(1 - \theta)I_pS_cE_c^*}{E_c} - \frac{\alpha d(S_p^*)^2I_c^*}{S_p} \\ &- \frac{a\omega E_cI_c^*}{I_c} - \frac{\alpha dS_pI_cE_p^*}{E_p} - \frac{b\beta E_pI_p^*}{I_p} + 2r_1S_c^* - \frac{r_1(S_c^*)^2}{S_c} - r_1S_c - \frac{\kappa(S_p^*)^2}{S_p} - \kappa S_p + 2\kappa S_p^* + \alpha dS_p^*I_c^*. \end{aligned}$$

Replacing a and b by their values and exploiting (24)–(29) gives

$$a\omega = \frac{\alpha dI_c^*S_p^*}{E_c^*}, \tag{30}$$

$$b\beta = \frac{\alpha a(1 - \theta)S_c^*I_p^*}{E_p^*}. \tag{31}$$

Using (24)–(31), we have

$$\begin{aligned} \dot{V} &= r_1S_c^* \left(2 - \frac{S_c^*}{S_c} - \frac{S_c}{S_c^*}\right) + 3\alpha a(1 - \theta)S_c^*I_p^* - \frac{\alpha a(1 - \theta)(S_c^*)^2I_p^*}{S_c} - \frac{\alpha a(1 - \theta)S_cI_pE_c^*}{E_c} \\ &- \frac{\alpha dS_p^*E_c(I_c^*)^2}{I_cE_c^*} + \kappa S_p^* \left(2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*}\right) + 3\alpha dS_p^*I_c^* - \frac{\alpha d(S_p^*)^2I_c^*}{S_p} - \frac{\alpha dI_cS_pE_p^*}{E_p} \\ &- \frac{\alpha a(1 - \theta)(I_p^*)^2E_pS_c^*}{E_p^*I_p} \left(3 - \frac{S_c^*}{S_c} - \frac{S_cE_c^*I_c}{S_c^*E_cI_c^*} - \frac{E_cI_c^*}{E_c^*I_c}\right) \\ &= r_1S_c^* \left(2 - \frac{S_c^*}{S_c} - \frac{S_c}{S_c^*}\right) + \kappa S_p^* \left(2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*}\right) + \alpha a(1 - \theta)I_p^*S_c^* \left(3 - \frac{S_c^*}{S_c} - \frac{S_cI_pE_c^*}{S_c^*I_p^*E_c}\right) \\ &- \frac{dS_p^*E_c(I_c^*)^2}{E_c^*I_c a(1 - \theta)I_p^*S_c^*} + \alpha dS_p^*I_c^* \left(3 - \frac{S_p^*}{S_p} - \frac{I_cS_pE_p^*}{S_p^*I_c^*E_p} - \frac{a(1 - \theta)(I_p^*)^2E_pS_c^*}{E_p^*I_p dS_p^*I_c^*}\right). \end{aligned}$$

Using arithmetic-geometric means inequality, i.e., $n - (a_1 + a_2 + \dots + a_n) \leq 0$, where $a_1.a_2\dots a_n = 1$ and $a_1, a_2, \dots, a_n > 0$, it follows that $\dot{V} \leq 0$ with $V = 0$ if and only if $S_c = S_c^*, E_c = E_c^*, I_c = I_c^*, S_p = S_p^*, E_p = E_p^*, I_p = I_p^*$. Hence, the largest compact invariant subset of the set where $\dot{V} = 0$ is $(S_c, E_c, I_c, S_p, E_p, I_p) = (S_c^*, E_c^*, I_c^*, S_p^*, E_p^*, I_p^*)$ and by classical stability theorem of Lyapunov and LaSalle’s Invariance Principle, it follows that every solution in Γ approaches E_1 for $R_0 > 1$ as $t \rightarrow \infty$. ■

Remark 2 *The epidemiological implication of Theorem 6 is that Huanglongbing disease will invade a population if each infected host passes on, the infection to more than one other host (i.e., $R_0 > 1$). This further implies that citrus tree population that starts with either small or large Huanglongbing infection (i.e., irrespective of the initial sizes of the infectious population) increases and turns into epidemic whenever $R_0 > 1$.*

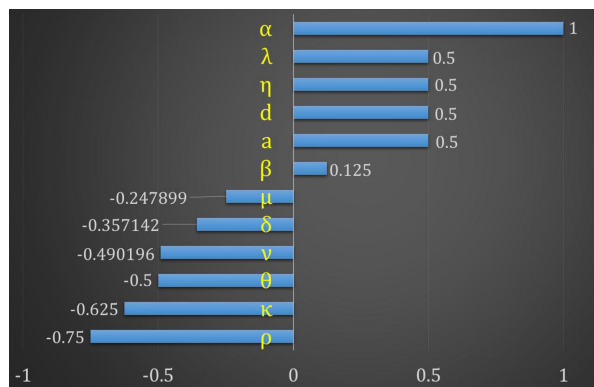


Figure 2: Sensitivity analysis between R_0 and its associated parameters.

3.6 Sensitivity Analysis

In this section, a sensitivity analysis of parameters of the model system (8)–(13) is carried out so as to determine the relative importance of model parameters on the disease infection. The rationale is to consider and to manage several factors responsible for huanglongbing infection.

Sensitivity indices are computed numerically to find out parameters that have reasonable impact on basic reproduction number R_0 and which of the parameters is most sensitive, which can help in combating the disease.

The normalized forward sensitivity index of a variable to a parameter is the ratio of the relative change in the variable to the relative change in the parameter. When the variable is a differentiable function of the parameter, the sensitivity index may be alternatively defined using partial derivatives.

The normalized forward sensitivity index of a variable, $u(p)$, that depends differentially on a parameter, p , is defined as

$$N_p^u = \frac{\partial u}{\partial p} \times \frac{p}{u} \text{ for } u \neq 0.$$

Thus to investigate the sensitivity of R_0 to parameter p_i , we have

$$N_{p_i}^{R_0} = \frac{\partial R_0}{\partial p_i} \times \frac{p_i}{R_0},$$

where $p_i, i \in \mathbb{N}$ denotes each parameter involved in computing R_0 . We compute sensitivity index of each parameter with respect to R_0 using parameters values in Table 2. For instance:

$$N_{\alpha}^{R_0} = \frac{\partial R_0}{\partial \alpha} \times \frac{\alpha}{R_0} = 1.000000.$$

Figure 2 gives the sensitivity indices on R_0 with respect to each parameter. Figure 2 indicates that R_0 is most sensitive to ACP biting rate (α) in the positive sense and R_0 is most sensitive to the fraction of susceptible ACP who are sensitive to insecticides (ρ), in a negative sense. This suggests that a 10% increase (or decrease) of ACP biting rate, α , would correspond to a 10% increase or (decrease) of R_0 . Likewise, a 10% increase (or decrease) of the fraction of susceptible ACP who are sensitive to insecticides ρ , would correspond to a 7.5% decrease (or increase) in the value of R_0 . Another striking parameter of interest is the fraction of susceptible citrus trees that undergo irrigation and nutrition θ . That is, a 10% increase (or decrease) of θ would correspond to a 5% decrease (or increase) in the value of R_0 . Other relevant parameters responsible for the transmission and spread of HLB are emergence and replanting rate of citrus trees λ , transmission probability from I_p to S_c a , recruitment rate of ACP η , transmission probability from I_c to S_p , d and progression rate from E_p to I_p , β and disease progression rate of exposed citrus tree to infectious citrus trees ω . Therefore, the above interpretations recommend that intervention strategies that can effectively decrease

the ACP biting rate α , emergence and replanting of citrus trees, a , recruitment rate of ACP η , transmission probability from I_c to S_p , d , and progression rate from E_p to I_p , β , should be used to control the disease transmission effectively. In addition, irrigation and nutrition of fraction of susceptible citrus trees have been shown to have negative impact on R_0 . This suggests that farmers should make irrigation and nutrition their first priority in citrus trees plantation. This method will make the susceptible citrus trees to be less attackable by ACP, as it boosts their immune response. Further, as we know that insecticide is the most widely followed option for reducing ACP, alternation and rotation of insecticides with different chemistry is needed to overcome the incidence of fraction of susceptible ACP who are resistant to insecticides since sensitivity analysis has revealed that fraction of susceptible ACP that are sensitive to insecticides reduces R_0 the most.

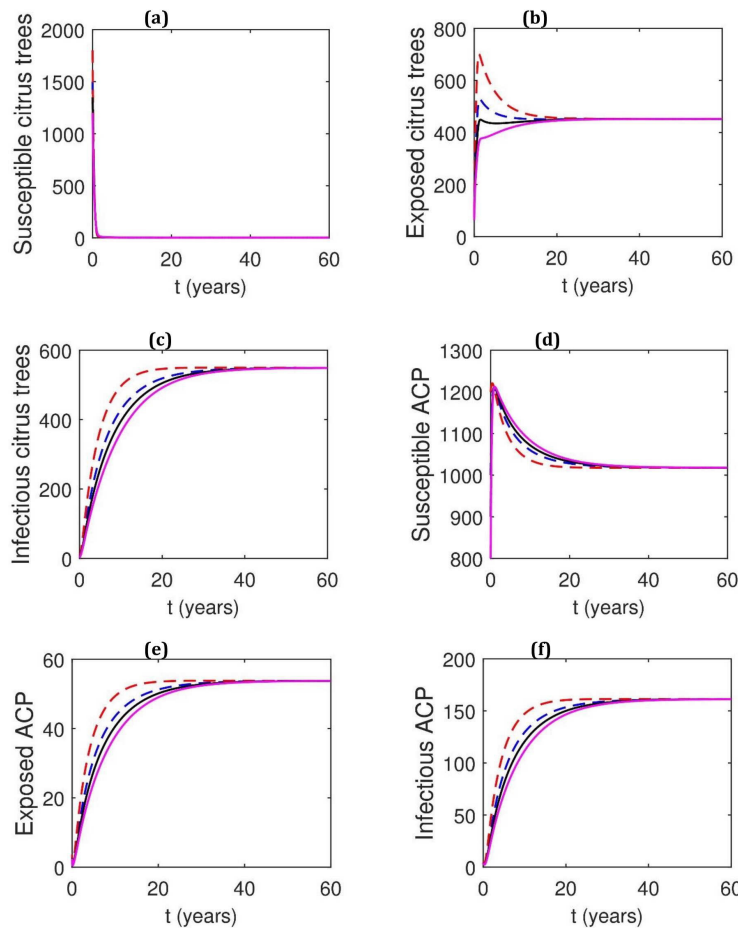


Figure 3: Simulation results showing the global stability of the endemic equilibrium.

4 Numerical Simulations

In order to understand the picture of the disease behaviour, this section presents numerical simulations for huanglongbing model using the parameter values in Table 2. Model (8)–(13) is solved using classical Runge-Kutta method on Matlab platform.

We investigate, numerically, the stability of the equilibrium solutions. For this purpose, we pick different initial values for the variables and observe the solution trajectories over time. Figure 3 represents the solution

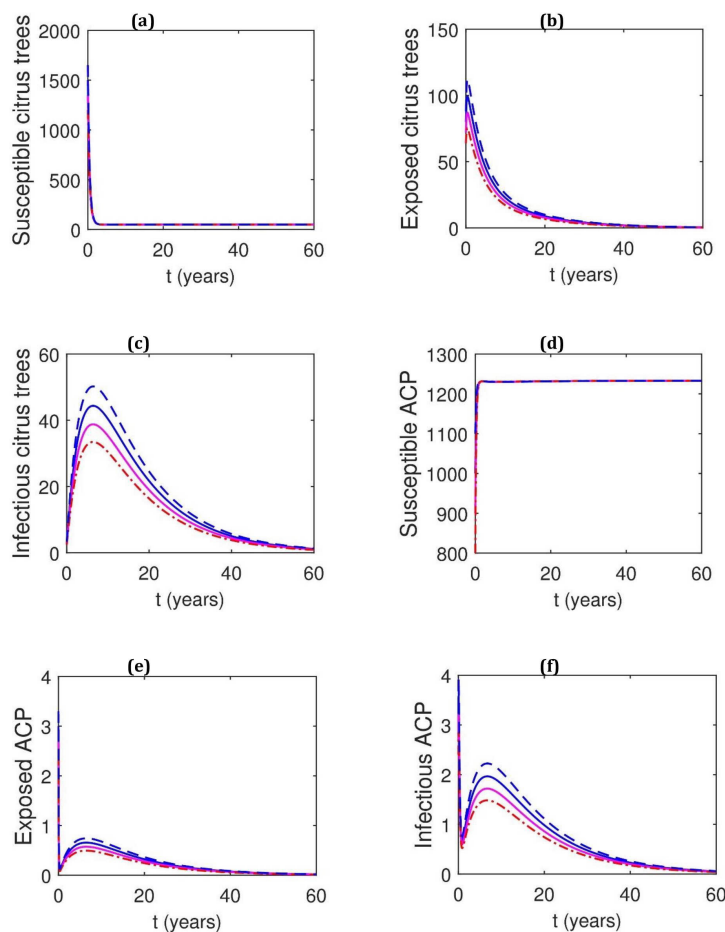


Figure 4: Simulation results showing the global stability of the disease-free equilibrium.

trajectories of the model using the parameter values in Table 2. With these parameter values, $R_0 = 4.9189$. It is observed, in Figure 3, that the population of the susceptible citrus trees reduces whereas the populations of exposed and infectious citrus trees increase and settle at the equilibrium position. In ACP population, the susceptible ACP population reduces while the exposed and infectious ACP increase. In the both citrus tree and ACP populations, the susceptible classes decrease and settle at equilibrium positions while the disease classes increase and also settle at equilibrium positions. The equilibrium point for which all the classes settle is the endemic equilibrium point. This confirms Theorem 6.

Figure 4 represents the solution trajectories of the model taking $\theta = 0.95$, $\rho = 0.95$ and other parameter values as contained in Table 2. With these parameter values, $R_0 = 0.55$. From Figure 4, it can be seen that exposed citrus tree population E_c and infectious citrus tree population I_c converge to their corresponding disease-free equilibrium solution. This indicates that the infected citrus trees disappear from the population whenever $R_0 < 1$. In ACP population, we can observe that there are always susceptible ACP in the population but the exposed and infectious ACP shrink to zero for $R_0 < 1$. This implies that huanglongbing infection is cleared from the population for $R_0 < 1$. This confirms Theorem 4.

Next we investigate possible intervention strategies towards mitigating the spread of huanglongbing infection. For this purpose, we choose initial values as $S_c(0) = 1500$, $E_c(0) = 80$, $I_c(0) = 3$, $Q_c(0) = 20$, $S_p(0) = 1000$, $E_p(0) = 3$, $I_p(0) = 3$. Since the condition $R_0 < 1$ is sufficient for disease eradication, we therefore seek parameter combination such that $R_0 < 1$. From the sensitivity analysis, R_0 is most sensitive to ACP biting rate α in the positive sense and sensitive to the fraction of susceptible ACP that are sensitive

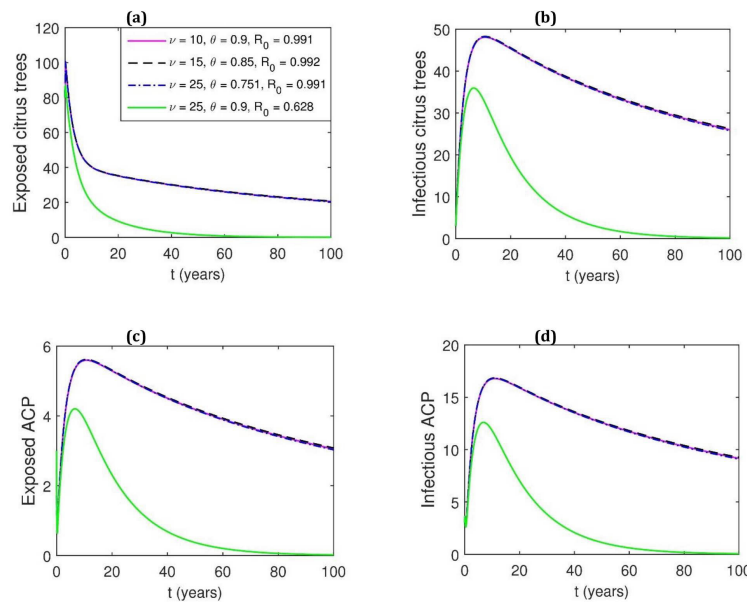


Figure 5: Impact of irrigation and nutrition together with vaccination of susceptible citrus trees to control the spread of huanglongbing.

to insecticides, natural mortality rate of ACP, fraction of susceptible citrus trees that undergo irrigation and nutrition, vaccination rate, in the negative sense. We therefore build control strategies about these parameters.

Next we investigate the impact of irrigation in controlling the spread of huanglongbing. Since the condition $R_0 < 1$ is sufficient for disease eradication, we find the fraction of susceptible citrus trees that undergo irrigation and nutrition, θ , such that $R_0 < 1$. Taking $\theta = 0.9795$ makes $R_0 = 0.996$. Therefore to effectively control the spread of huanglongbing infection with the use of irrigation and nutrition, minimum of 98% of the susceptible citrus trees must be irrigated and the irrigation must be 100% effective. This appears unrealistic, we therefore consider the impact of this intervention strategy together with vaccination of citrus trees. This is shown in Figure 5.

In Figure 5, it is seen that there is no significant difference in the trend of the infection once the value of basic reproduction number remains unchanged. However, when θ and ν are combined such that $R_0 = 0.628$, there is a significant reduction in population of infected classes. Thus efforts must be made towards improving greatly on vaccination as well as irrigation of susceptible citrus trees in order to have a rapid decrease in the spread of huanglongbing infection.

Next we investigate the influence of the use of insecticide in controlling the spread of huanglongbing. Taking $\rho = 0.985$, while other parameters remain as contained in Table 2, makes $R_0 = 0.952$. Therefore to effectively control the spread of huanglongbing infection with the use of insecticide, the insecticide used must be upward of 98.5% effective. This appears unrealistic as the efficacy of insecticide may decrease with time, we therefore consider the impact of using insecticide together with vaccination of citrus trees. Combination of the use of insecticide and vaccination of citrus trees such that $R_0 < 1$ is shown in Figure 6. Figure 6 shows that all the infected classes tend to zero over time however, increase in vaccination rate lowers the peak of infection in citrus trees while increase in the efficacy of insecticide lowers the peak of infected ACP.

5 Conclusion

Our aim is to investigate the impact of irrigation and nutrition on HLB dynamics. This leads us to formulate a mathematical model of HLB transmission that takes into account a fraction of susceptible citrus trees that

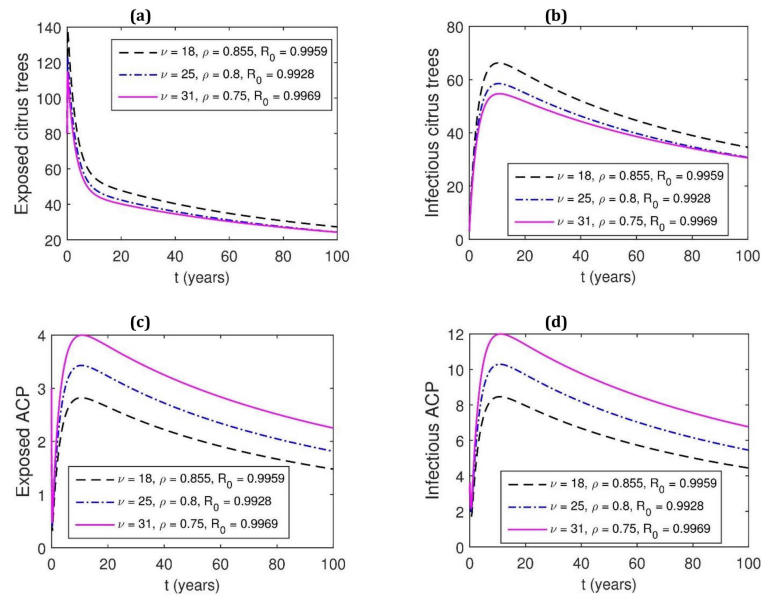


Figure 6: Impact of using insecticide together with vaccination of susceptible citrus trees to control the spread of haunglongbing.

undergo irrigation and nutrition. We determine the basic reproduction number, R_0 , and prove that if $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable. We prove the global asymptotic stability of disease-free equilibrium points. We further establish the existence condition for endemic equilibrium points for $R_0 > 1$. A global asymptotic stability of endemic equilibrium point is established and validated using numerically, simulations.

From our numerical results, we found that effective prevention of citrus trees through irrigation and nutrition has a positive impact in reducing the spread of HLB. However, this alone might not be sufficient unless it is implemented in conjunction with other intervention strategies such as vaccination susceptible citrus trees. It is therefore noted that using a single strategy will place a heavy demand on the strategy and as such may not be effective. Therefore combining at least two intervention strategies effectively will help in curtailing the spread of the infection.

Irrigation and nutrition practice as proposed in the present study is important for every citrus growers and production managers as it minimizes environmental stresses, especially closer to the harvesting window. Also, it increases profitability and enhances sustainability and worldwide competitiveness. This is because citrus trees supplied with sufficient water and nutrients grow stronger, better tolerates pests and stresses as a result of boosted immunity and yield more consistently to produce good quality fruits. This study will help the citrus tree industries to guide future research needs. This study can be applied to some regions with realistic data on HLB transmission that would allow to show the impact of irrigation and nutrition practice on citrus trees.

We have assumed the use of perfect vaccine in this work. That is, a vaccinated susceptible citrus tree is perfectly protected against HLB infection. This may not be true in reality as studies have shown that the efficacy of vaccine reduces with time [2]. The use of imperfect vaccine and the influence of weather on the dynamics of HLB infection may be considered in the future work. Further, this study can be extended by introducing fractional order into the formulated model. In this case, the model will be given fractional differential equations. All these directions need more investigation and therefore, they are left for future works.

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