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WZ Proofs of Identities From Chu and Kılıç, With Applications*

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Abstract

We systematically apply the WZ method to prove finite sum identities involving the Catalan numbers recently established by Chu and Kılıç via classical infinite series identities, and we make use of the WZ pairs from our alternate proofs to prove new and very non-trivial evaluations for fast-converging infinite series involving inverted binomial coefficients, such as an elegant series for π^2 that contains the inverses of cubed central binomial coefficients and that can only be expressed as an inevaluable ${}_5F_4\left(\frac{1}{4}\right)$-series by current CAS software.

1 Introduction

The determination of closed-form evaluations of infinite series involving inverted binomial coefficients forms an important aspect about the discipline that is referred to as experimental mathematics [1, §1.7]. As in [1, §1.7], we record the classical series evaluation

$$\frac{2\pi}{\sqrt{3}} = \sum_{n=1}^{\infty} \frac{18-9n}{\binom{2n}{n}} \tag{1}$$

along with Gosper's formula whereby

$$\pi = \sum_{n=0}^{\infty} \frac{50n-6}{\binom{3n}{n}2^n} \tag{2}$$

as two prominent and well-known instances of binomial sum evaluations, but this is not to mention Apéry's famously making use of the equality

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3\binom{2n}{n}} \tag{3}$$

in his unmitigatedly renowned proof of the irrationality of the constant $\zeta(3)$, i.e., Apéry's constant; we also find it pertinent to make reference to Ramanujan's work on infinite series containing inverted binomial coefficients, as in with the following formula involving the golden ratio ϕ [4, §11]:

$$\frac{\pi^2}{6} - 3\ln^2(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+1)^2}. \tag{4}$$

In this article, we systematically apply the Wilf–Zeilberger (WZ) method to finite sum identities involving Catalan numbers that were quite recently proved by Chu and Kılıç in [3] using special cases of classical identities for infinite hypergeometric series, and we make use of the WZ pairs obtained from our alternate proofs so as to formulate creative telescoping proofs for new and elegant infinite summations that involve inverted binomial coefficients and that can only be expressed via current Computer Algebra System (CAS)

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software as linear combinations of inevaluable ${}_pF_q$-series. Our new series evaluations are very much inspired by the famous results displayed above, as in (1), (2), (3), and (4). For example, consider how our formula

$$\boxed{\frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{16^n (n+1)(3n+1)}{n(2n+1)^2 \binom{2n}{n}^3}} \tag{5}$$

that is introduced in this article bears a resemblance to the above displayed equations, and, on the other hand, considerably extends previously known series formulas involving "non-powered" expressions as in $\frac{1}{\binom{2n}{n}}$, in contrast to the cubed, inverted binomial coefficients in (5), which, informally, make the symbolic computation of the infinite summation in (5) virtually impossible with CAS software: Mathematica is only able to express this summation as a linear combination of inevaluable ${}_4F_3\left(\frac{1}{4}\right)$-series, and Maple is only able to express (5) as an inevaluable ${}_5F_4\left(\frac{1}{4}\right)$-series.

1.1 Background and Preliminaries

The famous WZ method [9] has, of course, been applied widely in number theory, special functions theory, combinatorics, computer science, and many other areas. For the time being, we only review basics about the WZ method that are directly needed for this article.

A function $A(n,k)$ in two variables is hypergeometric if both $A(n+1,k)/A(n,k)$ and $A(n,k+1)/A(n,k)$ are rational functions, and a WZ pair is a pair (F,G) of functions such that F and G are two-variable hypergeometric functions that are such that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k), \tag{6}$$

and we also enforce the conditions whereby $\lim_{k\to\infty} G(n,k) = 0$ and $G(n,0) = 0$ [9]. Letting $F(n,k)$ be hypergeometric, the WZ method determines a function $G(n,k)$ such that F and G form a WZ pair, i.e., provided that such an expression G exists [9]. Summing each side of (6) with respect to k, we obtain that:

$$\sum_k F(n+1,k) = \sum_k F(n,k). \tag{7}$$

Summing both sides of the equality in (6) over n, we find that

$$\sum_{n=0}^{\infty} G(n,k) - \sum_{n=0}^{\infty} G(n,k+1) = F(0,k) - \lim_{n\to\infty} F(n,k), \tag{8}$$

i.e., if the above limit exists and if the above series are convergent.

As stated in [7], an important property concerning WZ pairs is given by the identity whereby

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{k=0}^{\infty} F(0,k). \tag{9}$$

Guillera, in [7], used this identity, as applied to the WZ pair obtained by setting

$$U(n,k) := (-1)^{k+n} \left(\frac{4^n \binom{2k}{k}}{\binom{k+n}{k}\binom{2k+2n}{k+n}} \right)^3$$

and

$$F(n,k) := U(n,k) \frac{n+2k+1}{(2n+2k+1)^3},$$

so as to obtain the following formula for Catalan's constant:

$$2G = \sum_{n=0}^{\infty} \frac{\left(-2^6\right)^n (4n+3)}{(2n+1)^3 \binom{2n}{n}^3}. \tag{10}$$

This may be written as a linear combination of ${}_4F_3(-1)$-series. In contrast, we employ extensions of the WZ identity in (9) to prove the ${}_5F_4\left(\frac{1}{4}\right)$-series identity in (5) along with the π^2 formula highlighted in (16) below, and we improve upon Guillera's G-formula in (10) by proving a new formula for G that also involves inverted central binomial coefficients but that may be expressed as a linear combination of ${}_5F_4\left(-\frac{1}{4}\right)$-series and thus converges extremely quickly, especially compared to (10). Our new series for G is highlighted in (17) below, with a similarly fast-convergent series for Apéry's constant provided in later in this article; for the time being, let us illustrate the extent to which our G-series in (17) improves upon Guillera's formula shown in (10).

Numerically computing the finite sum

$$\frac{1}{2} \sum_{n=0}^{10000} \frac{\left(-2^6\right)^n (4n+3)}{(2n+1)^3 \binom{2n}{n}^3} = 0.922925...,$$

we find that (10) is so slowly convergent that the second digit after the decimal point of Catalan's constant 0.915965... is not even reached by summing the first ten thousand entries in the sequence given by the summand of (10). In contrast, computing

$$-\frac{1}{2} - \frac{1}{16} \sum_{n=1}^{8} \frac{\left(-2^8\right)^n \left(40n^2+4n-1\right)}{n^2(4n+1)\binom{2n}{n}\binom{4n}{2n}^2} = 0.915901..., \tag{11}$$

we see that it only takes *eight* terms to reach *four* decimal places of accuracy past the decimal point, in our approximating G.

We are to provide "one-line" WZ proofs as in [9] for Chu and Kılıç's identities [3], so it is worthwhile to review what we mean by the concept of a WZ proof certificate. Mimicking notation used in [9, §2.3], suppose that we want to prove an identity as in

$$\sum_{k} \text{summand}(n,k) = \text{rhs}(n,k), \tag{12}$$

where the above summand is such that: For all n, the above summand vanishes for all k outside of some finite interval. We then set $F(n,k) := \frac{\text{summand}(n,k)}{\text{rhs}(n,k)}$, so that the WZ method gives us a rational function $R(n,k)$ such that F and $G(n,k) = R(n,k)F(n,k)$ form a WZ pair [9, §2.3]. Recalling (7), we find that $\sum_k F(n,k)$ is constant, since it is independent of n, so it remains to check that the aforementioned constant is 1. So, simply "having" the function $R(n,k)$ satisfying all of the desired properties to form a WZ pair is a proof, by itself: Just by evaluating such a function $R(n,k)$ that satisfies the aforementioned properties, this, by itself, is a proof for (12), so that we may simply present the rational function $R(n,k)$ as a "one-line proof" (cf. [5, 12, 13]). A particularly famous instance of a one-line WZ proof is that for the famous identity whereby the level-n row sum of Pascal's triangle is 2^n [13], as below.

Proof. $R(n,k) = \frac{-k}{2(n-k+1)}$. ∎

One-line WZ proofs that dramatically simplify proofs for Catalan number identities recently due to Chu and Kılıç are given below, in Section 2, and applied using extensions of (9). However, before we move onto our main proofs/results, it is necessary to recall two basic properties concerning the famous function $\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du$ known as the Γ-function. In particular, we are to later make use of the famous *reflection formula* $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$, along with the Legendre duplication formula:

$$\Gamma\left(k+\frac{1}{2}\right) = \sqrt{\pi}\left(\frac{1}{4}\right)^k \binom{2k}{k}\Gamma(k+1).$$

Our formula for Catalan's constant suggested in (11) is a natural companion to the formula whereby

$$-64G=\sum_{k=1}^{\infty}\frac{\left(-2^{8}\right)^{k}\left(40k^{2}-24k+3\right)}{k^{3}\binom{4k}{2k}^{2}\binom{2k}{k}(2k-1)} \tag{13}$$

that was proved by Lupaş in [8] using the WZ method in a fundamentally different way compared to our methods. As in [2], we note that Guillera [6] had employed the WZ method to prove the formula

$$\frac{\pi^{2}}{2}=\sum_{k=1}^{\infty}\frac{(3k-1)16^{k}}{k^{3}\binom{2k}{k}^{3}}, \tag{14}$$

which bears a resemblance to our formulas for π^2, as in (5) and (16); the WZ pair that Guillera had employed to prove (14) is also not equivalent to our WZ-based methods. The formula in (13) was also recorded in [10], in which it is noted that a proof of this same formula was given in [11]; this was proved in [11] using the Marko-Wilf-Zeilberger method, in contrast to the techniques applied in this article. We see that our Catalan formula in (17) is inequivalent to (13), in consideration as to the degrees of the polynomial factors in the respective numerators and denominators. The formula in (13) was reproduced in an unpublished note by Lima, in which Lima used hypergeometric series manipulations to prove a series for G involving inverted, cubed central binomial coefficients, resembling Guillera's formula

$$\sum_{k=1}^{\infty}(2k-1)\frac{(-8)^{k}}{k^{3}\binom{2k}{k}^{3}}=-2G$$

proved in [6], again in an inequivalent way compared to our techniques.

2 Main Proofs and Results

Letting $C_m=\binom{2m}{m}/(m+1)$ denote the m^{th} entry in the sequence of Catalan numbers, part (a) of Theorem 1 from [3] gives us that the identity whereby $\sum_{k=0}^{n}\binom{2n}{2k}C_kC_{n-k}=C_nC_{n+1}$ holds for $n\in\mathbb{N}_0$, and this is proved in a direct way through an application of Gauss' summation theorem, i.e., by writing C_n times

$${}_2F_1\left[\begin{matrix}-n,-n-1\\2\end{matrix}\,\middle|\,1\right]$$

as the finite sum under consideration, and by then writing this ${}_2F_1(1)$-expression as a quotient of Γ-expressions. We offer, as below, a one-line proof for the convolution-type identity under consideration, and we then apply the WZ pair associated with this summation identity to prove the remarkable ${}_5F_4\left(\frac{1}{4}\right)$-series evaluation highlighted as a motivating example in (5).

Theorem 1 *Part (a) of Theorem 1 from [3] holds.*

Proof. $R(n,k)=\frac{k(2k-3n-5)(k+1)}{2(2n+3)(k-n-2)(k-n-1)}$. ∎

We proceed to make use of an extension of (9), as applied to the WZ pair given by our proof of Theorem 1. More explicitly, this WZ pair is such that

$$F(n,k):=\frac{(n+1)(n+2)\binom{2k}{k}\binom{2n}{2k}\binom{2(n-k)}{n-k}}{(k+1)\binom{2n}{n}\binom{2n+2}{n+1}(n-k+1)},$$

with $G(n,k)=R(n,k)F(n,k)$. This leads us to the below proof.

Theorem 2 *The closed-form evaluation in (5), as reproduced below, must hold:*

$$\boxed{\frac{\pi^2}{4} = \sum_{n=0}^{\infty} \frac{16^n (n+1)(3n+1)}{n(2n+1)^2 \binom{2n}{n}^3}.}$$

Proof. Letting (F, G) denote the WZ pair indicated above, it is easily seen that $\lim_{a\to\infty} F(a,r) = 0$ for all r. So, in view of the WZ identity in (8), we may easily establish that

$$-F(0,r) = \sum_{n=0}^{\infty} (G(n, r+1) - G(n,r))$$

for suitably bounded real r. Setting the variable r to be equal to b, $b+1$, $b+2$, etc., and then adding the resultant identities, so as to form a telescoping sum of infinite series, we can show that

$$-\sum_{n=0}^{m} F(0, b+n) = \sum_{n=0}^{\infty} (G(n, b+m+1) - G(n,b)) \tag{15}$$

for nonnegative integers m. Since $G(n, 0) = 0$, we see that

$$\sum_{n=0}^{\infty} G(n, b+m+1) + \sum_{n=0}^{m} F(0, b+n)$$

vanishes as $b \to 0$. According to the reflection formula for the Γ-function, this gives us that:

$$\sum_{n=0}^{\infty} G(n, b+m+1)$$

approaches

$$\frac{2\left(b^2+b+1\right)\sin^2(\pi b)}{\pi^2 b^2 (b+1)^2 \left(b^2+b-2\right)}$$

as $b \to 0$. This is easily seen to give us that $\lim_{c\to\infty} \sum_{n=0}^{\infty} G(n,c) = -1$ for real c. So, this leads us, according to (15), to the following, for real b, giving us an extension of (9):

$$-\sum_{n=0}^{\infty} F(0, b+n) = -1 - \sum_{n=0}^{\infty} G(n, b).$$

In particular, setting $b = \frac{1}{2}$ and applying the reflection formula, this gives us that

$$\sum_{n=0}^{\infty} \frac{16\cos^2(\pi n)}{\pi^2 (2n-1)(2n+1)^2 (2n+3)}$$

equals

$$-1 + \sum_{n=0}^{\infty} \frac{2^{-2n-3}(n+1)(3n+4)\Gamma^2(n+1)\Gamma(n+3)}{\sqrt{\pi}\,\Gamma\left(n+\frac{3}{2}\right)\Gamma^2\left(n+\frac{5}{2}\right)}.$$

Evaluating this second-to-last infinite series as $-\frac{1}{2}$ only involves basic calculus. Rewriting the summand for this latter series according to the Legendre duplication formula, this gives us that:

$$\frac{\pi^2}{256} = \sum_{n=0}^{\infty} \frac{2^{4n}(3n+4)}{(n+1)(n+2)\binom{2n+2}{n+1}\binom{2n+4}{n+2}^2}.$$

The desired result then easily follows by applying an index shift. ■

We find that the evaluation in the above theorem is equivalent to the following ${}_5F_4\left(\frac{1}{4}\right)$-series evaluation:

$$\boxed{\frac{9\pi^2}{64} = {}_5F_4\left[\begin{matrix}1,1,2,\frac{7}{3},3\\ \frac{4}{3},\frac{3}{2},\frac{5}{2},\frac{5}{2}\end{matrix}\,\middle|\,\frac{1}{4}\right].}$$

There is not much known about ${}_{q+1}F_q\left(\frac{1}{4}\right)$-series, especially for higher-order values q, motivating our interest in the above formula.

Part (b) of Theorem 1 from [3] states that $\sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}kC_k$ equals $(-1)^n$, and this is proved using the Gauss summation theorem. Instead of using this classical hypergeometric identity, we present the following one-line WZ proof.

Theorem 3 *Part (b) of Theorem 1 from [3] holds.*

Proof. $R(n,k) = \frac{2(n+1)(k-1)(k+1)}{(n-k+1)n(n+2)}$. ■

The techniques that we had used in our proof of Theorem 2 cannot be applied to the WZ pair corresponding to the above theorem, since the limit of $F(a,b)$ as $a \to \infty$ is not, in general, finite. So we move onto part (c) of Theorem 1 from [3], which states that $\sum_{k=0}^{n}\binom{2n}{n+k}\binom{n+k}{2k}C_k$ equals $\binom{2n+1}{n}C_n$. Again, the Gauss summation identity is used in [3] to establish this result, in contrast to our one-line proof below.

Theorem 4 *Part (c) of Theorem 1 from [3] holds.*

Proof. $R(n,k) = \frac{k\left(2kn-3n^2+3k-8n-5\right)(k+1)}{2(k-n-1)^2(2n+3)(n+1)}$. ■

Letting (F,G) denote the WZ pair associated with our proof of the above theorem, we can, using exactly the same approach as in our proof of Theorem 2, show that $\sum_{n=0}^{\infty} -F(0,r+n) = -1 + \sum_{n=0}^{\infty} -G(n,r)$ for real r. Setting $r = \frac{1}{2}$ and applying an index shift, this is easily seen to give us that:

$$\boxed{2 + \frac{\pi^2}{2} = \sum_{n=1}^{\infty} \frac{16^n\left(6n^2+2n-1\right)}{n^2(2n+1)\binom{2n}{n}^3}.} \tag{16}$$

Again, Mathematica is only able to evaluate the above series as a linear combination of inevaluable ${}_4F_3\left(\frac{1}{4}\right)$-series. Maple is only able to evaluate the above series as a ${}_6F_5\left(\frac{1}{4}\right)$-series that contains non-rational parameters.

Part (d) of Theorem 1 from [3] gives us that $\sum_{k=0}^{n}\binom{2n}{2k}\binom{2n-2k}{n-k}C_k$ equals $(2n+1)C_n^2$. We proceed to offer the following one-line WZ proof of this result.

Theorem 5 *Part (d) of Theorem 1 from [3] holds.*

Proof. $R(n,k) = \frac{k(k+1)\left(2kn+3k-3n^2-8n-5\right)}{2(n+1)(2n+3)(k-n-1)^2}$. ■

Letting (F,G) denote the WZ pair corresponding to the above proof, we again have that

$$-\sum_{n=0}^{\infty} F(0,r+n) = -1 - \sum_{n=0}^{\infty} G(n,r)$$

for real r. Setting $r = \frac{1}{2}$ and applying reindexing, this gives us another proof of (16).

We move on to Theorem 2 from [3], which leads us to a proof for our remarkable G-series discussed in Section 1.1. Part (a) of Theorem 2 from [3] states that

$$\sum_{k=0}^{\ell}(-1)^k\binom{2\ell}{2k}C_kC_{\ell-k} = (-1)^{\binom{\ell}{2}}C_\ell C_{\ell/2}$$

if ℓ is even, and that this finite sum vanishes otherwise. This is proved in [3] using the Kummer summation theorem together with limiting properties about the Γ-function. In contrast, setting $\ell = 2n$, we may obtain a one-line proof for the non-vanishing case for the finite sum identity under consideration.

Theorem 6 *The first case for part (a) of Theorem 2 from [3] holds.*

Proof. $R(n,k) = \frac{k(k+1)\left(k^2-6kn-7k+10n^2+24n+14\right)}{2(k-2n-3)(k-2n-2)^2(k-2n-1)}$. ■

We let (F,G) denote the WZ pair corresponding to the above proof, giving us that

$$-\sum_{n=0}^{m} F(0,r+n) = \sum_{n=0}^{\infty}\left(G(n,r+m+1)-G(n,r)\right),$$

and, by letting $m\to\infty$, we may apply almost exactly the same approach as in our proof of Theorem 2 to prove that:

$$\boxed{-16G-8 = \sum_{n=1}^{\infty}\frac{\left(-2^8\right)^n\left(40n^2+4n-1\right)}{n^2(4n+1)\binom{2n}{n}\binom{4n}{2n}^2}.} \tag{17}$$

We now turn our attention toward Theorem 3 from [3]. Part (a) of this Theorem states that

$$\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\binom{2\ell}{2k}C_kC_{\ell-k} = (-1)^{\binom{\ell}{2}}\binom{1+3\ell/2}{\ell/2}C_\ell C_{\ell/2}$$

if ℓ is even, and that this finite sum vanishes otherwise. This is proved in [3] by expressing this finite sum as a ${}_3F_2$-series, and by then applying Dixon's theorem and limiting properties about the Γ-function. In contrast, for the non-vanishing case, we may obtain the simplified WZ proof below.

Theorem 7 *The first case for part (a) of Theorem 3 from [3] holds.*

Proof. The proof certificate, in this case, is given by the following verbatim output.

```
(k^2*(1 + k)*(584 - 816*k + 460*k^2 - 120*k^3 + 12*k^4 + 2880*n - 3140*k*n + 1281*k^2*n -
210*k^3*n + 9*k^4*n + 5560*n^2 - 4440*k*n^2 + 1166*k^2*n^2 - 90*k^3*n^2 + 5264*n^3 -
2740*k*n^3 + 348*k^2*n^3 + 2448*n^4 - 624*k*n^4 + 448*n^5))/(6*(-3 + k - 2*n)*(-2 + k -
2*n)^3*(-1 + k - 2*n)^2*(2 + 3*n)*(4 + 3*n))
```
■

By mimicking our proof of Theorem 2, as applied to the WZ pair corresponding to the above theorem, we may prove the remarkable result below; for the sake of brevity, we leave this to the reader.

$$\boxed{-448\zeta(3)-128 = \sum_{n=1}^{\infty}\frac{(-2^{12})^n\left(7168n^5-1664n^4-1328n^3+212n^2+49n-9\right)}{n^4(2n-1)(3n+1)(4n+1)\binom{2n}{n}\binom{3n}{n}\binom{4n}{2n}^3}}$$

Mathematica is only able to evaluate the above series as a linear combination of inevaluable ${}_9F_8\left(-\frac{1}{27}\right)$-series, and similarly for Maple. Again, for the sake of brevity, we refer the reader to [3] for the Catalan sum identity referred to below.

Theorem 8 *The first case for part (b) of Theorem 3 from [3] holds.*

Proof. $R(n,k) = \frac{k(2k-1)\left(4k^2n+8k^2-24kn^2-64kn-32k+32n^3+120n^2+118n+33\right)}{2(n+2)(2n-3)(2k-4n-3)(2k-4n-1)(k-2n-1)}$. ■

In this case, $\sum_{n=0}^{m}(G(n,r+1)-G(n,r))$ does not converge, so it is unclear as to how to go about mimicking our proof of Theorem 2.

A number of the results given within the Theorems from [3] are not proved in this article, since the WZ method does not seem to apply in these cases with readily available implementations of the WZ method, e.g., through Maple's *SumTools*[*Hypergeometric*] subpackage. We encourage extending our results using these "leftover cases" and extending the results given in this paper using variants or generalizations of our WZ pairs.

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