Applied Mathematics E-Notes, 22(2022), 585-591 ©
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ISSN 1607-2510

On The Solutions Of Boundary Value Problem Arising In Mixed Convection*

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Received 6 August 2021

Abstract

In this article, we are interested in studying solutions of the differential equation $f''' + ff'' + \beta f'(f' - 1) = 0$ on $[0, +\infty)$ with $\beta \geq 1$, satisfying the boundary conditions $f(0) = a \in \mathbb{R}$, $f'(0) = b < 0$ and $f'(t) \to \lambda$ as $t \to +\infty$, where $\lambda \in \{0, 1\}$. This problem arises in the study of mixed convection adjacent to surfaces embedded in a porous medium using the boundary layer approximation.

1 Introduction

Let $\beta \in \mathbb{R}$. We consider the third order autonomous nonlinear differential equation:

$$f''' + ff'' + \beta f'(f' - 1) = 0, \qquad t \in [0, +\infty). \tag{1}$$

Associated with the above equation, we have the following boundary value problem:

$$\begin{cases} f''' + ff'' + \beta f'(f' - 1) = 0, \\ f(0) = a, \\ f'(0) = b, \\ f'(t) \to \lambda \text{ as } t \to +\infty, \end{cases} \qquad (\mathcal{P}_{\beta;a,b,\lambda})$$

where $a, b, \beta \in \mathbb{R}$ and $\lambda \in \{0, 1\}$ with $b < 0$ and $\beta \geq 1$. The problem $(\mathcal{P}_{\beta;a,b,\lambda})$ in which the mix of three parameters β, a and b is of capital importance, where β have a law profile of power, a prescribed power law of the distance from the leading edge for the temperature and $b = \frac{R_a}{P_e} - 1$ is the mixed convection parameter, with R_a being the Rayleigh number and P_e is the Péclet number (see [2, 4]). Let us notice that, if $\lambda \notin \{0, 1\}$, then the problem $(\mathcal{P}_{\beta;a,b,\lambda})$ does not admit a solution (see [6, 8]).

The problem $(\mathcal{P}_{\beta;a,b,\lambda})$ was considered in [9] with $\beta < 0$, the reference contains also some results concerning the existence and uniqueness of the convex and concave solution of $(\mathcal{P}_{\beta;a,b,1})$ where $-2 < \beta < 0$ and $b > 0$. The results of [6] generalize the ones of [9] and some of [10]. In [11] and [12], some results were found for the problem $(\mathcal{P}_{\beta;0,b,1})$ with $-2 < \beta < 0$ and $b < 0$, the method used by the authors allows them to prove the existence of a convex solution by introducing a singular integral equation obtained from Eq (1) by a crocco-type transformation. The problem $(\mathcal{P}_{\beta;a,b,\lambda})$ with $\beta = 0$ is more commonly known as the Blasius problem (see [7]). The case $0 < \beta \leq 1$, where $a \geq 0$, $b \geq 0$, was treated in [1]. In [2], the authors studied the $(\mathcal{P}_{\beta;a,b,\lambda})$ with $0 < \beta < 1$, $a \in \mathbb{R}$ and $b < 0$.

We have only partial results in [1] about the case $\beta > 1$, $a \geq 0$ and $b \geq 0$. The problem in [3] and [5] come from the study of free convection boundary layer. Our goal, in this paper is to investigate the problem $(\mathcal{P}_{\beta;a,b,\lambda})$, with $\beta \geq 1$, $a \in \mathbb{R}$ and $b < 0$. We demonstrate some existence, non-existence and sign of concave,

*Mathematics Subject Classifications: 34B15, 34C11, 76D10.

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convex and convex-concave solutions. In what follows, we denote by f_c a solution of the initial value problem below and by $[0, T_c)$ the right maximal interval of its existence.

$$\begin{cases} f''' + ff'' + \beta f'(f'-1) = 0, \\ f(0) = a, \\ f'(0) = b, \\ f''(0) = c. \end{cases} \qquad (\mathcal{Q}_{\beta;a,b,c})$$

To study the boundary value problem $(\mathcal{P}_{\beta;a,b,\lambda})$, we will use the shooting method, which consists of finding the values of a real parameter c for which f_c is the solution of $(\mathcal{Q}_{\beta;a,b,c})$, such that $T_c = +\infty$ and $f_c'(t) \to \lambda$ as $t \to +\infty$.

2 On Blasius Equation

In this section, we recall some results about subsolutions and supersolutions of the Blasius equation ($f''' + ff'' = 0$) (see [6]). Remark that the constant function and the function $h_\tau : t \mapsto \frac{3}{t-\tau}$ for any $\tau \in \mathbb{R}$ with $t \neq \tau$, are solutions of the Blasius equation.

Definition 1 ([6]) *Let $I \subset \mathbb{R}$ be an interval, We say that a function $f : I \to \mathbb{R}$ is a subsolution (resp. a supersolution) of the Blasius equation if f is of class C^3 and if $f''' + ff'' \leq 0$ on I (resp. $f''' + ff'' \geq 0$ on I).*

Definition 2 ([6]) *Let $\epsilon > 0$, We say that f is a ϵ-subsolution (resp. a ϵ-supersolution) of the Blasius equation if f is of class C^3 and if $f''' + ff'' \leq -\epsilon$ on I (resp. $f''' + ff'' \geq \epsilon$ on I).*

Proposition 1 *Let $t_0 \in \mathbb{R}$. There does not exist nonpositive concave subsolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. See [6], Proposition 2.11. ■

Proposition 2 *Let $t_0 \in \mathbb{R}$. There does not exist nonpositive convex supersolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. See [6], Proposition 2.5. ■

Proposition 3 *Let $t_0 \in \mathbb{R}$. There does not exist ϵ-subsolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. See [6], Proposition 2.18. ■

3 Preliminary Results

Proposition 4 *Let f be a solution of Eq. (1) on some maximal interval $I = (T_-, T_+)$.*

1. *If F is any primitive function of f on I, then $(f''e^F)' = -\beta f'(f'-1)e^F$.*
2. *Assume that $T_+ = +\infty$ and that $f'(t) \to \lambda \in \mathbb{R}$ as $t \to +\infty$. If, moreover, f is of constant sign at infinity, then $f''(t) \to 0$ as $t \to +\infty$.*
3. *If $T_+ = +\infty$ and if $f'(t) \to \lambda \in \mathbb{R}$ as $t \to +\infty$, then $\lambda = 0$ or $\lambda = 1$.*
4. *If $T_+ < +\infty$, then f'' and f' are unbounded near T_+.*
5. *If there exists a point $t_0 \in I$ satisfying $f''(t_0) = 0$ and $f'(t_0) = \mu$, where $\mu = 0$ or 1 then for all $t \in I$, we have $f(t) = \mu(t - t_0) + f(t_0)$.*

6. *If $f'(t) \to 0$ as $t \to +\infty$, then $f(t)$ does not tend to $-\infty$ or $+\infty$ as $t \to +\infty$.*

Proof. The first Statement follows immediately from Eq. (1). For the proof of Statements 2-5, see [6] and Statement 6, see [1]. ■

Proposition 5 *Let us suppose that f is a solution of Eq. (1) on the maximal interval $I = (T_-, T_+)$.*

1. *Let $H_1 = f'' + f(f' - 1)$. Then $H_1' = (1 - \beta)f'(f' - 1)$, for all $t \in I$;*
2. *Let $H_2 = 3f''^2 + \beta f'^2(2f' - 3)$. Then $H_2' = -6ff''^2$, for all $t \in I$;*
3. *Let $H_3 = 2ff'' - f'^2 + (2f'^2$. Then $H_3' = 2(2 - \beta)ff'^2$, for all $t \in I$;*
4. *Let $H_4 = f'' + ff'$. Then $H_4' = (1 - \beta)f'^2 + \beta f'$, for all $t \in I$;*
5. *Let $H_5 = f' + \frac{1}{2}f^2$. Then $H_5' = H_4$, for all $t \in I$.*

Proof. The Statements 1-4 follow immediately from Eq. (1), such that for Statements 1 and 4, by using the relation $ff'' = (ff')' - f'^2$ in Eq. (1) and we integrate it. For Statement 2, we multiply the Eq. (1) by f'' and we integrate it, the Statement 3, also we multiplying Eq. (1) by f and integrate it by parts, while the last it follows from Statement 4. ■

4 The Boundary Value Problem $(P_{\beta;a,b,\lambda})$

Consider the boundary value problem $(P_{\beta;a,b,\lambda})$. We are interested here in concave, convex and convex-concave solutions of this problem. Define the following sets:

$$\begin{aligned}
&C_0 = \{c \le 0 : f_c'' \le 0 \text{ on } [0, T_c)\},\\
&C_1 = \{c \ge 0 : f_c' \le 0 \text{ and } f_c'' \ge 0 \text{ on } [0, T_c)\},\\
&C_2 = \{c \ge 0 : \exists t_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f_c' < 0 \text{ on } (0, t_c),\\
&\ f_c' > 0 \text{ on } (t_c, t_c + \varepsilon_c) \text{ and } f_c'' > 0 \text{ on } (0, t_c + \varepsilon_c)\},\\
&C_3 = \{c \ge 0 : \exists s_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f_c'' > 0 \text{ on } [0, s_c),\\
&f_c'' < 0 \text{ on } (s_c, s_c + \varepsilon_c) \text{ and } f_c' < 0 \text{ on } (0, s_c + \varepsilon_c)\}.
\end{aligned}$$

Lemma 1 *Let $\beta > 0$. If $c \in C_0$, then $T_c < +\infty$. Moreover, f_c is concave solution, decreasing and $f_c' \to -\infty$ as $t \to T_c$.*

Proof. If $c \in C_0$, we have $f_c'(t) < 0$ and $f_c''(t) < 0$ for all $t \in [0, T_c)$. Then f_c is a nonpositive concave subsolution of the Blasius equation on $[0, T_c)$ if $a < 0$, and on $[t_0, T_c)$ such that $f_c(t_0) = 0$ if $a > 0$. Therefore, $f_c' \to -\infty$ as $t \to T_c$, and we deduce from Proposition 1 that $T_c < +\infty$. Thanks to Proposition 4, Statement 1, f_c is concave solution, decreasing and $f_c' \to -\infty$ as $t \to T_c$. ■

Remark 1 *We note that C_0, C_1, C_2 and C_3 are disjoint nonempty subsets of $\mathbb{R}$, and we have $C_1 \cup C_2 \cup C_3 = (0, +\infty)$ (see Appendix A of [6] with $g(x) = \beta x(x - 1)$ and $\beta > 0$) and thanks to Lemma 1, we have $C_0 = (-\infty, 0]$.*

Lemma 2 *Let $\beta > 0$. Then f_c is a convex solution of the boundary value problem $(\mathcal{P}_{\beta,a,b,0})$ if and only if $c \in C_1$.*

Proof. See Appendix A of [6] with $g(x) = \beta x(x - 1)$ and $\beta > 0$. ■

Lemma 3 *Let $\beta > 0$. If $c \in C_3$, then $T_c < +\infty$. Moreover, f_c is convex-concave, decreasing and $f_c'(t) \to -\infty$ as $t \to T_c$.*

Proof. See [2], Lemma 9. ■

Remark 2 ([2]) *From Proposition 4, Statements 1,3 and 5, if $c \in C_2$, there are only three possibilities for the solution of the initial value problem $(\mathcal{Q}_{\beta;a,b,c})$. More precisely,*

1. *f_c is convex and $f_c'(t) \to +\infty$ as $t \to T_c$;*
2. *there exists a point $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$ and $f_c'(t_0) > 1$;*
3. *f_c is a convex solution of $(\mathcal{P}_{\beta;a,b,1})$.*

The next Proposition shows that Case (1) cannot hold.

Proposition 6 *Let $\beta \geq 1$. There does not exist a convex solution of $(\mathcal{P}_{\beta;a,b,+\infty})$.*

Proof. Assume that f_c is convex solution of $(\mathcal{P}_{\beta;a,b,+\infty})$. There exists $t_0 \in [0, T_c)$, such that $f_c'(t) > 1$ for all $t \in [t_0, T_c)$, then f_c is a ϵ-subsolution of the Blasius equation on $[t_0, T_c)$. Therefore, from Proposition 1, we have $T_c < +\infty$. Furthermore the function H_1 is decreasing for $t > t_0$. Hence for all $t \in [t_0, T_c)$, $H_1(t) < H_1(t_0)$, then we have

$$f_c(t)(f_c'(t) - 1) < f_c''(t) + f_c(t)(f_c'(t) - 1) < H_1(t_0) < f_c''(t_0) + f_c(t_0)f_c'(t_0)$$

which is a contradiction with the fact that $f_c' \to +\infty$ as $t \to T_c$. ■

Proposition 7 *Let $\beta \geq 1$. If there exists $t_0 \in [0, T_c)$ such that $f_c'(t_0) = 0$ and $f_c''(t_0) < 0$, then for all $t > t_0$, $f_c''(t)) < 0$.*

Proof. Let f_c be a solution of $(\mathcal{Q}_{\beta;a,b,c})$ on its right maximal interval of existence $[0, T_c)$. Let $t_0 \in [0, T_c)$ such that $f_c'(t_0) = 0$ and $f_c''(t_0) < 0$. We suppose that there exists $t_1 > t_0$, where t_1 is the first point after t_0 such that $f_c''(t_1) \geq 0$. Thanks to Proposition 4, Statement 1, the function $t \mapsto f_c'' e^F$ is strictly decreasing on $[t_0, t_1]$, it follows that $f_c''(t_0)e^{F(t_0)} > f_c''(t_1)e^{F(t_1)}$, which is a contradiction. ■

4.1 The Case $a \leq 0$

Lemma 4 *Let $1 \leq \beta \leq 2$ and $b < -1$. If $c \in C_2$; and if there exists $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$, then $f_c'(t_0) > 1$.*

Proof. Let $1 \leq \beta \leq 2$ and $b < -1$. If $c \in C_2$ and if there exists $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$, then the function H_3 is decreasing on $[0, t_0)$, so we have $H_3(0) \geq H_3(t_0)$. This implies that $-b^2 \geq -f_c'^2(t_0)$, and we obtain $f_c'(t_0) \geq -b > 1$. ■

The following Proposition generalizes the previous Lemma.

Proposition 8 *Let $\beta \geq 1$. The boundary value problem $(P_{\beta;a,b,1})$ has no convex solution.*

Proof. Let f_c be a convex solution of the boundary value problem $(P_{\beta;a,b,1})$. Then there exists $t_0 \in [0, +\infty)$, such that $f_c(t_0) = 0$ and $0 < f_c'(t) < 1$ for $t > t_0$. Thus the function H_1 is increasing for all $t > t_0$, i.e. $H_1(t) \geq H_1(t_0)$ for $t > t_0$. Hence we have $f_c''(t) - f_c''(t_0) \geq -f_c(t)\,(f_c'(t) - 1) > 0$, which is a contradiction for t large enough because $f_c''(t) \to 0$ and $f_c(t) > 0$. ■

Proposition 9 *The boundary value problem $(P_{\beta;a,b,0})$ has no negative convex-concave solution.*

Proof. Let f_c be a convex-concave solution of the boundary value problem $(P_{\beta;a,b,0})$. There exists $t_c \in [0, +\infty)$ such that $f_c'(t_c) = 0$, so the function H_2 is strictly increasing for all $t > t_c$. Hence $3f_c''^2(t_c) < H_2(t)$ for all $t > t_c$. $H_2(t) \to 0$ as $t \to +\infty$, a contradiction. ■

Remark 3 *If the boundary value problem* $(P_{\beta;a,b,0})$ *has a convex-concave solution, then this solution changes its sign.*

Lemma 5 *If* $c \in C_1$*, then there exists* c_* *such that* $c < c_*$*,* $T_c = +\infty$*, and the solution* f_c *is negative on* $[0, +\infty)$*.*

Proof. Let $c \in C_1$. From proposition 4, Statement 4, we have $T_c = +\infty$, and the function H_2 is strictly increasing on $[0, T_c)$. It follows that $3c^2 + \beta b^2(2b - 3) < 0$, we obtain $c < -b\sqrt{\frac{\beta(3-2b)}{3}}$. Therefore, the solution f_c is negative because $a \leq 0$ and $f_c' < 0$. ■

Lemma 6 *If* $c \in C_3$*, then there exists* c_* *such that* $c < c_*$*,* $T_c < +\infty$ *and the solution* f_c *is negative on* $[0, T_c)$*.*

Proof. If $c \in C_3$, then $f_c' \to -\infty$, and $T_c < +\infty$. Apply the same proof as that of Lemma 5. ■

Remark 4 *It follows from Lemma 5 and Lemma 6, that there exists* $c_* > 0$ *such that* $c \geq c_*$*,* C_2 *is not empty and here the solution* f_c *changes convexity.*

Lemma 7 *Let* $1 \leq \beta \leq 2$*, if* $c \in C_2$*. There does not exist a nonpositive solution of the problem* $(P_{\beta;a,b,-\infty})$*.*

Proof. Let $1 \leq \beta \leq 2$, $c \in C_2$ and f_c is a nonpositive solution of the problem $(P_{\beta;a,b,-\infty})$. From Remark 2 and Propositions 7, 9, there exists $t_c, t_0 \in [0, T_c)$, such that $t_c < t_0$, $f_c''(t_c) > 0$ and $f_c''(t_0) < 0$. Thus the funtion H_3 is decreasing on $[t_c, t_0]$, we get

$$-\beta f_c^2(t_c) > 2f_c(t_c)f_c''(t_c) - \beta f_c^2(t_c) > 2f_c(t_0)f_c''(t_0) - \beta f_c^2(t_0) > -\beta f_c^2(t_0),$$

it follows that $f_c(t_c) > f_c(t_0)$, which is a contradiction. ■

Theorem 1 *Let* $\beta \geq 1$*,* $a \leq 0$ *and* $b < 0$*.*

1. *The boundary value problem* $(P_{\beta;a,b,-\infty})$ *has infinitely many negative concave solutions on* $[0, T_c)$*, with* $T_c < +\infty$*.*
2. *The boundary value problem* $(P_{\beta;a,b,0})$ *has at least one negative convex solution on* $[0, +\infty)$*.*
3. *The boundary value problem* $(P_{\beta;a,b,1})$ *has no convex solution on* $[0, +\infty)$*.*
4. *The boundary value problem* $(P_{\beta;a,b,+\infty})$ *has no convex solution on* $[0, T_c)$*, with* $T_c < +\infty$*.*

Proof. The first result follows from Lemma 1. The second follows from Remark 1 and Lemma 2. The third Statement follows from Proposition 8. The last result follows from Proposition 6. ■

4.2 The Case $a > 0$

Let us divide the sets C_2 and C_3 into the following two subsets:

$$\begin{aligned}
C_{2.1} &= \{c \in C_2; f_c' > 0 \text{ on } [t_c, T_c)\}, \\
C_{2.2} &= \{c \in C_2; \exists r_c > t_c \text{ s.t } f_c' > 0 \text{ on } [t_c, r_c) \text{ and } f_c'(r_c) = 0\}, \\
C_{3.1} &= \{c \in C_3; f_c(s_c) < 0\}, \\
C_{3.2} &= \{c \in C_3; f_c(s_c) > 0\}.
\end{aligned}$$

Proposition 10 *If* $c \in C_1 \cup C_2 \cup C_{3.1}$*, then* $c > -ab$

Proof. From Proposition 4, Statement 4, if $c \in C_1$ then $T_c = +\infty$, $f_c'(t) \to 0$ as $t \to +\infty$. The function H_4 is strictly decreasing on $[0, +\infty)$, and so we have $c + ab > 0$. If $c \in C_2 \cup C_{3.1}$, there exists $t_c \in [0, T_c)$ such that $f_c'(t_c) = 0$ or there exists $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$. Thus $c + ab \geq f_c''(t_0) > 0$. ■

Remark 5 *If $c \leq -ab$, then $c \in C_{3.2}$ and $T_c < +\infty$. Thus $C_{3.2}$ is not empty and the convex part of the solution f_c is positive.*

Proposition 11 *If $c \in C_1 \cup C_{2.1}$ and $b \geq -\frac{1}{2}a^2$, then $T_c = +\infty$ and there exists $c_* > 0$ such that $c > c_*$. Moreover, the solution f_c is positive.*

Proof. Let $c \in C_1 \cup C_{2.1}$. By the definition of C_1 and $C_{2.1}$ and thanks to Proposition 4, Statement 4 and Proposition 6, we have $T_c = +\infty$. Otherwise the function H_2 is decreasing for $t > 0$. Thus we obtain $3c^2 + \beta b^2(2b-3) > 0$, which implies that $c > -b\sqrt{\frac{\beta(3-2b)}{3}}$. Now if we suppose that there exists $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$, the function H_4 is decreasing for all $t > 0$. We have $H_4(t_0) = f_c''(t_0)$. Therefore H_5 is strictly increasing on $[0, t_0)$ and so we obtain $b + \frac{1}{2}a^2 < f_c'(t_0) < 0$. This is a contradiction. ■

Remark 6 *If $c \in C_{2.2}$ and $b \geq -\frac{1}{2}a^2$, then the solution f_c is positive on $[0, t_0)$, t_0 is the point such that $t_0 > s_c$ with $f_c(t_0) = 0$ and s_c be as in definition of $C_{2.2}$.*

Lemma 8 *Let f_c be a solution of the initial value problem $(Q_{\beta;a,b,c})$, on the right maximal interval of existence $[0, T_c)$ with $b \geq -\frac{1}{2}a^2$. If there exists $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$ and $f_c'(t_0) < 0$, then $f_c''(t_0) < 0$.*

Proof. For the sake of contradiction, let us assume that $t_0 \in [0, T_c)$ with $f_c(t_0) = 0$ and $f_c'(t_0) < 0$. Since the function H_4 is decreasing on $[0, t_0)$ and $H_4(t_0) = f_c''(t_0) > 0$, for all $t \in [0, t_0)$, $H_4 > 0$, and H_5 is strictly increasing on $[0, t_0)$, we have $b + \frac{1}{2}a^2 < f_c'(t_0) < 0$, this is a contradiction. ■

Proposition 12 *Let $1 \leq \beta \leq 2$ and $b \geq -\frac{1}{2}a^2$. Then $C_{2.2} = \emptyset$.*

Proof. Let $1 \leq \beta \leq 2$, $b \geq -\frac{1}{2}a^2$ and $c \in C_{2.2}$. There exists $t_c \in [0, T_c)$, such that $t_c < s_c$ with $f_c(t_c) > 0$, $f_c'(t_c) = 0$ and $f_c''(t_c) > 0$, where t_c be as in definition of C_2 and s_c be as in definition of $C_{2.2}$. Therefore, since the function H_3 is increasing on $[t_c, s_c]$, we then have

$$-\beta f_c^2(t_c) < 2f_c(t_c)f_c''(t_c) - \beta f_c^2(t_c) \leq 2f_c(s_c)f_c''(s_c) - \beta f_c^2(s_c) < -\beta f_c^2(s_c),$$

which implies that $f_c(t_c) > f_c(s_c)$, this is a contradiction. ■

If $c \in C_{2.1}$ then $T_c = +\infty$. So, let us divide the set $C_{2.1}$ into the following two subsets:

$$C_{2.1.1} = \{c \in C_{2.1}; f_c'(t) \to 0 \text{ as } t \to +\infty\},$$
$$C_{2.1.2} = \{c \in C_{2.1}; f_c'(t) \to 1 \text{ as } t \to +\infty\}.$$

Proposition 13 *Let $1 \leq \beta \leq 2$. If $b \geq -\frac{1}{2}a^2$, then $C_{2.1.1} = \emptyset$.*

Proof. Let $1 \leq \beta \leq 2$, $b \geq -\frac{1}{2}a^2$ and $c \in C_{2.1.1}$, we deduce from Proposition 11 that the function H_3 is increasing on $[t_c, +\infty)$, where t_c be as in definition of C_2, we then have for $t > t_c$,

$$-\beta f_c^2(t_c) < 2f_c(t_c)f_c''(t_c) - \beta f_c^2(t_c) \leq 2f_c(t)f_c''(t) - f_c'^2(t) + (2f_c'(t) - \beta)f_c^2(t).$$

From Proposition 4, Statements 2, 4 and 6, it follows that $f_c(t) \to l < +\infty$ as $t \to +\infty$, which implies that $f_c(t_c) > l$ as $t \to +\infty$, this is a contradiction. ■

Proposition 14 *Let $1 \leq \beta \leq 2$, and let $c \in C_1 \cup C_3 \cup C_{2.2} \cup C_{2.1.1}$. Then there exists $c^* > 0$ such that $c < c^*$.*

Proof. Let $c \in C_1 \cup C_3 \cup C_{2.2} \cup C_{2.1.1}$. Either there exists $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$ or $f_c'(t_0) = 0$ if $T_c < +\infty$, and if $T_c = +\infty$, we have $f_c'(t) \to 0$ as $t \to +\infty$. From Proposition 4, Statement 6, it follows that the function H_3 is increasing on $[0, t_0)$ or $[0, +\infty)$. We then get $2ac - b^2 + (2b - \beta)a^2 < 0$. We implies that $c < \frac{b^2 + (\beta - 2b)a^2}{2a}$. ■

Remark 7 *From the previous Proposition, there exists $c^* > 0$, such that for $c \geq c^*$, then $c \in C_{2.1.2}$. Thus $C_{2.1.2}$ is not empty. If, moreover, $b \geq -\frac{1}{2}a^2$, from Propositions 11 and 14, we have $C_1 \subset]c_*, c^*[$.*

Theorem 2 *Let $\beta \geq 1$, $a > 0$ and $b < 0$.*

1. *The boundary value problem $(P_{\beta;a,b,-\infty})$ has infinity convex-concave solutions on the maximal interval of existence $[0, T_c)$ with $T_c < +\infty$. If, in addition, $b \geq -\frac{1}{2}a^2$, then the convex part of these solutions will be non-negative.*
2. *The boundary value problem $(P_{\beta;a,b,0})$ has at least one convex solution on $[0, +\infty)$. If, in addition, $b \geq -\frac{1}{2}a^2$, then this solution becomes non-negative convex solution.*
3. *If $\beta \leq 2$, the boundary value problem $(P_{\beta;a,b,1})$ has infinitely many positive solutions on $[0, +\infty)$.*
4. *The boundary value problem $(P_{\beta;a,b,+\infty})$ has no convex solution on $[0, T_c)$, with $T_c < +\infty$.*

Proof. The first follows from Proposition 4, Proposition 7 and Remark 5, the second result follows from Remark 1, Lemma 2 and Proposition 11, while the third follows from Proposition 14 and Remark 7 . The last result follows from Proposition 6. ■

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