

Existence Theory For A Self-Adjoint Coupled System Of Nonlinear Ordinary Differential Equations With Nonlocal Integral Multi-Point Boundary Conditions*

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Abstract

In this paper we develop criteria ensuring the existence and uniqueness of solutions for a self-adjoint coupled system of nonlinear second-order ordinary differential equations equipped with nonlocal integral multi-point coupled boundary conditions on an arbitrary domain. The existence results are proved via Leray-Schauder alternative and Schauder fixed point theorem, while the existence of a unique solution is obtained by applying the Banach contraction mapping principle. Finally some examples are constructed for illustration of the obtained results.

1 Introduction

The study of boundary value problems constitutes an interesting and important area of investigation in view of occurrence of such problems in several disciplines such as fluid mechanics, mathematical physics, etc. For examples and details, we refer the reader to [1, 2] and the references cited therein. Much of the work on boundary value problems is concerned with classical boundary conditions. However, these conditions cannot be used to formulate the physical and chemical processes taking place at arbitrary positions of the given domain. This situation gives rise to the concept of nonlocal conditions, which specify the data at some interior positions of the domain. The advent of nonlocal boundary conditions inspired many researchers to work on nonlocal boundary value problems. One can find a variety of interesting results on such problems in the works [3]-[17]. On the other hand, there are fewer results on nonlocal self-adjoint boundary value problems. It is imperative to mention that self-adjoint differential equations appear in the study of Schrödinger operators [18], stability of periodic delay systems [19], oscillation of impulsive systems [20], Hamiltonian systems [21], etc.

Recently, in [22], the authors studied the following self-adjoint coupled system of nonlinear second-order ordinary differential equations on an arbitrary domain:

$$\begin{cases} (p(t)u'(t))' = f(t, u(t), v(t)), & t \in [a, b], \\ (q(t)v'(t))' = g(t, u(t), v(t)), & t \in [a, b], \end{cases} \quad (1)$$

subject to nonlocal multi-point coupled boundary conditions:

$$\begin{cases} u'(a) = 0, & u(b) = \sum_{j=1}^m \alpha_j v(\eta_j), \\ v'(a) = 0, & v(b) = \sum_{k=1}^n \beta_k u(\xi_k), \end{cases} \quad (2)$$

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where $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions, $a < \eta_1 < \dots < \eta_m < \xi_1 < \dots < \xi_n < b$, $\alpha_j \in \mathbb{R}^+$ ($j = 1, 2, \dots, m$), $\beta_k \in \mathbb{R}^+$ ($k = 1, 2, \dots, n$), and $p, q \in C([a, b], \mathbb{R}^+)$.

Motivated by the paper [22], in the present research we consider the following self-adjoint coupled system of nonlinear second-order ordinary differential equations on an arbitrary domain:

$$\begin{cases} (p(t)u'(t))' = \mu_1 f(t, u(t), v(t)), & t \in [a, b], \\ (q(t)v'(t))' = \mu_2 g(t, u(t), v(t)), & t \in [a, b], \end{cases} \tag{3}$$

supplemented with nonlocal integral multi-point coupled boundary conditions of the form:

$$\begin{cases} u'(a) = 0, & \int_a^b u(s)ds - \sum_{j=1}^m \alpha_j v(\eta_j) = \lambda_1, \\ v'(a) = 0, & \int_a^b v(s)ds - \sum_{k=1}^n \beta_k u(\xi_k) = \lambda_2, \end{cases} \tag{4}$$

where $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions, $a < \eta_1 < \dots < \eta_m < \xi_1 < \dots < \xi_n < b$, $\alpha_j \in \mathbb{R}^+$ ($j = 1, 2, \dots, m$), $\beta_k \in \mathbb{R}^+$ ($k = 1, 2, \dots, n$), $p, q \in C([a, b], \mathbb{R}^+)$, and $\lambda_i, \mu_i \in \mathbb{R}^+, i = 1, 2$.

Here it is worthwhile to notice that the nonlocal integral multi-point coupled boundary conditions in (4) imply that the distribution of one unknown function on the given domain differs from the sum of the values of the other unknown function at arbitrary positions within the given domain by a constant, and these contributions coincide for a zero value of the constant. The main objective of the present study is to develop the existence theory for the problem (3)–(4).

The rest of the paper is organized as follows. In section 2, we prove an auxiliary lemma, which deals with a linear variant of the problem (3)–(4) and it is useful to transform the problem (3)–(4) into an equivalent fixed point problem. In Section 3, we establish the main existence and uniqueness results. The existence results are proved via Leray-Schauder alternative and Schauder fixed point theorem, while the uniqueness of solutions is established by applying the Banach contraction mapping principle. In section 4, we construct some illustrative examples for the main results.

2 An Auxiliary Lemma

In this section, we solve a linear variant of the problem (3)–(4) which plays a key role in obtaining the desired results.

Lemma 1 For $f_1, g_1 \in C([a, b], \mathbb{R})$ and $Q \neq 0$, the solution of the linear system of differential equations

$$(p(t)u'(t))' = \mu_1 f_1(t), \quad (q(t)v'(t))' = \mu_2 g_1(t), \quad t \in [a, b], \tag{5}$$

supplemented with the boundary conditions (4), can be expressed in the formulas:

$$\begin{aligned} u(t) = & \int_a^t \left(\frac{\mu_1}{p(z)} \int_a^z f_1(\tau) d\tau \right) dz + \frac{1}{Q} \left[\lambda_1(b-a) + \lambda_2 \sum_{j=1}^m \alpha_j \right. \\ & - \int_a^b \int_a^s \frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \left(\int_a^z g_1(\tau) d\tau \right) dz ds \\ & + \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{\mu_1}{p(z)} \left(\int_a^z f_1(\tau) d\tau \right) dz \right) \\ & - (b-a) \int_a^b \int_a^s \frac{\mu_1}{p(z)} \left(\int_a^z f_1(\tau) d\tau \right) dz ds \\ & \left. + (b-a) \int_a^{\eta_j} \left(\frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \int_a^z g_1(\tau) d\tau \right) dz \right], \tag{6} \end{aligned}$$

and

$$\begin{aligned}
 v(t) = & \int_a^t \left(\frac{\mu_2}{q(z)} \int_a^z g_1(\tau) d\tau \right) dz + \frac{1}{Q} \left[\lambda_1 \sum_{k=1}^n \beta_k + \lambda_2(b-a) \right. \\
 & - \int_a^b \int_a^s \frac{\mu_1 \sum_{k=1}^n \beta_k}{p(z)} \left(\int_a^z f_1(\tau) d\tau \right) dz ds \\
 & + \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \int_a^{\eta_j} \frac{\mu_2}{q(z)} \left(\int_a^z g_1(\tau) d\tau \right) dz \right) \\
 & - (b-a) \int_a^b \int_a^s \frac{\mu_2}{q(z)} \left(\int_a^z g_1(\tau) d\tau \right) dz ds \\
 & \left. + (b-a) \int_a^{\xi_k} \left(\frac{\mu_1 \sum_{k=1}^n \beta_k}{p(z)} \int_a^z f_1(\tau) d\tau \right) dz \right], \tag{7}
 \end{aligned}$$

where

$$Q = \left[(b-a)^2 - \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \right) \right]. \tag{8}$$

Proof. Integrating the linear differential equations (5) twice from a to t , and using the conditions $u'(a) = 0$, $v'(a) = 0$, we obtain

$$u(t) = u(a) + \int_a^t \left(\frac{\mu_1}{p(z)} \int_a^z f_1(\tau) d\tau \right) dz \tag{9}$$

and

$$v(t) = v(a) + \int_a^t \left(\frac{\mu_2}{q(z)} \int_a^z g_1(\tau) d\tau \right) dz. \tag{10}$$

By using the coupled boundary conditions (4) in (9) and (10), we get the following system of equations:

$$\begin{aligned}
 (b-a)u(a) - \left(\sum_{j=1}^m \alpha_j \right) v(a) = & \lambda_1 - \int_a^b \int_a^s \left(\frac{\mu_1}{p(z)} \int_a^z f_1(\tau) d\tau \right) dz ds \\
 & + \int_a^{\eta_j} \frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \left(\int_a^z g_1(\tau) d\tau \right) dz, \tag{11}
 \end{aligned}$$

$$\begin{aligned}
 - \left(\sum_{k=1}^n \beta_k \right) u(a) + (b-a)v(a) = & \lambda_2 - \int_a^b \int_a^s \left(\frac{\mu_2}{q(z)} \int_a^z g_1(\tau) d\tau \right) dz ds \\
 & + \int_a^{\xi_k} \frac{\mu_1 \sum_{k=1}^n \beta_k}{p(z)} \left(\int_a^z f_1(\tau) d\tau \right) dz. \tag{12}
 \end{aligned}$$

Solving (11) and (12) for $u(a)$ and $v(a)$, and using the notation Q given by (8), we obtain

$$\begin{aligned}
 u(a) = & \frac{1}{Q} \left[\lambda_1(b-a) + \lambda_2 \sum_{j=1}^m \alpha_j - \int_a^b \int_a^s \left(\frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \int_a^z g_1(\tau) d\tau \right) dz ds \right. \\
 & + \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{\mu_1}{p(z)} \left(\int_a^z f_1(\tau) d\tau \right) dz \right) \\
 & \left. - (b-a) \int_a^b \int_a^s \left(\frac{\mu_1}{p(z)} \int_a^z f_1(\tau) d\tau \right) dz ds \right]
 \end{aligned}$$

$$+(b-a) \int_a^{\eta_j} \frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \left(\int_a^z g_1(\tau) d\tau \right) dz \Big], \tag{13}$$

and

$$\begin{aligned} v(a) = & \frac{1}{Q} \left[\lambda_1 \sum_{k=1}^n \beta_k + \lambda_2(b-a) - \int_a^b \int_a^s \left(\frac{\mu_1 \sum_{k=1}^n \beta_k}{p(z)} \int_a^z f_1(\tau) d\tau \right) dz ds \right. \\ & + \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \int_a^{\eta_j} \frac{\mu_2}{q(z)} \left(\int_a^z g_1(\tau) d\tau \right) dz \right) \\ & - (b-a) \int_a^b \int_a^s \left(\frac{\mu_2}{q(z)} \int_a^z g_1(\tau) d\tau \right) dz ds \\ & \left. + (b-a) \int_a^{\xi_k} \frac{\mu_1 \sum_{k=1}^n \beta_k}{p(z)} \left(\int_a^z f_1(\tau) d\tau \right) dz \right]. \tag{14} \end{aligned}$$

Substituting the values of $u(a)$ and $v(a)$ in (9) and (10) respectively, we get the solution (6) and (7). By direct computation, one can obtain the converse of the lemma. This completes the proof. ■

3 Main Results

Let $(\chi, \|\cdot\|)$ be a Banach space with $\chi = \{u(t) | u(t) \in C([a, b], \mathbb{R})\}$ equipped with norm $\|u\| = \sup\{|u(t)|, t \in [a, b]\}$. Note that the product space $(\chi \times \chi, \|(u, v)\|)$ is a Banach space with the norm given by

$$\|(u, v)\| = \|u\| + \|v\|$$

for $(u, v) \in \chi \times \chi$.

In view of Lemma 1, we transform the problem (3) and (4) into an equivalent fixed point problem as:

$$(u, v) = \mathcal{T}(u, v), \tag{15}$$

where the operator $\mathcal{T} : \chi \times \chi \rightarrow \chi \times \chi$ is defined by

$$\mathcal{T}(u, v)(t) := (\mathcal{T}_1(u, v)(t), \mathcal{T}_2(u, v)(t)), \tag{16}$$

$$\begin{aligned} \mathcal{T}_1(u, v)(t) = & \int_a^t \left(\frac{\mu_1}{p(z)} \int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz \\ & + \frac{1}{Q} \left[\lambda_1(b-a) + \lambda_2 \sum_{j=1}^m \alpha_j - \int_a^b \int_a^s \frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \left(\int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz ds \right. \\ & + \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{\mu_1}{p(z)} \left(\int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz \right) \\ & - (b-a) \int_a^b \int_a^s \frac{\mu_1}{p(z)} \left(\int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz ds \\ & \left. + (b-a) \int_a^{\eta_j} \left(\frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz \right], \tag{17} \end{aligned}$$

and

$$\mathcal{T}_2(u, v)(t) = \int_a^t \left(\frac{\mu_2}{q(z)} \int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz$$

$$\begin{aligned}
 & + \frac{1}{Q} \left[\lambda_1 \sum_{k=1}^n \beta_k + \lambda_2(b-a) - \int_a^b \int_a^s \frac{\mu_1 \sum_{k=1}^n \beta_k}{p(z)} \left(\int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz \, ds \right. \\
 & + \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \int_a^{\eta_j} \frac{\mu_2}{q(z)} \left(\int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz \right) \\
 & - (b-a) \int_a^b \int_a^s \frac{\mu_2}{q(z)} \left(\int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz \, ds \\
 & \left. + (b-a) \int_a^{\xi_k} \left(\frac{\mu_1 \sum_{k=1}^n \beta_k}{p(z)} \int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz \right]. \tag{18}
 \end{aligned}$$

Observe that the fixed points of operator $\mathcal{T} : \chi \times \chi \rightarrow \chi \times \chi$ will be solutions of the problem (3)–(4). For the sake of computational convenience, we set:

$$\mathcal{R}_1 = M_1 + M_3, \quad \mathcal{R}_2 = M_2 + M_4, \quad \mathcal{C} = C_1 + C_2, \tag{19}$$

where

$$\begin{aligned}
 M_1 &= \left| \frac{\mu_1}{Q\bar{p}} \right| \left[|Q| \frac{(b-a)^2}{2} + \frac{(b-a)^4}{6} + \left(\sum_{k=1}^n \beta_k \right) \left(\sum_{j=1}^m \alpha_j \frac{(\xi_k - a)^2}{2} \right) \right], \\
 M_2 &= \frac{|\mu_2| \sum_{j=1}^m \alpha_j}{|Q\bar{q}|} \left[\frac{(b-a)(\eta_j - a)^2}{2} + \frac{(b-a)^3}{6} \right], \\
 M_3 &= \frac{|\mu_1| \sum_{k=1}^n \beta_k}{|Q\bar{p}|} \left[\frac{(b-a)(\xi_k - a)^2}{2} + \frac{(b-a)^3}{6} \right], \\
 M_4 &= \left| \frac{\mu_2}{Q\bar{q}} \right| \left[|Q| \frac{(b-a)^2}{2} + \frac{(b-a)^4}{6} + \left(\sum_{k=1}^n \beta_k \right) \left(\sum_{j=1}^m \alpha_j \frac{(\eta_j - a)^2}{2} \right) \right], \\
 \bar{p} &= \inf_{z \in [a,b]} |p(z)|, \quad \bar{q} = \inf_{z \in [a,b]} |q(z)|, \\
 C_1 &= \frac{1}{|Q|} \left[|\lambda_1(b-a)| + |\lambda_2| \sum_{j=1}^m \alpha_j \right], \quad C_2 = \frac{1}{|Q|} \left[|\lambda_1| \sum_{k=1}^n \beta_k + |\lambda_2(b-a)| \right].
 \end{aligned} \tag{20}$$

3.1 Existence Results

Let us begin with our first existence result for problem (3)–(4) which is based on Leray-Schauder alternative [23] and is stated below.

Lemma 2 (Leray-Schauder alternative) *Let Ω be a Banach space, and $G : \Omega \rightarrow \Omega$ be a completely continuous operator (i.e., a map restricted to any bounded set in Ω is compact). Let $\Theta(G) = \{g \in \Omega : g = \epsilon G(g) \text{ for some } 0 < \epsilon < 1\}$. Then either the set $\Theta(G)$ is unbounded or G has at least one fixed point.*

Theorem 1 *Assume that:*

(S₁) (Growth conditions) *There exist real constants $\kappa_i, \gamma_i \geq 0$ ($i = 1, 2$), and $\kappa_0 > 0, \gamma_0 > 0$, such that $\forall u, v \in \mathbb{R}$, we have*

$$|f(t, u, v)| \leq \kappa_0 + \kappa_1|u| + \kappa_2|v|, \quad |g(t, u, v)| \leq \gamma_0 + \gamma_1|u| + \gamma_2|v|.$$

If

$$\mathcal{R}_1\kappa_1 + \mathcal{R}_2\gamma_1 < 1 \quad \text{and} \quad \mathcal{R}_1\kappa_2 + \mathcal{R}_2\gamma_2 < 1, \tag{21}$$

where \mathcal{R}_i ($i = 1, 2$) are given by (19). Then there exists at least one solution for the problem (3)–(4) on $[a, b]$.

Proof. The proof will be given in two steps:

Step 1: We show that the operator $\mathcal{T} : \chi \times \chi \rightarrow \chi \times \chi$ defined by (16) is completely continuous.

Obviously from the continuity of the functions f and g , the operators \mathcal{T}_1 and \mathcal{T}_2 are continuous and hence the operator \mathcal{T} is continuous. Furthermore, let $\mathcal{K} \subset \chi \times \chi$ be bounded. Then, there exist positive constants L_f and L_g such that $|f(t, u(t), v(t))| \leq L_f$, and $|g(t, u(t), v(t))| \leq L_g$, $\forall (u, v) \in \mathcal{K}$. Then, for any $(u, v) \in \mathcal{K}$, we obtain

$$\begin{aligned} |\mathcal{T}_1(u, v)(t)| &= \left| \int_a^t \left(\frac{\mu_1}{p(z)} \int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz \right. \\ &\quad + \frac{1}{Q} \left[\lambda_1(b-a) + \lambda_2 \sum_{j=1}^m \alpha_j - \int_a^b \int_a^s \frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \left(\int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz ds \right. \\ &\quad + \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{\mu_1}{p(z)} \left(\int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz \right) \\ &\quad \left. - (b-a) \int_a^b \int_a^s \frac{\mu_1}{p(z)} \left(\int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz ds \right. \\ &\quad \left. + (b-a) \int_a^{\eta_j} \left(\frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz \right] \Big| \\ &\leq L_f \left\{ \left| \frac{\mu_1}{Q\bar{p}} \right| \left[|Q| \frac{(b-a)^2}{2} + \frac{(b-a)^4}{6} + \left(\sum_{k=1}^n \beta_k \right) \left(\sum_{j=1}^m \alpha_j \frac{(\xi_k - a)^2}{2} \right) \right] \right\} \\ &\quad + L_g \left\{ \frac{|\mu_2| \sum_{j=1}^m \alpha_j}{|Q\bar{q}|} \left[\frac{(b-a)(\eta_j - a)^2}{2} + \frac{(b-a)^3}{6} \right] \right\} \\ &\quad + \left\{ \frac{1}{|Q|} \left[|\lambda_1(b-a)| + |\lambda_2| \sum_{j=1}^m \alpha_j \right] \right\}, \end{aligned}$$

which, on taking the norm for $t \in [a, b]$, yields $\|\mathcal{T}_1(u, v)\| \leq L_f M_1 + L_g M_2 + C_1$. In the same manner, it can be shown that $\|\mathcal{T}_2(u, v)\| \leq L_f M_3 + L_g M_4 + C_2$, where M_i ($i = 1, \dots, 4$) and C_i ($i = 1, 2$) are given by (20). In consequence, we obtain that

$$\|\mathcal{T}(u, v)\| \leq L_f \mathcal{R}_1 + L_g \mathcal{R}_2 + \mathcal{C},$$

where \mathcal{R}_i ($i = 1, 2$) and \mathcal{C} are given by (19). Therefore, it follows from the foregoing inequality that \mathcal{T} is uniformly bounded.

Next we will prove that \mathcal{T} is an equicontinuous operator. For $t_1, t_2 \in [a, b]$ with $t_1 < t_2$, we have

$$\begin{aligned} &|\mathcal{T}_1(u, v)(t_2) - \mathcal{T}_1(u, v)(t_1)| \\ &= \left| \int_a^{t_2} \left(\frac{\mu_1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds - \int_a^{t_1} \left(\frac{\mu_1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right| \\ &= \left| \int_a^{t_1} \left(\frac{\mu_1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_{t_1}^{t_2} \left(\frac{\mu_1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ &\quad \left. - \int_a^{t_1} \left(\frac{\mu_1}{p(s)} \int_a^s f(\tau, u(\tau), v(\tau)) d\tau \right) ds \right| \end{aligned}$$

$$\leq L_f \frac{|\mu_1|}{2p} |(t_2 - a)^2 - (t_1 - a)^2| \rightarrow 0 \text{ as } t_2 - t_1 \rightarrow 0 \text{ independent of } (u, v).$$

In a similar manner, one can find that

$$|\mathcal{T}_2(u, v)(t_2) - \mathcal{T}_2(u, v)(t_1)| \rightarrow 0 \text{ as } t_2 - t_1 \rightarrow 0 \text{ independent of } (u, v).$$

Thus we deduce that the operator \mathcal{T} is equicontinuous by Arzelá-Ascoli theorem.

Step 2: We verify that the set $\Theta = \{(u, v) \in \chi \times \chi | (u, v) = \epsilon \mathcal{T}(u, v), 0 < \epsilon < 1\}$ is bounded. Let $(u, v) \in \Theta$. Then $(u, v) = \epsilon \mathcal{T}(u, v)$, and for any $t \in [a, b]$, we have

$$u(t) = \epsilon \mathcal{T}_1(u, v)(t), \quad v(t) = \epsilon \mathcal{T}_2(u, v)(t).$$

Then, by the growth conditions (S_1) , we obtain

$$\begin{aligned} |u(t)| &\leq M_1(\kappa_0 + \kappa_1|u| + \kappa_2|v|) + M_2(\gamma_0 + \gamma_1|u| + \gamma_2|v|) + C_1 \\ &\leq M_1\kappa_0 + M_2\gamma_0 + (M_1\kappa_1 + M_2\gamma_1)\|u\| + (M_1\kappa_2 + M_2\gamma_2)\|v\| + C_1 \end{aligned}$$

and

$$\begin{aligned} |v(t)| &\leq M_3(\kappa_0 + \kappa_1|u| + \kappa_2|v|) + M_4(\gamma_0 + \gamma_1|u| + \gamma_2|v|) + C_2 \\ &\leq M_3\kappa_0 + M_4\gamma_0 + (M_3\kappa_1 + M_4\gamma_1)\|u\| + (M_3\kappa_2 + M_4\gamma_2)\|v\| + C_2. \end{aligned}$$

From the last two inequalities, we get

$$\begin{aligned} \|u\| + \|v\| &\leq (M_1 + M_3)\kappa_0 + (M_2 + M_4)\gamma_0 + [(M_1 + M_3)\kappa_1 + (M_2 + M_4)\gamma_1]\|u\| \\ &\quad + [(M_1 + M_3)\kappa_2 + (M_2 + M_4)\gamma_2]\|v\| + (C_1 + C_2), \end{aligned}$$

which, in view of (22) and (21), implies that

$$\|(u, v)\| \leq \frac{\mathcal{R}_1\kappa_0 + \mathcal{R}_2\gamma_0 + \mathcal{C}}{\mathcal{R}},$$

where

$$\mathcal{R} = \min\{1 - (\mathcal{R}_1\kappa_1 + \mathcal{R}_2\gamma_1), 1 - (\mathcal{R}_1\kappa_2 + \mathcal{R}_2\gamma_2)\}, \quad \kappa_i, \gamma_i \text{ are given in } (S_1). \tag{22}$$

This shows that the set Θ is bounded.

Thus the hypotheses of Lemma 2 are satisfied and hence its conclusion implies that the operator \mathcal{T} has at least one fixed point. Consequently, the problem (3)–(4) has at least one solution on $[a, b]$. This completes the proof. ■

In our second existence result we apply the Schauder fixed point theorem [23] to prove the existence of solutions for the problem (3)–(4).

Theorem 2 *Assume that:*

(S_2) (Sub-growth conditions) *There exist nonnegative functions $\omega(t), \lambda(t) \in L(a, b)$ such that*

$$|f(t, u, v)| \leq \omega(t) + \rho_1|u|^{z_1} + \rho_2|v|^{z_2}, \quad u, v \in \mathbb{R}, \quad \rho_1, \rho_2 > 0, \quad 0 < z_1, z_2 < 1,$$

$$|g(t, u, v)| \leq \lambda(t) + \nu_1|u|^{k_1} + \nu_2|v|^{k_2}, \quad u, v \in \mathbb{R}, \quad \nu_1, \nu_2 > 0, \quad 0 < k_1, k_2 < 1.$$

Then there exists at least one solution for the problem (3)–(4) on $[a, b]$.

Proof. Fixing

$$\delta \geq \max \left\{ 6\mathcal{R}_1\|\omega\|, 6\mathcal{R}_2\|\lambda\|, (6\rho_1\mathcal{R}_1)^{\frac{1}{1-z_1}}, (6\rho_2\mathcal{R}_1)^{\frac{1}{1-z_2}}, (6\nu_1\mathcal{R}_2)^{\frac{1}{1-k_1}}, (6\nu_2\mathcal{R}_2)^{\frac{1}{1-k_2}} \right\},$$

we introduce a ball defined by

$$\Lambda = \{(u, v) \in \chi \times \chi : \|(u, v)\| \leq \delta\},$$

and consider the operator $\mathcal{T} : \Lambda \rightarrow \Lambda$. For any $(u, v) \in \Lambda$, we have

$$\begin{aligned} |\mathcal{T}_1(u, v)(t)| &= \left| \int_a^t \left(\frac{\mu_1}{p(z)} \int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz \right. \\ &\quad + \frac{1}{Q} \left[\lambda_1(b-a) + \lambda_2 \sum_{j=1}^m \alpha_j - \int_a^b \int_a^s \frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \left(\int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz ds \right. \\ &\quad + \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{\mu_1}{p(z)} \left(\int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz \right) \\ &\quad - (b-a) \int_a^b \int_a^s \frac{\mu_1}{p(z)} \left(\int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz ds \\ &\quad \left. + (b-a) \int_a^{\eta_j} \left(\frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz \right] \Big| \\ &\leq \left(\omega(t) + \rho_1|u|^{z_1} + \rho_2|v|^{z_2} \right) \left\{ \left| \frac{\mu_1}{Qp} \right| \left[|Q| \frac{(b-a)^2}{2} + \frac{(b-a)^4}{6} \right. \right. \\ &\quad \left. \left. + \left(\sum_{k=1}^n \beta_k \right) \left(\sum_{j=1}^m \alpha_j \frac{(\xi_k - a)^2}{2} \right) \right] \right\} \\ &\quad + \left(\lambda(t) + \nu_1|u|^{k_1} + \nu_2|v|^{k_2} \right) \left\{ \frac{|\mu_2| \sum_{j=1}^m \alpha_j}{|Qq|} \left[\frac{(b-a)(\eta_j - a)^2}{2} + \frac{(b-a)^3}{6} \right] \right\} \\ &\quad + \left\{ \frac{1}{|Q|} \left[|\lambda_1(b-a)| + |\lambda_2| \sum_{j=1}^m \alpha_j \right] \right\}, \end{aligned}$$

which, on taking the norm for $t \in [a, b]$, yields

$$\|\mathcal{T}_1(u, v)\| \leq \left(\|\omega\| + \rho_1\|u\|^{z_1} + \rho_2\|v\|^{z_2} \right) M_1 + \left(\|\lambda\| + \nu_1\|u\|^{k_1} + \nu_2\|v\|^{k_2} \right) M_2 + C_1.$$

In the same way, we can find that

$$\|\mathcal{T}_2(u, v)\| \leq \left(\|\omega\| + \rho_1\|u\|^{z_1} + \rho_2\|v\|^{z_2} \right) M_3 + \left(\|\lambda\| + \nu_1\|u\|^{k_1} + \nu_2\|v\|^{k_2} \right) M_4 + C_2,$$

where M_i ($i = 1, \dots, 4$) and C_i ($i = 1, 2$) are given by (20). Therefore, we obtain

$$\|\mathcal{T}(u, v)\| \leq \left(\|\omega\| + \rho_1\|u\|^{z_1} + \rho_2\|v\|^{z_2} \right) \mathcal{R}_1 + \left(\|\lambda\| + \nu_1\|u\|^{k_1} + \nu_2\|v\|^{k_2} \right) \mathcal{R}_2 + \mathcal{C} \leq \delta,$$

where \mathcal{R}_1 and \mathcal{R}_2 are given by (19). Thus we deduce that $\mathcal{T} : \Lambda \rightarrow \Lambda$.

As in the proof of Theorem 1, one can show that the operator \mathcal{T} is completely continuous. Hence, by the Schauder fixed point theorem, there exists at least one solution for the problem (3) and (4) on $[a, b]$. The proof is now complete. ■

3.2 Uniqueness Result

In this subsection, we apply Banach's contraction mapping principle to show the uniqueness of solutions for the problem (3) and (4).

Theorem 3 *Assume that:*

(S₃) (Lipschitz conditions) *For all $t \in [a, b]$ and $u_i, v_i \in \mathbb{R}$ ($i = 1, 2$), there exist $\ell_i > 0$ ($i = 1, 2$) such that*

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \ell_1(|u_1 - u_2| + |v_1 - v_2|)$$

and

$$|g(t, u_1, v_1) - g(t, u_2, v_2)| \leq \ell_2(|u_1 - u_2| + |v_1 - v_2|).$$

Then the problem (3)–(4) has a unique solution on $[a, b]$, provided that

$$\mathcal{R}_1 \ell_1 + \mathcal{R}_2 \ell_2 < 1, \quad (23)$$

where \mathcal{R}_i ($i = 1, 2$) are given by (19).

Proof. Firstly, let us set $H_1 = \sup_{t \in [a, b]} |f(t, 0, 0)|$, $H_2 = \sup_{t \in [a, b]} |g(t, 0, 0)|$, and consider a set $B_r = \{(u, v) \in \mathcal{X} \times \mathcal{X} : \|(u, v)\| \leq r\}$ with

$$r \geq \frac{H_1 \mathcal{R}_1 + H_2 \mathcal{R}_2 + \mathcal{C}}{1 - (\ell_1 \mathcal{R}_1 + \ell_2 \mathcal{R}_2)},$$

and show that $\mathcal{T}B_r \subset B_r$. For any $(u, v) \in B_r$, $t \in [a, b]$, by the condition (S₃), we have

$$\begin{aligned} |f(t, u(t), v(t))| &= |f(t, u(t), v(t)) - f(t, 0, 0) + f(t, 0, 0)| \\ &\leq |f(t, u(t), v(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq \ell_1(\|u\| + \|v\|) + H_1 \leq \ell_1\|(u, v)\| + H_1 \leq \ell_1 r + H_1. \end{aligned}$$

Similarly, $|g(t, u(t), v(t))| \leq \ell_2\|(u, v)\| + H_2 \leq \ell_2 r + H_2$. Then, for $(u, v) \in B_r$, we obtain

$$\begin{aligned} |\mathcal{T}_1(u, v)(t)| &= \left| \int_a^t \left(\frac{\mu_1}{p(z)} \int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz \right. \\ &\quad + \frac{1}{Q} \left[\lambda_1(b-a) + \lambda_2 \sum_{j=1}^m \alpha_j - \int_a^b \int_a^s \frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \left(\int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz ds \right. \\ &\quad + \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{\mu_1}{p(z)} \left(\int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz \right) \\ &\quad - (b-a) \int_a^b \int_a^s \frac{\mu_1}{p(z)} \left(\int_a^z f(\tau, u(\tau), v(\tau)) d\tau \right) dz ds \\ &\quad \left. + (b-a) \int_a^{\eta_j} \left(\frac{\mu_2 \sum_{j=1}^m \alpha_j}{q(z)} \int_a^z g(\tau, u(\tau), v(\tau)) d\tau \right) dz \right] \Big| \\ &\leq (\ell_1 r + H_1) \left\{ \left| \frac{\mu_1}{Qp} \right| \left[|Q| \frac{(b-a)^2}{2} + \frac{(b-a)^4}{6} + \left(\sum_{k=1}^n \beta_k \right) \left(\sum_{j=1}^m \alpha_j \frac{(\xi_k - a)^2}{2} \right) \right] \right\} \\ &\quad + (\ell_2 r + H_2) \left\{ \frac{|\mu_2| \sum_{j=1}^m \alpha_j}{|Qq|} \left[\frac{(b-a)(\eta_j - a)^2}{2} + \frac{(b-a)^3}{6} \right] \right\} \\ &\quad + \left\{ \frac{1}{|Q|} \left[|\lambda_1(b-a)| + |\lambda_2| \sum_{j=1}^m \alpha_j \right] \right\}, \end{aligned}$$

which implies that

$$\|\mathcal{T}_1(u, v)\| \leq (\ell_1 r + H_1)M_1 + (\ell_2 r + H_2)M_2 + \mathcal{C}_1.$$

Analogously, we can get $\|\mathcal{T}_2(u, v)\| \leq (\ell_1 r + H_1)M_3 + (\ell_2 r + H_2)M_4 + \mathcal{C}_2$, where M_i ($i = 1, \dots, 4$) and \mathcal{C}_i ($i = 1, 2$) are defined in (20). Consequently, we have,

$$\|\mathcal{T}(u, v)\| \leq (\ell_1 r + H_1)\mathcal{R}_1 + (\ell_2 r + H_2)\mathcal{R}_2 + \mathcal{C} \leq r.$$

Therefore, $\mathcal{T}B_r \subset B_r$.

Next, we show that the operator \mathcal{T} is a contraction. Let $(u_1, v_1), (u_2, v_2) \in \chi \times \chi$. Then we have

$$\begin{aligned} & |\mathcal{T}_1(u_1, v_1)(t) - \mathcal{T}_1(u_2, v_2)(t)| \\ & \leq \left\{ \int_a^t \left(\left| \frac{\mu_1}{p(z)} \right| \int_a^z |f(\tau, u_1(\tau), v_1(\tau)) - f(\tau, u_2(\tau), v_2(\tau))| d\tau \right) dz \right. \\ & \quad + \frac{1}{|Q|} \left[\int_a^b \int_a^s \frac{|\mu_2| \sum_{j=1}^m \alpha_j}{|q(z)|} \left(\int_a^z |g(\tau, u_1(\tau), v_1(\tau)) - g(\tau, u_2(\tau), v_2(\tau))| d\tau \right) dz ds \right. \\ & \quad \left. \left. + \left(\sum_{j=1}^m \alpha_j \right) \left(\sum_{k=1}^n \beta_k \int_a^{\xi_k} \frac{\mu_1}{p(z)} \left(\int_a^z |f(\tau, u_1(\tau), v_1(\tau)) - f(\tau, u_2(\tau), v_2(\tau))| d\tau \right) dz \right) \right. \right. \\ & \quad \left. \left. + (b-a) \int_a^b \int_a^s \left| \frac{\mu_1}{p(z)} \right| \left(\int_a^z |f(\tau, u_1(\tau), v_1(\tau)) - f(\tau, u_2(\tau), v_2(\tau))| d\tau \right) dz ds \right. \right. \\ & \quad \left. \left. + (b-a) \int_a^{\eta_j} \left(\frac{|\mu_2| \sum_{j=1}^m \alpha_j}{|q(z)|} \int_a^z |g(\tau, u_1(\tau), v_1(\tau)) - g(\tau, u_2(\tau), v_2(\tau))| d\tau \right) dz \right\} \right. \\ & \leq \ell_1 (|u_1 - u_2| + |v_1 - v_2|) \left\{ \left| \frac{\mu_1}{Q\bar{p}} \right| \left[|Q| \frac{(b-a)^2}{2} + \frac{(b-a)^4}{6} + \left(\sum_{k=1}^n \beta_k \right) \left(\sum_{j=1}^m \alpha_j \frac{(\xi_k - a)^2}{2} \right) \right] \right\} \\ & \quad + \ell_2 (|u_1 - u_2| + |v_1 - v_2|) \left\{ \frac{|\mu_2| \sum_{j=1}^m \alpha_j}{|Q\bar{q}|} \left[\frac{(b-a)(\eta_j - a)^2}{2} + \frac{(b-a)^3}{6} \right] \right\}, \end{aligned}$$

which implies that

$$\|\mathcal{T}_1(u_1, v_1) - \mathcal{T}_1(u_2, v_2)\| \leq (\ell_1 M_1 + \ell_2 M_2)(|u_1 - u_2| + |v_1 - v_2|). \tag{24}$$

In the same fashion, we can find that

$$\|\mathcal{T}_2(u_1, v_1) - \mathcal{T}_2(u_2, v_2)\| \leq (\ell_1 M_3 + \ell_2 M_4)(|u_1 - u_2| + |v_1 - v_2|). \tag{25}$$

In consequence, from (24) and (25), it follows that

$$\|\mathcal{T}(u_1, v_1) - \mathcal{T}(u_2, v_2)\| \leq (\mathcal{R}_1 \ell_1 + \mathcal{R}_2 \ell_2)(|u_1 - u_2| + |v_1 - v_2|). \tag{26}$$

According to the assumption (23), it follows from (26) that the operator \mathcal{T} is a contraction. Thus the Banach contraction mapping principle applies and the operator \mathcal{T} has a unique fixed point, which corresponds to a unique solution of the problem (3)–(4) on $[a, b]$. The proof is finished. ■

4 Illustrative Examples

Example 1 Consider the following coupled system of second-order ordinary differential equations

$$\begin{cases} \left(\left(\frac{4t+7}{t^2+15} \right) u'(t) \right)' = \frac{9}{257} \left[\frac{t}{270} + \frac{29}{240\sqrt{t+1}} u(t) + \frac{v(t)}{65(t^2+1)} \right], & t \in [0, 2], \\ \left((\sqrt{8t^2+25}) v'(t) \right)' = \frac{56}{34} \left[\frac{32}{7e^t} + \frac{1}{180(t+1)} u(t) + \frac{\sqrt{t^3}}{250+t^2} v(t) \right], & t \in [0, 2], \end{cases} \tag{27}$$

supplemented with the following boundary conditions:

$$\begin{aligned} u'(0) &= 0, \quad \int_0^2 u(s) \, ds - \sum_{j=1}^3 \alpha_j v(\eta_j) = 10, \\ v'(0) &= 0, \quad \int_0^2 v(s) \, ds - \sum_{k=1}^3 \beta_k u(\zeta_k) = 25. \end{aligned} \tag{28}$$

Here $p(t) = (4t + 7)/(t^2 + 15)$, $q(t) = \sqrt{8t^2 + 25}$, $a = 0$, $b = 2$, $\lambda_1 = 10$, $\lambda_2 = 25$, $\mu_1 = \frac{9}{257}$, $\mu_2 = \frac{56}{34}$, $\eta_1 = 1/3$, $\eta_2 = 2/3$, $\eta_3 = 1$, $\xi_1 = 6/5$, $\xi_2 = 7/5$, $\xi_3 = 8/5$, $\alpha_1 = 3/8$, $\alpha_2 = 1/2$, $\alpha_3 = 5/8$, $\beta_1 = 2/7$, $\beta_2 = 4/7$, $\beta_3 = 6/7$. Using the given values, we found that $|Q| \approx 1.428571429 \neq 0$, where Q is given by (8), $\bar{p} \approx 0.4666667$, $\bar{q} = 5$, $M_1 \approx 0.1694622581$, $M_2 \approx 1.350413943$, $M_3 \approx 1.983975761$, $M_4 \approx 32.20915033$, (\bar{p} , \bar{q} and M_i ($i = 1, \dots, 4$) are defined in (20)), $\mathcal{R}_1 \approx 2.153438019$ and $\mathcal{R}_2 \approx 33.55956427$ (\mathcal{R}_1 and \mathcal{R}_2 are given by (19)). Obviously,

$$|f(t, u, v)| \leq \frac{1}{3855} + \frac{87}{20560} \|u\| + \frac{9}{16705} \|v\|, \quad |g(t, u, v)| \leq \frac{128}{17} + \frac{7}{765} \|u\| + \frac{14}{2125} \|v\|,$$

with $\kappa_0 = 1/3855$, $\kappa_1 = 87/20560$, $\kappa_2 = 9/16705$, $\gamma_0 = 128/17$, $\gamma_1 = 7/765$, $\gamma_2 = 28/425$. Moreover, $\mathcal{R}_1 \kappa_1 + \mathcal{R}_2 \gamma_1 \approx 0.3161932909 < 1$, $\mathcal{R}_1 \kappa_2 + \mathcal{R}_2 \gamma_2 \approx 0.2222584939 < 1$, which implies that (21) is satisfied. Clearly the hypotheses of Theorem 1 are satisfied. In consequence, by the conclusion of Theorem 1, the problem (27)–(28) has at least one solution on $[0, 2]$.

Example 2 Consider the following system:

$$\begin{cases} \left(\left(\frac{4t+7}{t^2+15} \right) u'(t) \right)^2 = (3t^2 + 2) + \frac{\arctan(t)}{4\pi} (u(t))^{\frac{1}{4}} + \frac{2 \sin t}{9} (v(t))^{\frac{3}{4}}, \quad t \in [0, 2], \\ \left(\sqrt{8t^2 + 25} v'(t) \right)' = \frac{3(t+2)}{10} + \frac{8}{3(t^2+3)} (u(t))^{\frac{2}{7}} + \frac{10}{3(5+t)} (v(t))^{\frac{5}{9}}, \quad t \in [0, 2], \end{cases} \tag{29}$$

subject to the coupled boundary conditions of Example 4.1.

Clearly, the condition (S_2) is satisfied with $\omega(t) = (3t^2 + 2)$, $\lambda(t) = 3(t + 2)/10$, $z_1 = 1/4$, $z_2 = 3/4$, $k_1 = 2/7$, $k_2 = 5/9$, $\rho_1 = 1/8$, $\rho_2 = 2/9$, $\nu_1 = 8/9$, $\nu_2 = 2/3$. Consequently, Theorem 2 applies to the system (29) with boundary conditions (28). So, there exists at least one solution of the problem (29) with the coupled boundary conditions (28) on $[0, 2]$.

Example 3 Consider the following system:

$$\begin{cases} \left(\left(\frac{4t+7}{t^2+15} \right) u'(t) \right)' = \frac{1}{21} \left(\frac{|u(t)|}{1+|u(t)|} + v(t) + \frac{81}{8} \sqrt{3t+7} \right), \quad t \in [0, 2], \\ \left(\sqrt{8t^2 + 25} v'(t) \right)' = \frac{1}{5\sqrt{t^3+81}} \left(\sin u(t) + \frac{\cos t}{t+1} \arctan v(t) + \frac{1}{5} e^t \right), \quad t \in [0, 2], \end{cases} \tag{30}$$

with the coupled boundary conditions (28).

Obviously, we have

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \frac{1}{21} (|u_1 - u_2| + |v_1 - v_2|)$$

and

$$|g(t, u_1, v_1) - g(t, u_2, v_2)| \leq \frac{1}{45} (|u_1 - u_2| + |v_1 - v_2|),$$

with $\ell_1 = 1/21$, $\ell_2 = 1/45$. By the data found in Example 27, we have

$$\mathcal{R}_1 \approx 19.84279771, \quad \mathcal{R}_2 \approx 1.058148148 \quad \text{and} \quad \mathcal{R}_1 \ell_1 + \mathcal{R}_2 \ell_2 \approx 0.4913390674 < 1.$$

Thus, by Theorem 3, the problem (30) supplemented with the boundary conditions (28) has a unique solution on $[0, 2]$.

5 Conclusions

We have presented sufficient criteria for the existence and uniqueness of solutions for a coupled system of self-adjoint nonlinear second-order ordinary differential equations supplemented with integral nonlocal multi-point coupled boundary conditions on an arbitrary domain. The given problem is transformed into an equivalent fixed point operator problem. Then the tools of the fixed point theory are applied to establish the fixed points of the involved operator, which correspond to the solutions of the original problem. The obtained results are well illustrated with the aid of numerical examples.

As a special case, we obtain the new results for a coupled system of self-adjoint nonlinear second-order ordinary differential equations (3) complemented with the integral boundary conditions of the form:

$$u'(a) = 0, \int_a^b u(s)ds = \lambda_1, v'(a) = 0, \int_a^b v(s)ds = \lambda_2,$$

when $\alpha_j = 0, \beta_k = 0$ for all $j = 1, \dots, m$ and $k = 1, \dots, n$.

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