Applied Mathematics E-Notes, 22(2022), 592-601 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Group Analysis Of The Second Order Linear Differential Equation With Variable Delay*

Jervin Zen Lobo†

Received 29 August 2021

Abstract

We perform group analysis of the second order linear differential equation with the most general variable time delay by developing a Lie type invariance condition using Taylor's theorem for a function of more than one variable. This condition is then used to obtain the symmetry algebra and make a complete group classification of this delay differential equation. We deduce certain compatibility conditions for the infinitesimals which lead to an extension of the symmetry algebra. Finally, we obtain a change of variables leading the differential equation with variable delay to be reduced to a differential equation with constant delay.

1 Introduction

In modeling several physical phenomena, we have to account for the fact that the unknown function may not only depend on an instant value of time t but also at earlier instants $t^* < t$. Such equations called delay differential equations are studied in [4, 5, 7] and have numerous applications which can be seen in [9]. Such equations cannot be easily solved owing to the presence of the delay terms and as such the properties of these equations can be effectively studied using group analysis. The theory of group analysis developed for ordinary and partial differential equations can be found in [2, 3, 8]. Literature on the stability, qualitative analysis and oscillation criteria for delay differential equations can be seen [1, 21, 22].

Lie's motivation to model the continuous symmetries of differential equations using Lie groups came from Galois who modeled the discrete symmetries of algebraic equations using Galois groups. While numerical methods are used in several problems such as [18], the invariance of such differential equations under these Lie groups is the only unified explanation for solving differential equations. As seen in [15], invariance laws are the essential requirement for reproducing experiments at different places and time.

Consider the global form of the Lie group, $\bar{t} = \phi(t, x; \epsilon)$, $\bar{x} = \psi(t, x; \epsilon)$. Let ϕ and ψ be analytic functions in t and x. Further, we assume that they have a convergent Taylor series in ϵ. Then, the infinitesimals are given by

$$\Phi(t, x) = \frac{\partial \phi(t, x; 0)}{\partial \epsilon} \quad \text{and} \quad \Psi(t, x) = \frac{\partial \psi(t, x; 0)}{\partial \epsilon}.$$

A definition of an admitted Lie group using Lie-Bäcklund operators for functional differential equations was proposed by Tanthanuch and Meleshko in [19, 20]. This definition was used in [11] to classify some first order delay differential equations. Later in [6], all classes of linear first-order delay ordinary differential systems having additional symmetries were identified and representatives of each class were provided. This definition helped Pue-on [17] to classify second order differential equations with constant delay. A group method is suggested in [10] to study functional differential equations, based on the search for symmetries of underdetermined systems of differential equations, using the principle of factorization. First-order linear neutral differential equations were classified in [12] by developing an invariance condition using Taylor's theorem. Group analysis of differential equations of fractional order are investigated in [16] and their exact

*Mathematics Subject Classifications: 34K06, 34C14, 22E99.

†Department of Mathematics, St. Xavier's College, Mapusa 403507, Goa, India

solutions have been found. An application of Lie symmetry analysis is seen in [13] to obtain invariant solutions of a model in the study of early carcinogenesis.

We will be developing an invariance condition for

$$f(t, x(t), x'(t), x(g(t)), x'(g(t)), x''(t)) = 0,$$

where f is defined on $I \times D^5$, for some open set D in $\mathbb{R}$ and an open interval I in $\mathbb{R}$. The function $g(t)$ is assumed to be sufficiently smooth with $g(t) < t$.

The invariance condition will be developed using Taylor's theorem for a function of several variables. An increase in delay is seen on using Lie-Bäcklund operators. No such increase in delay is observed using the technique developed here. Motivated by [14], which classifies a system of ordinary differential equations of the second order, we perform a symmetry analysis of second order differential equation with the most general time delay.

In the following section, we develop a novel approach to obtain the Lie invariance condition for second order differential equations with variable delay. This condition is used in Section 3 to obtain the symmetry algebra of the second order linear differential equation. Certain compatibility conditions are then developed in Section 4 to obtain the additional symmetries admitted by this differential equation. We then introduce a change of variables in Section 5 to transform the differential equation with variable delay to a differential equation with constant delay. We then present an example to illustrate our results.

2 Lie Type Invariance Condition for Second Order Delay Differential Equations

We develop a Lie type invariance condition for the most general time-delayed linear second order delay differential equation. The delay term is specified in order to determine this linear delay differential equation completely.

Theorem 1 *Let F be defined on a 6-dimensional space $I_1 \times D \times I_2 \times D^3$, where D is an open set in $\mathbb{R}$, and I_1, I_2 are any intervals in $\mathbb{R}$. Then the second order delay differential equation*

$$\frac{d^2x}{dt^2} = F(t, x, g(t), x(g(t)), x'(t), x'(g(t))), \tag{1}$$

with the notations, $\Phi^g = \Phi(g(t), x(g(t)))$ and $\Psi^g = \Psi(g(t), x(g(t)))$, has the Lie type invariance condition

$$\begin{aligned} & \Phi F_t + \Psi F_x + \Phi^g F_{g(t)} + \Psi^g F_{x(g(t))} + \Psi_{[t]} F_{x'(t)} + \Psi^g_{[t]} F_{x'(g(t))} \\ = \; & \Psi_{tt} + (2\Psi_{tx} - \Phi_{tt})x' + (\Psi_{xx} - 2\Phi_{tx})x'^2 - \Phi_{xx}x'^3 + (\Psi_x - 2\Phi_t)x'' - 3\Phi_x x' x''. \end{aligned}$$

where,

$$\Psi_{[t]} = D_t(\Psi) - x' D_t(\Phi),$$

$$\Psi_{[tt]} = D_t(\Psi_{[t]}) - x'' D_t(\Phi), \quad \text{where} \quad D_t = \frac{\partial}{\partial t} + x' \frac{\partial}{\partial x} + x'' \frac{\partial}{\partial x'} + \cdots ,$$

$$\Psi^g_{[t]} = (\Psi^g)_{g(t)} + ((\Psi^g)_{x(g(t))} - (\Phi^g)_{g(t)})x'(g(t)) - (x'(g(t)))^2 (\Phi^g)_{x(g(t))}.$$

Proof. We consider the infinitesimal form of the Lie group under which the delay differential equation is invariant, which is given by

$$\bar{t} = t + \epsilon\Phi(t, x) + O(\epsilon^2), \quad \bar{x} = x + \epsilon\Psi(t, x) + O(\epsilon^2),$$

where Φ and Ψ are defined in the introduction. It follows that

$$\overline{g(t)} = g(t) + \epsilon\Phi(g(t), x(g(t))) + O(\epsilon^2),$$

and

$$\overline{x(g(t))} = x(g(t)) + \epsilon\Psi(g(t), x(g(t))) + O(\epsilon^2).$$

Then

$$\begin{aligned}\frac{d\bar{x}}{d\bar{t}} &= \left[\frac{dx}{dt} + (\Psi_t + \Psi_x x')\epsilon + O(\epsilon^2)\right]\left[1 - (\Phi_t + \Phi_x x')\epsilon + O(\epsilon^2)\right] \\ &= \frac{dx}{dt} + [\Psi_t + (\Psi_x - \Phi_t)x' - \Phi_x x'^2]\epsilon + O(\epsilon^2).\end{aligned}$$

With the notation $D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x}$, we can write

$$\begin{aligned}\frac{d\bar{x}}{d\bar{t}} &= \frac{dx}{dt} + (D_t(\Psi) - x' D_t(\Phi))\epsilon + O(\epsilon^2) \\ &= \frac{dx}{dt} + \Psi_{[t]}\epsilon + O(\epsilon^2),\end{aligned}$$

where $\Psi_{[t]} = D_t(\Psi) - x' D_t(\Phi) = \Psi_t + (\Psi_x - \Phi_t)x' - \Phi_x x'^2$. Now,

$$\begin{aligned}\frac{d^2\bar{x}}{d\bar{t}^2} &= \left(\frac{d^2x}{dt^2} + D_t(\Psi_{[t]})\epsilon + O(\epsilon^2)\right)(1 - \epsilon D_t(\Phi) + O(\epsilon^2)) \\ &= \frac{d^2x}{dt^2} + (D_t(\Psi_{[t]}) - D_t(\Phi)x'')\epsilon + O(\epsilon^2).\end{aligned}$$

So $\Psi_{[tt]} = D_t(\Psi_{[t]}) - x'' D_t(\Phi)$. We will require to extend the definition of D_t because $\Psi_{[t]}$ contains t, x and x'. Hence we have

$$D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + x''\frac{\partial}{\partial x'} + \cdots .$$

Expanding $\Psi_{[tt]}$, gives

$$\Psi_{[tt]} = \Psi_{tt} + (2\Psi_{tx} - \Phi_{tt})x' + (\Psi_{xx} - 2\Phi_{tx})x'^2 - \Phi_{xx}x'^3 + (\Psi_x - 2\Phi_t)x'' - 3\Phi_x x' x''.$$

With the notations $\Phi^g = \Phi(g(t), x(g(t)))$ and $\Psi^g = \Psi(g(t), x(g(t)))$, it follows that

$$\overline{x}'(g(t)) = x'(g(t)) + [(\Psi^g)_{g(t)} + \left((\Psi^g)_{x(g(t))} - (\Phi^g)_{g(t)}\right) x'(g(t)) - ((x'(g(t)))^2 (\Phi^g)_{x(g(t))}]\epsilon + O(\epsilon^2).$$

Let $\Psi^g_{[t]} = (\Psi^g)_{g(t)} + ((\Psi^g)_{x(g(t))} - (\Phi^g)_{g(t)})x'(g(t)) - (x'(g(t)))^2 (\Phi^g)_{x(g(t))}$. For invariance,

$$\frac{d^2\bar{x}}{d\bar{t}^2} = F(\bar{t}, \bar{x}, \overline{g(t)}, \overline{x(g(t))}, \frac{d\bar{x}}{d\bar{t}}, \frac{d\bar{x}}{d\bar{t}}(\overline{g(t)})).$$

This gives

$$\frac{d^2x}{dt^2} + \Psi_{[tt]}\epsilon + O(\epsilon^2) = F(t + \epsilon\Phi + O(\epsilon^2), x + \epsilon\Psi + O(\epsilon^2), g(t) + \epsilon\Phi^g + O(\epsilon^2), x(g(t)) + \epsilon\Psi^g + O(\epsilon^2),$$

$$\begin{aligned}&\frac{dx}{dt} + \epsilon\Psi_{[t]} + O(\epsilon^2), \frac{dx}{dt}(g(t)) + \Psi^g_{[t]}\epsilon + O(\epsilon^2)) \\ =\; & F(t, x, g(t), x(g(t)), x'(t), x'(g(t))) + (\Phi F_t + \Psi F_x + \Phi^g F_{g(t)} + \Psi^g F_{x(g(t))} + \Psi_{[t]} F_{x'(t)} \\ &+ \Psi^g_{[t]} F_{x'(g(t))})\epsilon + O(\epsilon^2).\end{aligned}$$

Equating the coefficient of ϵ, we get

$$\begin{aligned}&\Phi F_t + \Psi F_x + \Phi^g F_{g(t)} + \Psi^g F_{x(g(t))} + \Psi_{[t]} F_{x'(t)} + \Psi^g_{[t]} F_{x'(g(t))} \\ =\; & \Psi_{tt} + (2\Psi_{tx} - \Phi_{tt})x' + (\Psi_{xx} - 2\Phi_{tx})x'^2 - \Phi_{xx}x'^3 + (\Psi_x - 2\Phi_t)x'' - 3\Phi_x x' x''. \qquad (2)\end{aligned}$$

Equation (2) gives us the desired invariance condition. ■

The prolonged operator for equation (1) is

$$\zeta = \Phi\frac{\partial}{\partial t} + \Phi^g\frac{\partial}{\partial(g(t))} + \Psi\frac{\partial}{\partial x} + \Psi^g\frac{\partial}{\partial x(g(t))}.$$

Equation (1) has the following extended operator:

$$\zeta^{(1)} = \Phi\frac{\partial}{\partial t} + \Phi^g\frac{\partial}{\partial(g(t))} + \Psi\frac{\partial}{\partial x} + \Psi^g\frac{\partial}{\partial x(g(t))} + \Psi_{[t]}\frac{\partial}{\partial x'} + \Psi^g_{[t]}\frac{\partial}{\partial x'(g(t))} + \Psi_{[tt]}\frac{\partial}{\partial x''}. \tag{3}$$

Defining

$$\Delta = \frac{d^2x}{dt^2} - F(t, x(t), g(t), x(g(t)), x'(t), x'(g(t))) = 0,$$

we get

$$\zeta^{(1)}\Delta = \Psi_{[tt]} - \Phi F_t - \Psi F_x - \Phi^g F_{g(t)} - \Psi^g F_{x(g(t))} - \Psi_{[t]}F_{x'(t)} - \Psi^g_{[t]}F_{x'(g(t))}. \tag{4}$$

Comparing equation (4) with equation (2), we get

$$\Psi_{[tt]} = \Psi_{tt} + (2\Psi_{tx} - \Phi_{tt})x' + (\Psi_{xx} - 2\Phi_{tx})x'^2 - \Phi_{xx}x'^3 + (\Psi_x - 2\Phi_t)x'' - 3\Phi_x x'x''.$$

We get an invariance condition for equation (1) as $\zeta^{(1)}\Delta \mid_{\Delta=0} = 0$, on substituting $x'' = F$ into $\zeta^{(1)}\Delta = 0$. This condition will be used in computing the determining equations.

3 Symmetries of the Second Order Differential Equation with the Most General Time Delay

We now turn to obtain the equivalent symmetries of

$$x''(t) + \alpha(t)x'(g(t)) + \beta(t)x(t) + \gamma(t)x'(t) + \rho(t)x(g(t)) = 0, \tag{5}$$

where $\alpha(t), \beta(t), \gamma(t)$ and $\rho(t)$ are twice differentiable functions. Applying the operator defined by equation (3) to the delay term, $t_1 = g(t)$, we get

$$\Phi_1 = g'(t)\Phi. \tag{6}$$

Applying the operator defined by equation (3) to equation (5), we get

$$\begin{aligned}&\Psi_{tt} + (2\Psi_{tx} - \Phi_{tt})x' + (\Psi_{xx} - 2\Phi_{tx})x'^2 - \Phi_{xx}x'^3 + (\Psi_x - 2\Phi_t)x'' - 3\Phi_x x'x''\\ &+\Phi\left[\alpha'(t)x'(t_1) + \beta'(t)x(t) + \gamma'(t)x'(t) + \rho'(t)x(t_1)\right] + \Psi\beta(t) + \rho(t)\Psi^g + \alpha(t)\Big[\Psi^g_{t_1}\\ &+((\Psi^g)_{x(t_1)} - (\Phi^g)_{t_1})x'(t_1) - (\Phi^g)_{x(t_1)}x'(t_1)^2\Big] + \gamma(t)\left[\Psi_t + (\Psi_x - \Phi_t)x' - \Phi_x x'^2\right] = 0.\end{aligned}$$

Using equation (5), we can substitute for x'' to get the following determining equations

$$\begin{aligned}&\Psi_{tt} + (2\Psi_{tx} - \Phi_{tt})x' + (\Psi_{xx} - 2\Phi_{tx})x'^2 - \Phi_{xx}x'^3 + (\Psi_x - 2\Phi_t)[-\alpha(t)x'(t_1) - \beta(t)x(t) - \gamma(t)x'(t)\\ &-\rho(t)x(t_1)] - 3\Phi_x x'[-\alpha(t)x'(t_1) - \beta(t)x(t) - \gamma(t)x'(t) - \rho(t)x(t_1)] + \Phi\Big[\alpha'(t)x'(t_1)\\ &+\beta'(t)x(t) + \gamma'(t)x'(t) + \rho'(t)x(t_1)\Big] + \Psi\beta(t) + \rho(t)\Psi^g\\ &+\alpha(t)[\Psi^g_{t_1} + ((\Psi^g)_{x(t_1)} - (\Phi^g)_{t_1})x'(t_1) - (\Phi^g)_{x(t_1)}x'(t_1)^2] + \gamma(t)[\Psi_t + (\Psi_x - \Phi_t)x' - \Phi_x x'^2] = 0.\end{aligned} \tag{7}$$

Splitting equation (7) with respect to $x'(t_1)^2$ and solving the resulting equation we get

$$\Phi(t, x) = A(t). \tag{8}$$

Splitting equation (7) with respect to ${x'}^2$, and using equation (8), we get

$$\Psi(t,x) = B(t)x + C(t). \tag{9}$$

Further, the Lie invariance condition on the delay equation $t_1 = g(t)$ gives

$$\Phi^g = g'(t)\Phi = g'(t)A(t) = A(t_1). \tag{10}$$

Substituting equations (8), (9) and (10) in equation (7), we get

$$\begin{aligned}
&B''(t)x + C''(t) + (2B'(t) - A''(t))x' + (B(t) - 2A'(t))[-\alpha(t)x'(t_1) - \beta(t)x(t) - \gamma(t)x'(t) - \rho(t)x(t_1)]\\
&+\Phi[-\alpha'(t)x'(t_1) - \beta'(t)x(t) - \gamma'(t)x'(t) - \rho'(t)x(t_1)] + \beta(t)(B(t)x + C(t)) + \gamma(t)[B'(t)x + C'(t)\\
&+(B(t) - A'(t))x'] + \rho(t)(B(t_1)x(t_1) + C(t_1)) + \alpha(t)[B'(t_1)x(t_1) + C'(t_1) + (B(t_1) - A'(t_1))x'(t_1)]\\
&= 0.
\end{aligned} \tag{11}$$

Splitting equation (11) with respect to x, we get

$$B''(t) + 2A'(t)\beta(t) + A(t)\beta'(t) + \gamma(t)B'(t) = 0. \tag{12}$$

Splitting equation (11) with respect to x', we get

$$2B'(t) - A''(t) + A(t)\gamma'(t) + A'(t)\gamma(t) = 0. \tag{13}$$

Splitting equation (11) with respect to $x(t_1)$, we get

$$2A'(t)\rho(t) + A(t)\rho'(t) + \rho(t)B(t_1) - \rho(t)B(t) + \alpha(t)B'(t_1) = 0. \tag{14}$$

Splitting equation (11) with respect to $x'(t_1)$, we get

$$2A'(t)\alpha(t) + A(t)\alpha'(t) - \alpha(t)B(t) + \alpha(t)B(t_1) - \alpha(t)A'(t_1) = 0. \tag{15}$$

Splitting equation (11) with respect to the constant term, we get

$$C''(t) + \alpha(t)C'(t_1) + \beta(t)C(t) + \gamma(t)C'(t) + \rho(t)C(t_1) = 0, \tag{16}$$

From equation (16), we see that $C(t)$ satisfies equation (5). Solving equation (13), we get

$$B(t) = \frac{1}{2}A'(t) - \frac{1}{2}\gamma(t)A(t) + c_1, \tag{17}$$

where c_1 is an arbitrary constant. Substituting equation (17) into equation (12), we get

$$A'''(t) + [4\beta(t) - 2\gamma'(t) - \gamma^2(t)]A'(t) + [2\beta'(t) - \gamma(t)\gamma'(t) - \gamma''(t)]A(t) = 0. \tag{18}$$

From equation (17), we have

$$B(t_1) = \frac{1}{2}A'(t_1) - \frac{1}{2}\gamma(t_1)A(t_1) + c_1, \tag{19}$$

and consequently,

$$B'(t_1) = \frac{1}{2}A''(t_1) - \frac{1}{2}(\gamma'(t_1)A(t_1) + \gamma(t_1)A'(t_1)). \tag{20}$$

Differentiating equation (10) with respect to t, we get

$$A'(t_1) = A'(g(t)) = \frac{g''(t)A(t)}{g'(t)} + A'(t), \tag{21}$$

and differentiating equation (21) with respect to t, we get

$$A''(t_1) = A''(g(t)) = \frac{g'''(t)}{[g'(t)]^2}A(t) + \frac{g''(t)}{[g'(t)]^2}A'(t) - \frac{[g''(t)]^2}{[g'(t)]^3}A(t) + \frac{A''(t)}{g(t)}. \tag{22}$$

Substituting equation (17) into equation (14) and using equations (19)–(22), we get

$$\begin{aligned}
&\frac{\alpha(t)}{g'(t)}A''(t) + \left[4\rho(t) + \alpha(t)\left(\frac{g''(t)}{[g'(t)]^2} - \gamma(g(t))\right)\right]A'(t) \\
&+\Big[2\rho'(t) + \alpha(t)\left(\frac{g'''(t)}{[g'(t)]^2} - \frac{[g''(t)]^2}{[g'(t)]^3} - \gamma'(g(t))g'(t) - \frac{\gamma(g(t))g''(t)}{g'(t)}\right) \\
&+\rho(t)\left(\frac{g''(t)}{g'(t)} - \gamma(g(t))g'(t) + \gamma(t)\right)\Big]A(t) = 0.
\end{aligned} \tag{23}$$

Substituting equation (17) into equation (14) and using equations (10) and equation (17), we get

$$\alpha(t)A'(t) + \left[\alpha'(t) + \frac{\alpha(t)}{2}\left(\gamma(t) - \gamma(g(t))g'(t) - \frac{g''(t)}{g'(t)}\right)\right]A(t) = 0. \tag{24}$$

Since $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\rho(t)$ and $g(t)$ are arbitrary, equations (18), (23) and (24) together with equation (10) have only one solution, namely $A(t) = 0$. Thus, we get

$$\Phi(t,x) = 0, \quad \Psi(t,x) = c_1 x + D(t).$$

Therefore, we obtain the infinitesimal generator of the corresponding Lie group to be

$$\zeta^* = \Phi(t,x)\frac{\partial}{\partial t} + \Psi(t,x)\frac{\partial}{\partial x} = (c_1 x + D(t))\frac{\partial}{\partial x}. \tag{25}$$

We have just established the following result:

Theorem 2 *The second order linear differential equation with the most general variable time delay given by*

$$x''(t) + \alpha(t)x'(g(t)) + \gamma(t)x'(t) + \beta(t)x(t) + \rho(t)x(g(t)) = 0,$$

for sufficiently smooth functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\rho(t)$ admits the two dimensional group generated by $\zeta_1^* = x\frac{\partial}{\partial x}$, $\zeta_2^* = D(t)\frac{\partial}{\partial x}$, where $D(t)$ solves equation (5).

It should be noted that if in place of equation (5) we had the corresponding nonhomogeneous delay differential equation given by

$$x''(t) + \alpha(t)x'(g(t)) + \gamma(t)x'(t) + \beta(t)x(t) + \rho(t)x(g(t)) = h(t), \tag{26}$$

for sufficiently smooth functions $\alpha(t), \beta(t), \gamma(t), \rho(t)$, and $h(t)$, then we can establish the following corollary:

Corollary 1 *The second order linear differential equation with the most general variable time delay given by equation (26) admits the two dimensional group generated by, $\zeta_1^* = (x - x_1(t))\frac{\partial}{\partial x}$, $\zeta_2^* = D(t)\frac{\partial}{\partial x}$, where $D(t)$ solves equation (5) and $x_1(t)$ is the particular solution of equation (26).*

4 Extensions of the Symmetry Algebra

In this section, we will discuss certain conditions on the obtained equations which result in non-trivial solutions and consequently make the symmetry algebra larger.

We can write equation (24) as

$$A'(t) = M(t)A(t), \tag{27}$$

where

$$M(t) = -\frac{1}{\alpha(t)}\left[\alpha'(t) + \frac{\alpha(t)}{2}\left(\gamma(t) - \gamma(g(t))g'(t) - \frac{g''(t)}{g'(t)}\right)\right].$$

Substituting equation (27) in equation (21) and using equation (10), we get

$$\left[M(g(t))\left[g'(t)\right]^2 - g''(t) - g'(t)M(t)\right]A(t) = 0.$$

Therefore, either $A(t) = 0$, or

$$M(g(t))\left[g'(t)\right]^2 = g''(t) + g'(t)M(t). \tag{28}$$

Since $A(t)$ must satisfy equation (27), we can write the general solution as

$$A(t) = c_2 \exp\left(\int_0^t M(y)dy\right), \tag{29}$$

where c_2 is an arbitrary non-zero constant.

We are now in a position to prove our result.

Theorem 3 *Under the hypothesis that equations (10), (18), (23) and (24) have nontrivial solutions, the delay differential equation given by (5) admits a symmetry algebra larger than the one given in Theorem 2. The symmetry algebra in this case is given by*

$$\zeta^{**} = A(t)\frac{\partial}{\partial t} + \frac{1}{2}(A'(t) - \gamma(t)A(t) + c_1)x\frac{\partial}{\partial x} + D(t)\frac{\partial}{\partial x}.$$

Proof. We prove the result in cases:
<u>Case 1:</u> Let $\alpha(t) \neq 0$, then equations (10) and (27) have a nontrivial solution if equation (28) holds. In addition, if equations (18) and (23) are satisfied by $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\rho(t)$ and $A(t)$, then

$$\Phi(t,x) = A(t), \quad \Psi(t,x) = \frac{1}{2}\left[A'(t) - \gamma(t)A(t) + c_1\right]x + D(t).$$

Therefore, we obtain the infinitesimal generator of the corresponding Lie group to be

$$\zeta^{*} = \Phi(t,x)\frac{\partial}{\partial t} + \Psi(t,x)\frac{\partial}{\partial x} = A(t)\frac{\partial}{\partial t} + \left(\frac{1}{2}\left[A'(t) - \gamma(t)A(t) + c_1\right]x + D(t)\right)\frac{\partial}{\partial x}. \tag{30}$$

We see that the symmetry algebra is larger.
<u>Case 2:</u> Let $\alpha(t) = 0,\ \rho(t) \neq 0$.
Then equation (24) identically holds. We can then rewrite equation (23) as

$$A'(t) = M_1(t)A(t), \tag{31}$$

where

$$M_1(t) = \left[-\frac{1}{2}\frac{\rho'(t)}{\rho(t)} - \frac{1}{4}\frac{g''(t)}{g'(t)} + \frac{1}{4}\gamma(g(t))g'(t) - \frac{1}{4}\gamma(t)\right].$$

Equations (10) and (31) have nontrivial solutions if $g(t)$ and $M_1(t)$ satisfy equation (28). The additional symmetry is seen when $\beta(t)$ and $\gamma(t)$ satisfy equation (18) and when $A(t) = c_3\int_0^t M(y)dy$, where c_3 is an arbitrary constant.

Once again the corresponding infinitesimal generator of the Lie group is given as in equation (30). ■

Remark 1 *The additional symmetries admitted are found to be*

$$^{a}\zeta^{**} = A(t)\frac{\partial}{\partial t} + \frac{1}{2}(A'(t) - \gamma(t)A(t))x\frac{\partial}{\partial x}. \tag{32}$$

5 A Reduction Result

In this section, we will use the additional symmetries obtained in Section 4 to transform the second order differential equation with the most general variable delay to a second order differential equation with constant delay.

Theorem 4 *Consider the second order differential equation*

$$x''(t) + \alpha(t)x'(g(t)) + \beta(t)x(t) + \gamma(t)x'(t) + \rho(t)x(g(t)) = 0,$$

studied in Section 3. Assume that this differential equation admits the additional symmetry given by equation (32). Then this differential equation can be transformed to

$$x''(t) + ax'(t-r) + bx(t) + cx(t-r) = 0, \tag{33}$$

where a, b, c and r are constants with $r > 0$, and $a^2 + c^2 \neq 0$. The Lie group is three dimensional and generated by

$${}^a\zeta_1^{**} = \frac{\partial}{\partial t}, \quad {}^a\zeta_2^{**} = D(t)\frac{\partial}{\partial x}, \quad {}^a\zeta_3^{**} = x\frac{\partial}{\partial x},$$

where $D(t)$ solves equation (5).

Proof. We find a change of variables $(t, x) \to (\tilde{t}, \tilde{x})$ that will "straighten out" the vector field ${}^a\zeta^{**}$ into ${}^a\tilde{\zeta}^{**}$ and also preserve the linearity of the delay differential equation. In order to preserve the linearity of the transformations, we choose

$$\tilde{t} = P(t), \quad \tilde{x} = U(t)x + V(t), \quad U(t) \neq 0.$$

In terms of the new variables, equation (32) takes the form

$$\begin{aligned} {}^a\zeta^{**} &= A(t)\left[P'(t)\frac{\partial}{\partial \tilde{t}} + U'(t)x\frac{\partial}{\partial \tilde{x}} + V'(t)\frac{\partial}{\partial \tilde{x}}\right] + \frac{1}{2}(A'(t) - \gamma(t)A(t))\left(xU(t)\frac{\partial}{\partial \tilde{x}}\right) \\ &= A(t)P'(t)\frac{\partial}{\partial \tilde{t}} + \left(\left(A(t)U'(t) + \frac{1}{2}(A'(t) - \gamma(t)A(t))U(t)\right)x + A(t)V'(t)\right)\frac{\partial}{\partial \tilde{x}}. \end{aligned} \tag{34}$$

Imposing,

$$A(t)P'(t) = 1, \quad A(t)U'(t) + \frac{1}{2}(A'(t) - \gamma(t)A(t))U(t) = 0, \quad V'(t) = 0,$$

we get

$$P(t) = \int \frac{1}{A(t)}dt + P_0, \quad U(t) = \frac{U_0}{\sqrt{A(t)}}\exp\left(\frac{1}{2}\int \gamma(t)dt\right), \quad V(t) = V_0,$$

where P_0, U_0, V_0 are constants.

Choose $P_0 = 0$, $U_0 = 1$, $V_0 = 0$, we obtain the new variables

$$\tilde{t} = \int \frac{1}{A(t)}dt, \quad \tilde{x} = \frac{x}{\sqrt{A(t)}}\exp\left(\frac{1}{2}\int \gamma(t)dt\right).$$

The obtained delay differential equation is linear and invariant under the above transformations. We can rewrite this delay differential equation (by dropping the the tilde on t and x) as

$$x''(t) + ax'(t-r) + bx(t) + cx(t-r) + dx'(t) = 0, \quad r > 0, \tag{35}$$

where d is a constant. The translational invariance coressponding to $\zeta^{**} = \frac{\partial}{\partial t}$ together with the delay condition would impose that a, b, c are constants. A final transformation $\tilde{x} = \exp\left(-\frac{1}{2}\tilde{d}t\right)x$ to equation (35) transforms it into equation (33) with the symmetry algebra given by

$${}^a\zeta_1^{**} = \frac{\partial}{\partial t}, \quad {}^a\zeta_2^{**} = D(t)\frac{\partial}{\partial x}, \quad {}^a\zeta_3^{**} = x\frac{\partial}{\partial x}.$$

■

6 An Illustrative Example

We know turn to see the symmetries and the Lie group admitted by

$$x''(t) = x(t - \pi). \tag{36}$$

In this example $g(t) = t - r$, where $r = \pi > 0$ is a constant delay. It can be readily seen that $x(t) = \cos t$ is a solution of this delay differential equation. Acting the operator given by equation (3) to the delay condition $g(t) = t - r$, we get $\Phi^r = \Phi$, where $\Phi^r = \Phi(t - r, x(t - r))$. Acting the operator given by equation (3) to equation (36), we get

$$\Psi_{tt} + (2\Psi_{tx} - \Phi_{tt})x' + (\Psi_{xx} - 2\Phi_{tx}){x'}^2 - \Phi_{xx}{x'}^3 + (\Psi_x - 2\Phi_t)x_r - 3\Phi_x x' x_r = \Psi^r,$$

where $x_r = x(t-r)$ and $\Psi^r = \Psi(t-r, x(t-r))$. Splitting this equation with respect to the arbitrary elements and solving the resulting system of partial differential equations, we get $\Phi(t, x) = c$, $\Psi(t, x) = \cos t$, where c is an arbitrary constant. The Lie group is generated by

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \cos t \frac{\partial}{\partial x}.$$

Solving the system,

$$\frac{d\bar{t}}{d\epsilon} = \Phi(\bar{t}, \bar{x}) = c, \quad \frac{d\bar{x}}{d\epsilon} = \Psi(\bar{t}, \bar{x}) = \cos \bar{t},$$

subject to the conditions, $\bar{t} = t$, $\bar{x} = x$, when $\epsilon = 0$, we see that the delay differential equation given by equation (33) is invariant under the Lie group

$$\bar{t} = t + c\epsilon, \quad \bar{x} = x + \epsilon \cos(t + c\epsilon).$$

7 Conclusion

We have obtained a Lie type invariance condition for differential equations of the second order and with the most general variable delay by using Taylor's theorem for a function of more than one variable. Using this condition, we have computed the symmetry algebra admitted by the delay differential equation. Further, we have obtained certain compatibility conditions under which the delay differential equation possess additional symmetries and we have found these symmetries. Finally, we obtain a suitable change of variables reducing the differential equation with variable delay to a differential equation with constant delay. We have also found the symmetry algebra in this case.

References

[1] Y. Altun, Stability of fractional neutral systems with time varying delay based on delay decomposition method, New Trends in Mathematical Sciences, 4(2020), 1–8.

[2] D. Arrigo, Symmetry Analysis of Differential Equations, John Wiley and sons, New York, 2015.

[3] G. Bluman and S. Kumei, Symmetries and Differential Equations, Springer-Verlag, New York, 1989.

[4] R. D. Driver, Ordinary and Delay Differential Equations, Springer-Verlag, New York, 1977.

[5] S. G. Deo, V. Lakshmikantham and V. Raghavendra, Textbook of Ordinary Differential Equations, McGraw Hill Education Private Limited, India, 2013.

[6] V. A. Dorodnitsyn, R. Kozlov, S. V. Meleshko and P. Winternitz, Lie group classification of first-order delay ordinary differential equations, Journal of Physics A: Mathematical and Theoretical, 51(20)(2018), 33p.

[7] J. Hale, Functional Differential Equations, Springer-Verlag, New York, 1977.

[8] N. H. Ibragimov, CRC Handbook of Lie Group Analysis of differential equations, CRC Press, Boca Raton, 1996.

[9] Y. Kyrychko and S. J. Hogan, On use of delay equations in engineering applications, Journal of Vibrations and Control, 16(2010), 943–960.

[10] L. V. Linchuk, On group analysis of functional differential equations, Proceedings of the International Conference MOGRAN, Ufa. USATU Publishers, (2001), 111–115.

[11] J. Z. Lobo and Y. S. Valaulikar, On defining an admitted Lie group for first order delay differential equations with constant coefficients, J. Appl. Sci. Comput., 5(2018), 1301–1307.

[12] J. Z. Lobo and Y. S. Valaulikar, Lie Symmetries of first order neutral differential equations, J. Appl. Math. Comput. Mech., 18(2019), 29–40.

[13] M. B. Matadi, Lie symmetry analysis of early carcinogenesis model, Appl. Math. E-Notes, 18(2018), 238–249.

[14] S. V. Meleshko, S. Moyo and G. F. Oguis, On the group classification of systems of second-order linear ordinary differential equations with constant coefficients, J. Math. Anal. Appl., 410(2014), 341–347.

[15] F. Oliveri, Lie symmetries of differential equations: classical results and recent contributions, Symmetry Journal, 2(2010), 658–706.

[16] P. Ponnusamy and R. Sahadevan, Lie symmetry analysis and exact solution of certain fractional ordinary differential equations, Nonlinear Dyn., 89(2017), 305–319.

[17] P. Pue-on, Group classification of second-order delay differential equations, Commun. Nonlinear Sci. Numer. Simul., 15(2009), 1444–1453.

[18] M. A. Rasheed, On blow-up solutions of a parabolic system coupled in both equations and boundary conditions, Baghdad Science Journal, 18(2021), 315–321.

[19] J. Tanthanuch and S. V. Meleshko, Application of group analysis to delay differential equations, Proceedings of the 16th International Symposium on Nonlinear Acoustic, Moscow, Russia, (2002), 607–610.

[20] J. Tanthanuch and S, V. Meleshko, On definition of an admitted Lie group for functional differential equations, Commun. Nonlinear Sci. Numer. Simul., 9(2004), 117–125.

[21] C. Tunç and O. Tunç, Qualitative analysis for a variable delay system of differential equations of second order, Journal of Taibah University for Science, 13(2019), 468–477.

[22] E. Tunç, L. Coraklik and O. Özdemir, Oscillation results for second order functional differential equations, J. Comput. Anal. Appl., 18(2015), 61–70.