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On A Class Of Non-Cooperative Singular p-Laplacian Systems With Multiple Parameters*

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Abstract

We establish existence of positive solutions to the p-Laplacian singular systems with multiple parameters of the form

$$\begin{cases} -\Delta_p u = \lambda_1 f(v) + \mu_1 h(u), & x \in \Omega, \\ -\Delta_p v = \lambda_2 g(u) + \mu_2 k(v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div\,(|\nabla z|^{p-2}\nabla z)$, $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\lambda_1, \lambda_2, \mu_1$ and μ_2 are nonnegative parameters. Here $f, g, h, k : [0, \infty) \to \mathbb{R}$ are possibly singular at 0 and are not required to be positive or nondecreasing.

1 Introduction

The aim of this paper is to provide existence results for positive solutions of the following p-Laplacian singular system with multiple parameters

$$\begin{cases} -\Delta_p u = \lambda_1 f(v) + \mu_1 h(u), & x \in \Omega, \\ -\Delta_p v = \lambda_2 g(u) + \mu_2 k(v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div\,(|\nabla z|^{p-2}\nabla z)$, $p > 1$, Ω is a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary $\partial\Omega$, λ_1, λ_2, μ_1 and μ_2 are nonnegative parameters. Here $f, g, h, k : [0, \infty) \to \mathbb{R}$ are possibly singular at 0 and are not required to be positive or nondecreasing.

Elliptic equations involving the p-Laplace operator arise in some physical models like the flow of non-Newtonian fluids: pseudo-plastic fluids correspond to $p \in (1, 2)$ while dilatant fluids correspond to $p > 2$. The case $p = 2$ expresses Newtonian fluids [2]. On the other hand, systems like (1) describe various nonlinear phenomena such as chemical reactions, pattern formation, population evolution where, for example, u and v represent the concentrations of two species in the process. As a consequence, positive solutions of (1) are of interest.

Several methods have been used to treat p-Laplacian systems. In the scalar case, weak solutions can be obtained through variational methods which provide critical points of the corresponding energy functional (see [5], [15], [14]), an approach which is also fruitful in the case of potential systems i.e, the nonlinearities on the right hand side are the gradient of a C^1-functional [16], [13], [20]. However, due to the loss of the variational structure, the treatment of nonvariational systems like (1) is more complicated and is based mostly on topological methods [6].

Dalmasso in [4] discussed the system (1) when $p = q = 2$, $\mu_1 = \mu_2 = 0$, $\lambda_1 = \lambda_2$ and f, g are increasing and $f, g \geq 0$. In [10], Hai and Shivaji extended the study of [4] to the case when no sign conditions on $f(0)$ or $g(0)$ were required, and in [9] they extended this study to the case when $p = q > 1$. In [1], authors establish

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an existence result for the system (1) when $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ are large. In addition, for the case when $f(0) = h(0) = g(0) = \gamma(0) = 0$, authors discuss a multiplicity result for $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ large. Recently in [11], the author investigates the singular p-Laplacian system

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega, \\ -\Delta_p v = \lambda g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $f, g : [0,\infty) \to [0,\infty)$ are possibly singular at 0 and are not required to be positive or nondecreasing. Here we further extend this study to classes of systems with multiple parameters of the form (1). Our approach is based on the method of sub-and supersolutions.

One can refer to [3, 7, 8, 18, 19] for some recent existence results of elliptic and parabolic problems. In [7], the authors extended the results of [1, 9] to generalized elliptic systems involving $(p_1, p_2, ...p_m)$ Laplacian operator with zero Dirichlet boundary condition in bounded domain by using sub-super solutions method. Further, the existence of at least three solutions for a fractional system involving the p-Laplacian with Lipschitz nonlinearity is demonstrated in [8], by means of the three critical points theorem. Also in [3], the authors studied the existence of weak positive solutions for a new class of the system of parabolic differential equations with respect to the symmetry conditions by using the sub-super solutions method.

This paper is organized as follows. After making assumptions, we state our existence result in Section 2, we prove Theorem 2.1 in Section 3.

2 Our Results

We make the following assumptions:

(H1) $f, g, h, k : [0,\infty) \to \mathbb{R}$ are continuous.

(H2) There exist numbers $L, A > 0$ such that $f(t)$, $g(t)$, $h(t)$, $k(t) > L$ for $t > A$ and

$$\lim_{t\to\infty} \frac{f^{\frac{1}{p-1}}(cg(t)^{\frac{1}{p-1}})}{t} = 0 \quad \text{for all } c > 0.$$

(H3) There exists a number $\delta \in (0,1)$ such that:

$$\lim_{t\to 0^+} \sup t^{\delta} |f(t)| < \infty, \quad \lim_{t\to 0^+} \sup t^{\delta} |g(t)| < \infty,$$

and

$$\lim_{t\to 0^+} \sup t^{\delta} |h(t)| < \infty, \quad \lim_{t\to 0^+} \sup t^{\delta} |k(t)| < \infty.$$

(H4)

$$\lim_{t\to\infty} \frac{h(t)}{t^{p-1}} = \lim_{t\to\infty} \frac{k(t)}{t^{p-1}} = 0.$$

Now we are ready to state our existence result.

Theorem 1 *Let (H1)–(H4) hold. Then (1) has a positive solution provided $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ are large.*

We shall establish our existence result via the method of sub and supersolutions. A pair of nonnegative functions (Φ_1, Φ_2), (Ψ_1, Ψ_2) are called a subsolution and supersolution of (1) if they satisfy $\Phi_i \le 0 \le \Psi_i$ on $\partial\Omega$ for $i = 1, 2$ and for all $w \in W = \{w \in C_0^\infty(\Omega) | w \ge 0,\ x \in \Omega\}$,

$$\int_\Omega |\nabla\Phi_1|^{p-2} \nabla\Phi_1 \cdot \nabla w \, dx \le \int_\Omega \lambda_1 f(\tilde{v}) w \, dx + \int_\Omega \mu_1 h(\Phi_1) w \, dx,$$

$$\int_\Omega |\nabla \Phi_2|^{p-2}\, \nabla \Phi_2 \cdot \nabla w\, dx \leq \int_\Omega \lambda_2 g(\tilde{u})\, w\, dx + \int_\Omega \mu_2 k(\Phi_2)\, w\, dx,$$

where $\tilde{u} \in [\Phi_1, \Psi_1]$, $\tilde{v} \in [\Phi_2, \Psi_2]$, and

$$\int_\Omega |\nabla\, \Psi_1|^{p-2}\, \nabla\, \Psi_1 \cdot \nabla w\, dx \geq \int_\Omega \lambda_1 f(\hat{v}) w\, dx + \int_\Omega \mu_1 h(\Psi_1) w\, dx,$$

$$\int_\Omega |\nabla\, \Psi_2|^{p-2}\, \nabla\, \Psi_2 \cdot \nabla w\, dx \geq \int_\Omega \lambda_2 g(\hat{u}) w\, dx + \int_\Omega \mu_2 k(\Psi_2)\, w\, dx$$

where $\hat{u} \in [\Phi_1, \Psi_1]$, $\hat{v} \in [\Phi_2, \Psi_2]$. Here $[\Phi_i, \Psi_i] = \{u \in C(\bar{\Omega}) : \Phi_i \leq u_i \leq \Psi_i,\ x \in \Omega\}$ for $i = 1, 2$.

Then the following result holds:

Lemma 1 (See [11, 17]) *Suppose there exist sub and super-solutions (Φ_1, Φ_2) and (Ψ_1, Ψ_2) respectively of (1) such that $(\Phi_1, \Phi_2) \leq (\Psi_1, \Psi_2)$. Then (1) has a solution (u, v) such that $(\Phi_1, \Phi_2) \leq (u, v) \leq (\Psi_1, \Psi_2)$.*

3 Preliminary Results and Proof of Theorem 1

We first need some preliminary results. We shall denote by $\|.\|_q$, $|.|_1$ and $|.|_{1,\alpha}$ the norms in $L^q(\Omega)$, $C^1(\bar{\Omega})$ and $C^{1,\alpha}(\bar{\Omega})$ respectively.

Lemma 2 (See [12]) *Let $h \in L^\infty_{loc}(\Omega)$ and suppose there exist numbers $\gamma \in (0, 1)$ and $C > 0$ such that*

$$|h(x)| \leq \frac{C}{d^\gamma(x)} \tag{2}$$

for a.e. $x \in \Omega$, where $d(x)$ denote the distance from x to the boundary of Ω. Let $u \in W_0^{1,p}(\Omega)$ be the solution of

$$\begin{cases} -\Delta_p u = h, & in\ \ \Omega, \\ \ \ u = 0, & on\ \ \partial\Omega. \end{cases}$$

Then there exist constants $\alpha \in (0, 1)$ and $M > 0$ depending only on C, γ, Ω such that $u \in C_0^{1,\alpha}(\bar{\Omega})$ and $|u|_{1,\alpha} < M$.

Corollary 1 (See [12]) *Let $\varepsilon > 0$ and $h, \tilde{h} \in L^\infty_{loc}(\Omega)$ satisfy (2) with $h \geq 0$, $h \neq 0$. Let $u, u_\varepsilon \in W_0^{1,p}(\Omega)$ be, respectively, the solutions of*

$$\begin{cases} -\Delta_p u = h, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

and

$$-\Delta_p u_\varepsilon = \begin{cases} h, & if\ d(x) > \varepsilon, \\ \tilde{h}, & if\ d(x) < \varepsilon. \end{cases}$$

Then $u_\varepsilon \geq u/2$ in Ω for ε small enough.

Proof of Theorem 1. Let $\varepsilon > 0$ and $z, \psi, \psi_\varepsilon$ satisfy

$$\begin{cases} -\Delta_p z = \frac{1}{z^\delta} & \text{in } \ \Omega, \\ z = 0 & \text{on } \ \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_p \psi = 1 & \text{in } \ \Omega, \\ \psi = 0 & \text{on } \ \partial\Omega, \end{cases}$$

$$-\Delta_p \psi_\varepsilon = \begin{cases} L & \text{if } d(x) > \varepsilon, \\ -\frac{1}{z^\delta} & \text{if } d(x) < \varepsilon, \end{cases} \quad \text{and} \quad \psi_\varepsilon = 0 \ \text{ on } \ \partial\Omega,$$

respectively. Then by Corollary 1,

$$\psi_\varepsilon \geq \left(L^{\frac{1}{p-1}}/2\right)\psi, \quad \text{in } \Omega,$$

if ε is small enough, which we shall assume. By (H2) and (H3), there exists $b > 0$ such that

$$|f(t)| \leq \frac{b}{t^\delta}, \ |g(t)| \leq \frac{b}{t^\delta}$$

for $t < A$, and

$$f(t) \geq -\frac{b}{t^\delta}, \ g(t) \geq -\frac{b}{t^\delta}$$

for $t > 0$. Define

$$\tilde{f}(t) = \begin{cases} \sup\limits_{A \leq s \leq t} f(s) & \text{if } t \geq A \\ f(A) & \text{if } t < A \end{cases}, \tilde{g}(t) = \begin{cases} \sup\limits_{A \leq s \leq t} g(s) & \text{if } t \geq A \\ g(A) & \text{if } t < A. \end{cases}$$

Then $\tilde{f}$ and $\tilde{g}$ are nondecreasing and

$$\lim_{t \longrightarrow \infty} \frac{\tilde{f}^{\frac{1}{p-1}}\left(c\tilde{g}^{\frac{1}{p-1}}(t)\right)}{t} = 0$$

for each $c > 0$. Hence there exists $M \gg 1$ so that

$$(\lambda_1 + \mu_1)\left[b + \|z\|_\infty^\delta \tilde{f}\left((\lambda_2 + \mu_2)^{\frac{1}{p-1}}\|z\|_\infty\left(b + \|z\|_\infty^\delta \tilde{g}(M\|z\|_\infty)\right)^{\frac{1}{p-1}}\right)\right] \leq \frac{1}{2}M^{p-1}. \tag{3}$$

Define

$$\Phi_i = (\lambda_i + \mu_i)^{\frac{1}{p-1}}\psi_\varepsilon, \ \ i = 1, 2, \ \ \Psi_1 = Mz, \ \ \Psi_2 = (\lambda_2 + \mu_2)^{\frac{1}{p-1}}\left(b + \|z\|_\infty^\delta \tilde{g}(M\|z\|_\infty)\right)^{\frac{1}{p-1}} z.$$

We shall verify that $\Phi = (\Phi_1, \Phi_2)$ and $\Psi = (\Psi_1, \Psi_2)$ form a system of sub- and supersolutions for (1) if $(\lambda_i + \mu_i)$ for $i = 1, 2$ are large enough.

By increasing b, we can assume that

$$\psi_\varepsilon \leq b^{\frac{1}{p-1}} z \ \ \text{in } \Omega.$$

Next, take $(\lambda_i + \mu_i) > 0$ for $i = 1, 2$ large enough so that

$$(\lambda_i + \mu_i)^{\frac{1}{p-1}}\left(L^{\frac{1}{p-1}}/2\right)\psi(x) > A$$

for $d(x) > \varepsilon$, and

$$\Phi_i \geq \max\left(1, b^{1/\delta}\right) z \ \ \text{in } \Omega.$$

Then, for $M \gg (\lambda_1 + \mu_1)^{\frac{1}{p-1}}$, we have $\Phi_i \leq \Psi_i$ in Ω, $i = 1, 2$. Let $w \in W_0^{1,p}(\Omega)$ with $w \geq 0$. A calculation shows that

$$\int |\nabla\Phi_1|^{p-2}\nabla\Phi_1.\nabla w dx = (\lambda_1 + \mu_1) L \int_{d > \varepsilon} w dx - (\lambda_1 + \mu_1)\int_{d < \varepsilon} \frac{w}{z^\delta} dx. \tag{4}$$

For $\tilde{v} \in [\Phi_2, \Psi_2]$ and $d(x) > \varepsilon$, we have

$$\tilde{v} \geq \Phi_2 \geq (\lambda_2 + \mu_2)^{\frac{1}{p-1}}\left(L^{\frac{1}{p-1}}/2\right)\psi(x) > A,$$

which implies

$$\lambda_1 \int_{d > \varepsilon} f(\tilde{v}) w dx + \mu_1 \int_{d > \varepsilon} h(\Phi_1) w dx \geq \lambda_1 L \int_{d > \varepsilon} w dx + \mu_1 L \int_{d > \varepsilon} w dx = (\lambda_1 + \mu_1) L \int_{d > \varepsilon} w dx. \tag{5}$$

On the other hand,

$$\begin{aligned}\lambda_1 \int_{d<\varepsilon} f(\tilde{v})\, wdx + \mu_1 \int_{d<\varepsilon} h(\Phi_1)\, wdx &\geq -\lambda_1 b \int_{d<\varepsilon} \frac{w}{\tilde{v}^\delta} dx - \mu_1 b \int_{d<\varepsilon} \frac{w}{\Phi_1^\delta} dx \\ &\geq -\lambda_1 \int_{d<\varepsilon} \frac{w}{z^\delta} dx - \mu_1 \int_{d<\varepsilon} \frac{w}{z^\delta} dx \\ &= -(\lambda_1 + \mu_1) \int_{d<\varepsilon} \frac{w}{z^\delta} dx. \end{aligned} \tag{6}$$

Combining (4)–(6), we get

$$\int_\Omega |\nabla \Phi_1|^{p-2} \nabla \Phi_1 \cdot \nabla w \, dx \leq \int_\Omega \lambda_1 f(\tilde{v}) w \, dx + \int_\Omega \mu_1 h(\Phi_1) w \, dx.$$

Similarly

$$\int_\Omega |\nabla \Phi_2|^{p-2} \nabla \Phi_2 \cdot \nabla w \, dx \leq \int_\Omega \lambda_2 g(\tilde{u})\, w \, dx + \int_\Omega \mu_2 k(\Phi_2)\, w \, dx.$$

Next, since

$$f(t) \leq \frac{b}{t^\delta} + \tilde{f}(t),\ g(t) \leq \frac{b}{t^\delta} + \tilde{g}(t)$$

for $t > 0$, we deduce from (3) that

$$\int_\Omega |\nabla \Psi_1|^{p-2} \nabla \Psi_1 . \nabla w dx = M^{p-1} \int \frac{w}{z^\delta} dx. \tag{7}$$

On the other hand,

$$\begin{aligned}\lambda_1 \int_\Omega f(\hat{v})\, wdx &\leq \lambda_1 \int_\Omega \left(\frac{b}{z^\delta} + \tilde{f}\left((\lambda_2+\mu_2)^{\frac{1}{p-1}} \|z\|_\infty \left(b + \|z\|_\infty^\delta \tilde{g}(M \|z\|_\infty) \right)^{\frac{1}{p-1}} \right) \right) wdx \\ &\leq \frac{1}{2} M^{p-1} \int_\Omega \frac{w}{z^\delta} dx \end{aligned} \tag{8}$$

and by (H4), we have

$$\mu_1 \int_\Omega h(\Psi_1)\, wdx \leq \frac{1}{2} M^{p-1} \int_\Omega \frac{w}{\|z\|_\infty^\delta} dx \leq \frac{1}{2} M^{p-1} \int_\Omega \frac{w}{z^\delta} dx. \tag{9}$$

Combining (7)–(9), we get

$$\int_\Omega |\nabla\, \Psi_1|^{p-2} \nabla\, \Psi_1 \cdot \nabla w \, dx \geq \int_\Omega \lambda_1 f(\hat{v}) w \, dx + \int_\Omega \mu_1 h(\Psi_1) w \, dx.$$

Also

$$\int_\Omega |\nabla \Psi_2|^{p-2} \nabla \Psi_2 . \nabla w dx = (\lambda_2 + \mu_2) \int_\Omega \left(\frac{b + \|z\|_\infty^\delta \tilde{g}(M \|z\|_\infty)}{z^\delta} \right) wdx. \tag{10}$$

Then

$$\lambda_2 \int_\Omega g(\hat{u})\, wdx \leq \lambda_2 \int_\Omega \left(\frac{b}{z^\delta} + \tilde{g}(\hat{u}) \right) wdx \leq \lambda_2 \int_\Omega \left(\frac{b + \|z\|_\infty^\delta \tilde{g}(M \|z\|_\infty)}{z^\delta} \right) wdx. \tag{11}$$

Again by (H4), we have

$$\begin{aligned}\mu_2 \int_\Omega k(\Psi_2)\, wdx &\leq \mu_2 \int_\Omega \left(\frac{b + \|z\|_\infty^\delta \tilde{g}(M \|z\|_\infty)}{\|z\|_\infty^\delta} \right) wdx \\ &\leq \mu_2 \int_\Omega \left(\frac{b + \|z\|_\infty^\delta \tilde{g}(M \|z\|_\infty)}{z^\delta} \right) wdx. \end{aligned} \tag{12}$$

Combining (10)–(12), we get

$$\int_\Omega |\nabla \Psi_2|^{p-2} \nabla \Psi_2 \cdot \nabla w \, dx \geq \int_\Omega \lambda_2 g(\hat{u}) w \, dx + \int_\Omega \mu_2 k(\Psi_2) \, w \, dx.$$

We see that Φ and Ψ form a system of sub- and supersolutions for (1) which completes the proof of Theorem 2.1. ∎

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