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# Loxodromes On Semi-Riemannian Hypersurfaces Of Revolution In Minkowski Space-Time*

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**Abstract**

In this paper, we first introduce the notion of semi-Riemannian hypersurface of revolution of elliptic type and hyperbolic type, respectively in Minkowski space-time. Then, we find the equations of loxodromes on semi-Riemannian hypersurfaces of revolution. Also, we obtain the arc-lengths of the space-like or time-like loxodromes on semi-Riemannian hypersurfaces of revolution. Finally, we give two examples to strengthen our main results.

## 1 Introduction

Loxodrome is a special curve which makes a constant angle with the meridians on the Earth's surface. Loxodromes are usually used in navigation since they provide efficient navigation routes from one point to another. It is well known that the stereographic projections of loxodromes are logarithmic spirals and the Mercator projections of loxodromes are straight lines (see [3]). Loxodromes on rotational surfaces in different three- and four-dimensional ambient spaces were studied by a lot of authors (see [1, 2, 4, 5, 8, 9, 10, 12, 14, 15, 16]). In [6], the parametric equations of loxodromes on hypersurfaces of revolution in $n$-dimensional Euclidean space were investigated by using the notion of parallel transport in differential geometry.

In the theory of special relativity, Minkowski space-time $\mathbb{E}_1^4$ is a model for space-time (see [13]). In the present paper, we introduce the notion of semi-Riemannian hypersurface of revolution of elliptic type and hyperbolic type, respectively in $\mathbb{E}_1^4$. After that, we find the equations of loxodromes on semi-Riemannian hypersurfaces of revolution by using similar differential geometry methods in [6]. Also, we obtain the arc-lengths of the space-like or time-like loxodromes on semi-Riemannian hypersurfaces of revolution. Finally, we give examples of loxodromes on pseudo sphere and on pseudohyperbolic sphere in $\mathbb{E}_1^4$, respectively by using Mathematica.

## 2 Preliminaries

Minkowski space-time is a metric space $\mathbb{E}_1^4 = (\mathbb{R}^4, \langle\cdot,\cdot\rangle)$, where the scalar product is defined by

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3 - u_4v_4,$$

where $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$.

A causal character of an arbitrary vector $u \in \mathbb{E}_1^4$ is called space-like, time-like or light-like if $\langle u, u\rangle > 0$, $\langle u, u\rangle < 0$ or $\langle u, u\rangle = 0$, respectively.

The pseudo norm of a vector $u \in \mathbb{E}_1^4$ is defined by $\|u\| = \sqrt{|\langle u, u\rangle|}$ and $u$ is called a unit vector if $\|u\| = 1$.

Let $\alpha : I \to \mathbb{E}_1^4$ be a smooth and regular curve in $\mathbb{E}_1^4$, that is $\alpha'(t) \neq 0$ holds everywhere, where $I \subset \mathbb{R}$ is an open interval. Then, $\alpha$ is called space-like, time-like or light-like curve if all of velocity vectors $\alpha'(t)$ are space-like, time-like or light-like, respectively.

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The arc-length parameter of a space-like or a time-like curve $\alpha$ is introduced by

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| \, dt.$$

Also, $\alpha$ is a unit speed curve if $\|\alpha'(s)\| = 1$ for all $s \in I \subset \mathbb{R}$.

A semi-Riemannian manifold $M$ in $\mathbb{E}_1^4$ is a smooth manifold furnished with the scalar product $\langle \cdot, \cdot \rangle$. A semi-Riemannian hypersurface $S$ in $\mathbb{E}_1^4$ is just semi-Riemannian submanifold of codimension 1. In $\mathbb{E}_1^4$, an important family of semi-Riemannian hypersurfaces is hyperquadrics and they are defined as follows:

The pseudosphere is the hyperquadric

$$\mathbb{S}_1^3 = \{u \in \mathbb{E}_1^4 | \langle u, u \rangle = 1\}$$

with dimension 3 and index 1.

The pseudohyperbolic sphere is the hyperquadric

$$\mathbb{H}_0^3 = \{u \in \mathbb{E}_1^4 | \langle u, u \rangle = -1\}$$

with dimension 3 and index 0 (for more details, see [7, 11]).

## 3 Loxodromes on Semi-Riemannian Hypersurfaces of Revolution of Elliptic Type

In this section, we give the notion of semi-Riemannian hypersurface of revolution of elliptic type in $\mathbb{E}_1^4$. After that, we investigate the equation of loxodrome on semi-Riemannian hypersurface of revolution of elliptic type in $\mathbb{E}_1^4$. Also, we obtain the arc-length of the space-like or time-like loxodrome on semi-Riemannian hypersurface of revolution of elliptic type and as an example, we find the parametrization of an arbitrary loxodrome on pseudosphere $\mathbb{S}_1^3$.

Let $M_1$ be a 2-dimensional semi-Riemannian submanifold in $\mathbb{E}_1^4$. We now consider a local parametrization of $M_1$ as follows:

$$(x_1, 0, x_3, x_4) : U_1 \to \mathbb{E}_1^4,$$

where $U_1 \subseteq \{(x_1, 0, x_3, x_4) : x_1 > 0\}$ is an open subset. Then, semi-Riemannian hypersurface of revolution of elliptic type $S_1$ with the profile manifold $M_1$ is parametrized by

$$(x_1, 0, x_3, x_4, \psi) \to R_1(\mathbf{x}, \psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \end{bmatrix},$$

so

$$R_1(\mathbf{x}, \psi) = (x_1 \cos\psi, x_1 \sin\psi, x_3, x_4). \tag{1}$$

The meridian of $S_1$ is given by $R_1(\mathbf{x}, \psi)$, $\psi$=constant and it is denoted by $M_{1\psi}$.

Let us consider a smooth curve

$$\alpha_1(t) = (x_1(t) \cos\psi(t), x_1(t) \sin\psi(t), x_3(t), x_4(t)).$$

If $\alpha_1$ is a loxodrome on $S_1$, then its relative position to the meridians stays constant. To compare the relative position of $\alpha_1(t)$ to $M_{1\psi(t)}$ for different $t$-values, we can use a parallel translation in $\mathbb{E}_1^4$:

$$M_{1\psi} \to M_1$$

to bring the position and velocity vectors along $\alpha_1$ back on $M_1$. We can obtain the curve

$$\beta_1(t) = R_1(\alpha_1(t), -\psi(t)) = (x_1(t), 0, x_3(t), x_4(t))$$

on $M_1$ and the vector field

$$V_1(t) = R_1(\alpha_1'(t), -\psi(t)) = (x_1'(t), x_1(t)\psi'(t), x_3'(t), x_4'(t)) \tag{2}$$

along $\alpha_1$. Thus, we compare the relative positions of $\alpha_1(t)$ to $M_{1\psi(t)}$ for different $t$-values, we compare $V_1(t)$ along $\beta_1(t)$ on $M_1$ at different $t$-values instead.

Let $\nabla$ be the induced connection on $S_1 \subset \mathbb{E}_1^4$. We define a loxodrome $\alpha_1(t)$ as follows:

$$\nabla_{\beta_1'(t)} V_1(t) = 0, \ \forall t \in (a, b). \tag{3}$$

From (2), we have

$$V_1(t) = \beta_1'(t) + x_1(t)\psi'(t)N_1, \tag{4}$$

where $N_1 = (0, 1, 0, 0)$ is a unit normal vector to $M_1 \subset \mathbb{E}_1^4$. From (3) and (4), we obtain

$$\nabla_{\beta_1'(t)}\beta_1'(t) + \frac{d}{dt}(x_1(t)\psi'(t))N_1 = 0.$$

This equation is equivalent to

$$\begin{cases} \nabla_{\beta_1'(t)}\beta_1'(t) = 0, \\ x_1(t)\psi'(t) = c, \end{cases} \tag{5}$$

where $c$ is a constant. The first equation of (5) is the geodesic equation and the second equation gives the Lorentzian angle of rotation along the geodesic $\beta_1$, that is a loxodrome on $S_1$ is obtained by rotating a geodesic of $M_1$ by the Lorentzian angle

$$\psi(t) = c\int_{t_0}^{t} \frac{1}{x_1(t)}dt,$$

where $x_1(t)$ is 1th component of the geodesic. Thus, we have the following theorem.

**Theorem 1** *In $\mathbb{E}_1^4$, let $S_1$ be semi-Riemann hypersurface of revolution of elliptic type with the profile manifold $M_1$ given by (1). If $\beta_1(t) = (x_1(t), 0, x_3(t), x_4(t))$ is an arbitrary geodesic on $M_1$, then the parametrization of a loxodrome on $S_1$ is given by*

$$\alpha_1(t) = (x_1(t)\cos\psi(t), x_1(t)\sin\psi(t), x_3(t), x_4(t)),$$

*where $\psi(t) = c\int_{t_0}^{t} \frac{1}{x_1(t)}dt$.*

The arc-length of a space-like or a time-like loxodrome is given by the following result.

**Corollary 1** *The arc-length of the space-like or time-like loxodrome between $t_1$ and $t_2$ on the semi-Riemannian hypersurface of revolution of elliptic type in $\mathbb{E}_1^4$ is given by*

$$s = \Big| \int_{t_1}^{t_2} \sqrt{|x_1'^2(t) + x_3'^2(t) - x_4'^2(t) + c^2|}dt \Big|. \tag{6}$$

Also, we give the following remark.

**Remark 1** *If the semi-Riemannian hypersurfaces of revolution of elliptic type are reduced to the rotational surfaces of elliptic type in $\mathbb{E}_1^4$, then the equations of loxodromes on semi-Riemannian hypersurfaces of revolution of elliptic type coincide with the equations of loxodromes on rotational surfaces of elliptic type in $\mathbb{E}_1^4$ (see [5, 14]).*

Now we give the following example.

**Example 1** *Pseudosphere $\mathbb{S}^3_1$ in $\mathbb{E}^4_1$ is a semi-Riemannian hypersurface of revolution of elliptic type with profile manifold $\mathbb{S}^2_1$. To obtain the parametrization of an arbitrary space-like loxodrome on $\mathbb{S}^3_1$, we first find the parametrization of an arbitrary time-like geodesic on $\mathbb{S}^2_1$. Let $P_1 \subset \mathbb{E}^4_1$ be 2-plane which passes through the origin and $g|P_1$ be non-degenerate with index 1. Let us choose an orthonormal basis $\{e_1, e_4\}$ in $P_1$. Then, a time-like geodesic $\mathbb{S}^2_1 \cap P_1$ is parametrized by*

$$\beta_1(t) = \cosh t e_1 + \sinh t e_4, \; t \in \mathbb{R}$$

*(see [11]). From (5), we have*

$$\psi'(t) = \frac{c}{\cosh t}.$$

*It follows that*

$$\psi(t) = 2c \arctan\big(\tanh \frac{t}{2}\big)$$

*for $t_0 = 0$.*

*Consequently, the parametrization of the space-like loxodrome on $\mathbb{S}^3_1$ is given by*

$$\alpha_1(t) = (\cosh(t) \cos \psi(t), \cosh(t) \sin \psi(t), 0, \sinh(t)),$$

*where $\psi(t) = 2c \arctan\big(\tanh \frac{t}{2}\big)$.*

*By using the Lorentzian time-like angle between a space-like vector and a time-like vector in $\mathbb{E}^4_1$ and taking the angle as 1, we have $c = 1.31304$ (see [5, 13]). Then, the arc-length of space-like loxodrome on $\mathbb{S}^3_1$ between $t_1 = -2$ and $t_2 = 2$ is equal to 3.4037.*

*The graphs of the projections of an open part of $\mathbb{S}^3_1$, loxodrome and meridian ($\psi = 0$) in $\mathbb{E}^3_1$ can be drawn, see Figure 1.*

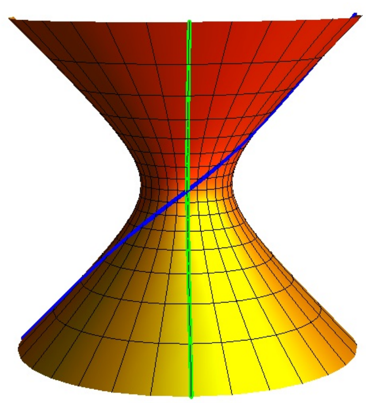

Figure 1: The projections of an open part of $\mathbb{S}^3_1$, loxodrome (blue), meridian (green).

# 4 Loxodromes on Semi-Riemannian Hypersurfaces of Revolution of Hyperbolic Type

In this section, we introduce the notion of semi-Riemannian hypersurface of revolution of hyperbolic type in $\mathbb{E}^4_1$. After that, we find the equation of loxodrome on semi-Riemannian hypersurface of revolution of hyperbolic type in $\mathbb{E}^4_1$. Also, we obtain the arc-length of the space-like or time-like loxodrome on semi-Riemannian hypersurface of revolution of hyperbolic type and as an example, we find the parametrization of an arbitrary loxodrome on pseudohyperbolic sphere $\mathbb{H}^3_0$.

Let $M_2$ be a 2-dimensional semi-Riemannian submanifold in $\mathbb{E}_1^4$. We consider a local parametrization of $M_2$ as

$$(x_1, x_2, 0, x_4) : U_2 \to \mathbb{E}_1^4,$$

where $U_2 \subseteq \{(x_1, x_2, 0, x_4) : x_4 > 0\}$ is an open subset. Then, semi-Riemannian hypersurfaces of revolution of hyperbolic type $S_2$ with the profile manifold $M_2$ is parametrized by

$$(x_1, x_2, 0, x_4, \theta) \to R_2(\mathbf{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\theta & \sinh\theta \\ 0 & 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \end{bmatrix},$$

so

$$R_2(\mathbf{x}, \theta) = (x_1, x_2, x_4 \sinh\theta, x_4 \cosh\theta). \tag{7}$$

The meridian of $S_2$ is given by $R_2(\mathbf{x}, \theta)$, $\theta$=constant and it is denoted by $M_{2\theta}$.

Let us consider a smooth curve

$$\alpha_2(t) = (x_1(t), x_2(t), x_4(t) \sinh\theta(t), x_4(t) \cosh\theta(t)).$$

If $\alpha_2$ is a loxodrome on $S_2$, then its relative position to the meridians stays constant. To compare the relative position of $\alpha_2(t)$ to $M_{2\theta(t)}$ for different $t$-values, we can use a parallel translation in $\mathbb{E}_1^4$:

$$M_{2\theta} \to M_2$$

to bring the position and velocity vectors along $\alpha_2$ back on $M_2$.

Let us consider the curve

$$\beta_2(t) = R_2(\alpha_1(t), -\theta(t)) = (x_1(t), x_2(t), 0, x_4(t))$$

on $M_2$ and the vector field

$$V_2(t) = R_2(\alpha_2'(t), -\theta(t)) = (x_1'(t), x_2'(t), x_4(t)\theta'(t), x_4'(t)) \tag{8}$$

along $\alpha_2$. Thus, we compare the relative positions of $\alpha_2(t)$ to $M_{2\theta(t)}$ for different $t$-values, we compare $V_2(t)$ along $\beta_2(t)$ on $M_2$ at different $t$-values instead.

Let $\nabla$ be the induced connection on $S_2 \subset \mathbb{E}_1^4$. We define a loxodrome $\alpha_2(t)$ as follows:

$$\nabla_{\beta_2'(t)} V_2(t) = 0, \ \forall t \in (a, b). \tag{9}$$

From (8), we have

$$V_2(t) = \beta_2'(t) + x_4(t)\theta'(t) N_2, \tag{10}$$

where $N_2 = (0, 0, 1, 0)$ is a unit normal vector to $M_2 \subset \mathbb{E}_1^4$. From (9) and (10), we obtain

$$\nabla_{\beta_2'(t)} \beta_2'(t) + \frac{d}{dt}(x_4(t)\theta'(t)) N_2 = 0.$$

This equation is equivalent to

$$\begin{cases} \nabla_{\beta_2'(t)} \beta_2'(t) = 0 \\ x_4(t)\theta'(t) = k, \end{cases} \tag{11}$$

where $k$ is a constant. The first equation of (11) is the geodesic equation and the second equation gives the Lorentzian angle of rotation along the geodesic $\beta_2$, that is a loxodrome on $S_2$ is obtained by rotating a geodesic of $M_2$ by the Lorentzian angle

$$\theta(t) = k \int_{t_0}^{t} \frac{1}{x_4(t)} dt, \tag{12}$$

where $x_4(t)$ is 4th component of the geodesic. Thus, we have the following theorem.

**Theorem 2** *In $\mathbb{E}_1^4$, let $S_2$ be semi-Riemann hypersurface of revolution of hyperbolic type with the profile manifold $M_2$ given by (7). If $\beta_2(t) = (x_1(t), x_2(t), 0, x_4(t))$ is an arbitrary geodesic on $M_2$, then the parametrization of a loxodrome on $S_2$ is given by*

$$\alpha_2(t) = (x_1(t), x_2(t), x_4(t)\sinh\theta(t), x_4(t)\cosh\theta(t)),$$

*where $\theta(t) = k\int_{t_0}^{t} \frac{1}{x_4(t)}dt$.*

The arc-length of a space-like or a time-like loxodrome can be given by the following result.

**Corollary 2** *The arc-length of the space-like or time-like loxodrome between $t_1$ and $t_2$ on the semi-Riemannian hypersurface of revolution of hyperbolic type in $\mathbb{E}_1^4$ is given by*

$$s = \Big| \int_{t_1}^{t_2} \sqrt{|x_1'^2(t) + x_2'^2(t) - x_4'^2(t) + k^2|}dt\Big|. \tag{13}$$

Also, we give the following remark.

**Remark 2** *If the semi-Riemannian hypersurfaces of revolution of hyperbolic type are reduced to the rotational surfaces of hyperbolic type in $\mathbb{E}_1^4$, then the equations of loxodromes on hypersurfaces of revolution of hyperbolic type coincide with the equations of loxodromes on rotational surfaces of hyperbolic type in $\mathbb{E}_1^4$ (see [5, 14]).*

Finally, we can give the following example.

**Example 2** *Pseudohyperbolic sphere $\mathbb{H}_0^3$ in $\mathbb{E}_1^4$ is a semi-Riemannian hypersurface of revolution of hyperbolic type with profile manifold $\mathbb{H}_0^2$. To obtain the parametrization of an arbitrary space-like loxodrome on $\mathbb{H}_0^3$, we first find the parametrization of an arbitrary space-like geodesic on $\mathbb{H}_0^2$. A geodesic on $\mathbb{H}_0^2$ is the intersection of $\mathbb{H}_0^2$ with a time-like 2-plane $P_2$ which passes through the origin (see [13]). Let us choose an orthonormal basis $\{e_1, e_4\}$ in $P_2$. Then, the space-like geodesic $H_0^2 \cap P_2$ is parametrized by*

$$\beta_2(t) = \sinh te_1 + \cosh te_4,\ t \in \mathbb{R}.$$

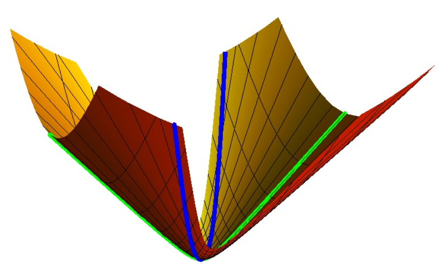

Figure 2: The projections of an open part of $\mathbb{H}_0^3$, loxodrome (blue), meridian (green).

*Similarly, from (12), we have*

$$\theta(t) = 2k\arctan\big(\tanh\frac{t}{2}\big)$$

*for $t_0 = 0$. As a result, the parametrization of the space-like loxodrome on $\mathbb{H}^3_0$ is given by*

$$\alpha_2(t) = (\sinh t, 0, \cosh t \sinh \theta(t), \cosh t \cosh \theta(t)),$$

*where $\theta(t) = 2k \arctan\left(\tanh \frac{t}{2}\right)$.*

*By using the Lorentzian space-like angle between two space-like vectors in $\mathbb{E}^4_1$ and taking the angle as $\pi/4$, we have $k = 1$ (see [13, 14]). Then, the arc-length of space-like loxodrome on $\mathbb{H}^3_0$ between $t_1 = -3$ and $t_2 = 3$ is equal to $6\sqrt{2}$.*

*The graphs of the projections of an open part of $\mathbb{H}^3_0$, loxodrome and meridian ($\theta = 0$) in $\mathbb{E}^3_1$ can be drawn, see Figure 2.*

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