Applied Mathematics E-Notes, 22(2022), 252-264 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Bifurcation Diagrams For Two-Point Boundary Value Problem With Quadratic Nonlinearity*

Shao-Yuan Huang[†], Ping-Han Hsieh[‡]

Received 6 June 2021

Abstract

In this paper, we provide the bifurcation diagrams of positive solutions of two-point boundary value problem

$$\begin{cases} -u'' = \lambda f(u), & \text{in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases}$$

where $f(u) = -au^2 + bu + c$, $a, b, c \in \mathbb{R}$ and $a \neq 0$. By these results, we obtain the exact multiplicity of positive solutions. In addition, there no references to completely solve this problem. Thus this research is important.

1 Introduction

In this paper, we study the shapes of bifurcation curve of positive solutions for two-point boundary value problem

$$\begin{cases} -u'' = \lambda(-au^2 + bu + c), & \text{in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \tag{1}$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$. On the $(\lambda, \|u\|_\infty)$-plane, we define the bifurcation curve S of (1) by

$$S \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1)}\}. \tag{2}$$

For the sake of convenience, we let

$$f(u) \equiv -au^2 + bu + c.$$

It is well-known that studying of the exact multiplicity of positive solutions of (1) is equivalent to studying the shape of the bifurcation curve S. Thus this research is important. For similar researches, we refer to [2, 3, 4, 5, 6, 7] and references therein.

The main motive is to study the problem with cubic nonlinearity

$$\begin{cases} -u'' = \lambda(-\varepsilon u^3 + \sigma u^2 + \tau u + \rho), & \text{in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \tag{3}$$

where $\varepsilon > 0$, $\sigma > 0$, $\tau > 0$ and $\rho > 0$. Hung and Wang [7, Theorem 2.1] proved that the bifurcation curve of (3) is from S-shaped to monotone increasing with varying $\varepsilon > 0$. However, there are no references to completely obtain the global bifurcation diagrams for problem (1) with general quadratic polynomial f. Thus we begin this research.

*2010 Mathematics Subject Classi cation: 34B15, 34B18, 34C23, 74G35.

[†]Department of Mathematics and Information Education National Taipei University of Education, Taipei 106, Taiwan
[‡]Department of Mathematics and Information Education National Taipei University of Education, Taipei 106, Taiwan

2 Main Results

It is well-known that (1) has no positive solutions when $f(u) < 0$ for all $u > 0$. So we require that $f(u) > 0$ for some $u > 0$. Then we consider the following eight conditions:

(C1) $a < 0$, $b > 0$ and $c = 0$.

(C2) either $a < 0$ and $\Delta < 0$, or $a < 0$, $b > 0$, $c > 0$ and $\Delta > 0$.

(C3) $a < 0$, $b < 0$ and $\Delta = 0$.

(C4) $a < 0$, $b < 0$, $c > 0$ and $\Delta > 0$.

(C5) either $a < 0$ and $c < 0$, or $a < 0$, $b < 0$ and $c = 0$.

(C6) $a > 0$ and $c > 0$;

(C7) $a > 0$, $b > 0$ and $c = 0$;

(C8) $a > 0$, $b > 0$, $c < 0$ and $\Delta > 0$

where $\Delta \equiv b^2 + 4ac$. See Figure 1.

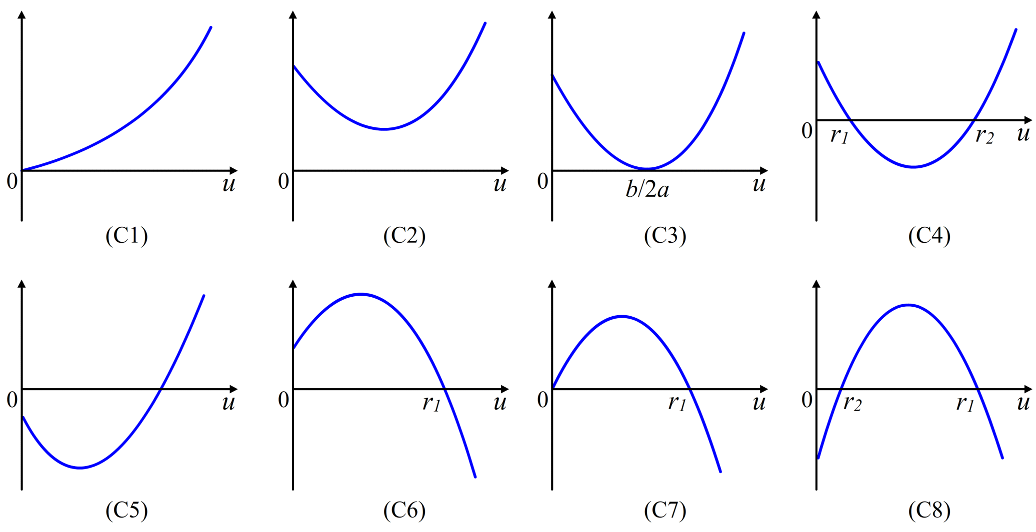

Figure 1: Graphs of $f(u)$ on $[0, \infty)$ when $f(u) > 0$ for some $u > 0$.

The following Theorem 1 is our main result.

Theorem 1 *Consider (1). Let*

$$r_1 \equiv \frac{b + \sqrt{b^2 + 4ac}}{2a}, \quad r_2 \equiv \frac{b - 2\sqrt{b^2 + 4ac}}{2a}, \tag{4}$$

$$\eta \equiv \frac{3b - \sqrt{9b^2 + 48ac}}{4a} \quad \text{and} \quad \bar{\eta} \equiv \frac{3b + \sqrt{9b^2 + 48ac}}{4a}. \tag{5}$$

Then the following statements hold:

(1) *If (C1) holds, then the bifurcation curve S is strictly decreasing, starts from $(0, \infty)$ and goes to the point $(\frac{\pi^2}{4b}, 0)$.*

(2) *If (C2) holds, then the bifurcation curve S is $\supset$-shaped, starts from the point $(0, 0)$ and goes to $(0, \infty)$.*

(3) *If (C3) holds, then the bifurcation curve S has two disjoint connected components such that*

(3a) the upper branch of S is strictly decreasing, starts from $(0, \infty)$ and goes to $(\infty, \frac{b}{2a})$;

(3b) the lower branch of S is strictly increasing, starts from the point $(0, 0)$ and goes to $(\infty, \frac{b}{2a})$.

(4) *If (C4) holds and*

$$b \leq -2\sqrt{\left(\frac{4}{3}\sqrt{2} - 3\right) ac} \approx -2.111\sqrt{-ac}, \tag{6}$$

then the bifurcation curve S has two disjoint connected components such that

(4a) the upper branch of S is strictly decreasing, starts from $(0, \infty)$ and goes to (∞, r_2);

(4b) the lower branch of S is strictly increasing, starts from the point $(0, 0)$ and goes to (∞, r_1).

(5) *If (C5) holds, then the bifurcation curve S is strictly decreasing, starts from $(0, \infty)$ and goes to (σ, η) where*

$$\sigma \equiv \int_0^1 \sqrt{\frac{3}{-2at\,(1-t)\,(\eta t - \bar{\eta})}} dt. \tag{7}$$

(6) *If (C6) holds, then the bifurcation curve S is strictly increasing, starts from the point $(0, 0)$ and goes to (∞, r_1).*

(7) *If (C7) holds, then the bifurcation curve S is strictly increasing, starts from the point $(\frac{\pi^2}{b}, 0)$ and goes to $(\infty, \frac{b}{a})$.*

(8) *If (C8) holds and $b \leq \frac{4}{\sqrt{3}}\sqrt{-ac}$, then the bifurcation curve S does not exist (i.e. (1) has no positive solutions for all $\lambda > 0$). If (C8) holds and $b > \frac{4}{\sqrt{3}}\sqrt{-ac}$, then the bifurcation curve S is $\subset$-shaped, starts from the point (σ, η) and goes to (∞, r_1) where σ is defined by (7).*

3 Proofs of Main Result

In order to study the shape of bifurcation curve S of (1), we use the time-map techniques. The time-map formula which we apply to study (1) takes the form as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{\sqrt{F(\alpha) - F(u)}} du \equiv T(\alpha), \tag{8}$$

where $F(u) \equiv \int_0^u f(t)dt = \frac{-a}{3}u^3 + \frac{b}{2}u^2 + cu$, see [1]. Observe that positive solutions u_λ for (1) correspond to

$$\|u_\lambda\|_\infty = \alpha \;\text{ and }\; T(\alpha) = \sqrt{\lambda}.$$

It implies that by (2),

$$S = \left\{(\lambda, \alpha) : \sqrt{\lambda} = T(\alpha)\right\} \tag{9}$$

Thus, studying the shapes of bifurcation curve S is equivalent to studying the shape of the time map $T(\alpha)$. In addition, we observe that

$$f(u) = -a(u - r_1)(u - r_0) \;\text{ and }\; F(u) = -\frac{a}{3}u\,(u - \eta)\,(u - \bar{\eta})\,, \tag{10}$$

where r_1 is defined by (4), η and $\bar{\eta}$ are defined by (5), and

$$r_0 \equiv \frac{b - \sqrt{b^2 + 4ac}}{2a}. \tag{11}$$

Next, we begin to prove Theorem 1.

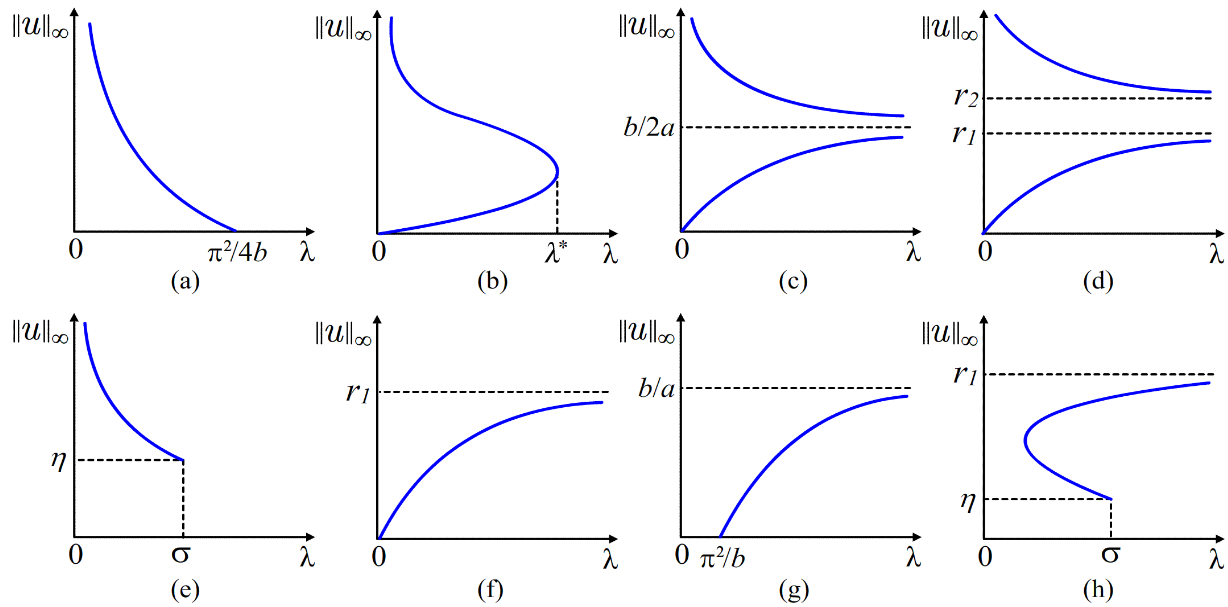

Figure 2: Graphs of bifurcation curves S. (a) (C1) holds. (b) (C2) holds. (c) (C3) holds. (d) (C4) and (6) hold. (e) (C5) holds. (f) (C6) holds. (g) (C7) holds. (h) (C8) holds and $b > \frac{4}{\sqrt{3}}\sqrt{-ac}$.

3.1 Proof of Theorem 1(1)

Assume that (C1) holds. Then $f(u) > 0$ for $u > 0$. So by (8), the domain of T is $(0, \infty)$. We compute

$$T'(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{[F(\alpha) - F(u)]^{3/2}} du, \tag{12}$$

where $\theta(u) \equiv 2F(u) - uf(u)$. Since $a < 0$ and $\theta(u) = \frac{1}{3}au^3$, and by (12), we see that

$$T'(\alpha) = \frac{a}{6\sqrt{2}\alpha} \int_0^\alpha \frac{\alpha^3 - u^3}{[F(\alpha) - F(u)]^{3/2}} du < 0 \quad \text{for } \alpha > 0. \tag{13}$$

In addition, we compute

$$f(0) = 0, \;\; f(u) - uf'(0) = -au^2 > 0 \;\text{ for } u > 0, \;\text{ and }\; \lim_{u\to\infty} \frac{f(u)}{u} = \infty.$$

So by [1, Theorem 2.5 and Corollary 2.10.1], we obtain

$$\lim_{\alpha\to 0^+} T(\alpha) = \frac{\pi}{2\sqrt{b}} \quad \text{and} \quad \lim_{\alpha\to\infty} T(\alpha) = 0. \tag{14}$$

By (9), (13) and (14), the proof of Theorem 1(1) is complete.

3.2 Proof of Theorem 1(2)

Assume that (C2) holds. Then $f(u) > 0$ for $u > 0$. So by (8), the domain of T is $(0, \infty)$. Clearly, we have $a < 0$ and $c > 0$. Then

$$\frac{d}{du}\frac{f(u)}{u} = \frac{f'(u)u - f(u)}{u^2} = \frac{-au^2 - c}{u^2} > 0 \;\text{ for eventually } u > 0. \tag{15}$$

Since $f(0) = c > 0$ and f is convex on $(0, \infty)$, and by (15) and [1, Theorem 3.2], we see that

$$T(\alpha) \text{ is strictly increasing and then strictly decreasing on } (0, \infty). \tag{16}$$

In addition, we compute

$$\lim_{u \to 0^+} \frac{f(u)}{u} = \lim_{u \to \infty} \frac{f(u)}{u} = \infty.$$

So by [1, Theorems 2.5 and 2.9], we obtain

$$\lim_{\alpha \to 0^+} T(\alpha) = \lim_{\alpha \to \infty} T(\alpha) = 0. \tag{17}$$

By (9), (16) and (17), the proof of Theorem 1(2) is complete.

3.3 Proof of Theorem 1(3)

Assume that (C3) holds. Then $f(u) = -a(u - \frac{b}{2a})^2 > 0$ for $u > 0$ and $u \neq \frac{b}{2a}$. So by (8), the domain of T is $(0, \infty) \backslash \{\frac{b}{2a}\}$. We compute

$$\lim_{u \to 0^+} \frac{f(u)}{u} = \lim_{u \to \infty} \frac{f(u)}{u} = \infty.$$

So by [1, Theorems 2.5 and 2.9], we obtain

$$\lim_{\alpha \to 0^+} T(\alpha) = \lim_{\alpha \to \infty} T(\alpha) = 0. \tag{18}$$

Since

$$F(\alpha) - F(u) = \int_u^\alpha f(t)dt = \frac{-a}{3}\left[(\alpha - \frac{b}{2a})^3 - (u - \frac{b}{2a})^3\right],$$

and by (8), we compute

$$\lim_{\alpha \to (\frac{b}{2a})^\pm} T(\alpha) = \sqrt{\frac{3}{-2a}} \int_0^{\frac{b}{2a}} \frac{1}{\left(\frac{b}{2a} - u\right)^{3/2}} du = \infty. \tag{19}$$

In addition, by [1, Lemma 3.1], (1) has at most two positive solutions for $\lambda > 0$. By (2), (18) and (19), we observe that $T(\alpha)$ is strictly increasing on $(0, \frac{b}{2a})$ and strictly decreasing on $(\frac{b}{2a}, \infty)$. So by (9), (18) and (19), the proof of Theorem 1(3) is complete.

3.4 Proof of Theorem 1(4)

Assume that (C4) and (6) hold. Recall r_0, r_1 and r_2 defined by (11) and (4), respectively. By (C4), we observe that $0 < r_1 < r_0 < r_2$. By (10), then

$$F'(u) = f(u) \begin{cases} > 0 & \text{for } 0 < u < r_1 \text{ or } u > r_0, \\ = 0 & \text{for } u = r_1 \text{ and } u = r_0, \\ < 0 & \text{for } r_1 < u < r_0. \end{cases} \tag{20}$$

Since we compute

$$F(r_1) = \frac{\left(b^2 + 6ac\right) b + \left(b^2 + 4ac\right) \sqrt{b^2 + 4ac}}{12a^2} = F(r_2), \tag{21}$$

and by (20), we observe that

$$F(\alpha) - F(u) > 0 \text{ for } 0 < u < \alpha \text{ and } \alpha \in (0, r_1] \cup (r_2, \infty). \tag{22}$$

Since $f(r_1) = 0$, and by (22), the domain of T is $(0, r_1) \cup (r_2, \infty)$. Recall the function $\theta(u)$ defined in the proof of Theorem 1(1). Clearly, $\theta(u) = \left(au^2 + 3c\right) u/3$. It follows that

$$\theta(0) = \theta(\sqrt{-3c/a}) = 0 \quad \text{and} \quad \theta'(u) = au^2 + c \begin{cases} > 0 & \text{for } 0 < u < \sqrt{\frac{c}{-a}}, \\ = 0 & \text{for } u = \sqrt{\frac{c}{-a}}, \\ < 0 & \text{for } u > \sqrt{\frac{c}{-a}}. \end{cases} \tag{23}$$

So we observe that

$$\theta(\alpha) - \theta(u) \begin{cases} > 0 & \text{for } 0 < u < \alpha \leq \sqrt{\frac{c}{-a}}, \\ < 0 & \text{for } 0 < u < \alpha \text{ and } \alpha \geq \sqrt{\frac{3c}{-a}}. \end{cases} \tag{24}$$

Next, we divide this proof into the following three steps.

Step 1. We prove that $T(\alpha)$ is strictly increasing on $(0, r_1)$. Since

$$\left(2\sqrt{-ac} + \sqrt{b^2 + 4ac} + b\right)\left(2\sqrt{-ac} + \sqrt{b^2 + 4ac} - b\right) = 4\sqrt{-ac\left(b^2 + 4ac\right)} > 0,$$

and $b < 0$, we see that $2\sqrt{-ac} + \sqrt{b^2 + 4ac} + b > 0$. It follows that

$$\sqrt{\frac{c}{-a}} - r_1 = \frac{2\sqrt{-ac} + \sqrt{b^2 + 4ac} + b}{2\,(-a)} > 0$$

because $a < 0$. By (24), we obtain $\theta(\alpha) - \theta(u) > 0$ for $0 < u < \alpha < r_1$. So by (12), $T'(\alpha) > 0$ on $(0, r_1)$. It implies that $T(\alpha)$ is strictly increasing on $(0, r_1)$.

Step 2. We prove that $T(\alpha)$ is strictly decreasing on (r_2, ∞). By (6), we have

$$-b \geq 2\sqrt{\left(\frac{4}{3}\sqrt{2} - 3\right) ac} \quad \text{and} \quad b^2 \geq 4\left(\frac{4}{3}\sqrt{2} - 3\right) ac.$$

Then we observe that

$$\begin{aligned} r_2 &= \frac{-b + 2\sqrt{b^2 + 4ac}}{-2a} \geq \frac{-2\sqrt{\left(\frac{4}{3}\sqrt{2} - 3\right) ac} + 2\sqrt{4\left(\frac{4}{3}\sqrt{2} - 3\right) ac + 4ac}}{-2a} \\ &= \frac{\left(\sqrt{8 - \frac{16}{3}\sqrt{2}} - \sqrt{3 - \frac{4}{3}\sqrt{2}}\right)\sqrt{-ac}}{-a} = \sqrt{\frac{3c}{-a}}. \end{aligned}$$

By (24), we obtain $\theta(\alpha) - \theta(u) < 0$ for $0 < u < \alpha$ and $\alpha > r_2$. So by (12), $T'(\alpha) < 0$ on (r_2, ∞). It implies that $T(\alpha)$ is strictly decreasing on (r_2, ∞).

Step 3. We prove

$$\lim_{\alpha \to 0^+} T(\alpha) = \lim_{\alpha \to \infty} T(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to r_1^-} T(\alpha) = \lim_{\alpha \to r_2^+} T(\alpha) = \infty.$$

Since $\lim_{u \to 0^+} f(u)/u = \infty$, and by [1, Theorem 2.9], we obtain $\lim_{\alpha \to 0^+} T(\alpha) = 0$. By (10), we compute

$$0 < \lim_{u \to r_1^-} \frac{f(u)}{r_1 - u} = -a(r_0 - r_1) < \infty.$$

So by [1, Theorem 2.6], we obtain $\lim_{\alpha\to r_1^-} T(\alpha)=\infty$. In addition, by (21), then

$$\begin{aligned}\lim_{\alpha\to r_2^+} T(\alpha) &= \lim_{\alpha\to r_2^+}\frac{1}{\sqrt{2}}\int_0^\alpha \frac{1}{\sqrt{F(r_2)-F(u)}}du \geq \lim_{\alpha\to r_1^-}\frac{1}{\sqrt{2}}\int_0^\alpha \frac{1}{\sqrt{F(r_2)-F(u)}}du \\ &= \lim_{\alpha\to r_1^-}\frac{1}{\sqrt{2}}\int_0^\alpha \frac{1}{\sqrt{F(r_1)-F(u)}}du = \lim_{\alpha\to r_1^-} T(\alpha)=\infty.\end{aligned}$$

Let $M>0$. Since $\lim_{u\to\infty} f(u)/u=\infty$, there exists $N>r_2$ such that $f(u)>Mu$ for $u\geq N$. Then

$$F(\alpha)-F(u)=\int_u^\alpha f(t)dt > M\int_u^\alpha tdt=\frac{M}{2}\left(\alpha^2-u^2\right)>0 \text{ for } \alpha>u\geq N. \tag{25}$$

By (22) and (25), then

$$\begin{aligned}F(\alpha)-F(u) &= [F(\alpha)-F(N)]+[F(N)-F(u)] \\ &> F(\alpha)-F(N)\geq \frac{M}{2}\left(\alpha^2-N^2\right) \quad \text{for } \alpha>2N>N>u>0.\end{aligned}$$

So for $\alpha>2N$,

$$\begin{aligned}T(\alpha) &= \frac{1}{\sqrt{2}}\int_0^N \frac{1}{\sqrt{F(\alpha)-F(u)}}du+\frac{1}{\sqrt{2}}\int_N^\alpha \frac{1}{\sqrt{F(\alpha)-F(u)}}du \\ &\leq \frac{1}{\sqrt{M}}\left(\int_0^N \frac{1}{\sqrt{\alpha^2-N^2}}du+\int_N^\alpha \frac{1}{\sqrt{\alpha^2-u^2}}du\right) \\ &= \frac{1}{\sqrt{M}}\left(\frac{N}{\sqrt{\alpha^2-N^2}}+\arcsin 1-\arcsin\frac{N}{\alpha}\right)\leq \frac{1}{\sqrt{M}}\left(\frac{1}{\sqrt{3}}+\arcsin\frac{u}{2N}\right).\end{aligned}$$

Since M is arbitrary, we see that $\lim_{\alpha\to\infty} T(\alpha)=0$.

So by (9) and Steps 1–3, the proof of Theorem 1(4) is complete.

3.5 Proof of Theorem 1(5)

Assume that (C5) holds. Recall r_0 and r_1 defined by (11) and (4), respectively. By (C5) and (10), we observe that $r_1<0<r_0$. By (10), then

$$F'(u)=f(u)\begin{cases} <0 & \text{for } 0<u<r_0, \\ =0 & \text{for } u=r_0, \\ >0 & \text{for } u>r_0.\end{cases} \tag{26}$$

Recall η and $\bar{\eta}$ defined by (5). By (C5), we observe that $\bar{\eta}<0<\eta$. So by (10) and (26), $F(\alpha)-F(u)>0$ for $0<u<\alpha$ and $\alpha>\eta$. It implies that the domain of T is (η,∞). Since $a<0$ and $c\leq 0$, we see that

$$\theta(\alpha)-\theta(u)=\frac{a\left(\alpha^3-u^3\right)+3c\left(\alpha-u\right)}{3}<0 \text{ for } 0<u<\alpha,$$

from which it follows that by (12), $T'(\alpha)<0$ on (η,∞).

By (8) and (10), we observe that

$$\begin{aligned}\lim_{\alpha\to\eta^+} T(\alpha) &= \lim_{\alpha\to\eta^+}\frac{1}{\sqrt{2}}\int_0^1 \frac{\alpha}{\sqrt{F(\alpha)-F(\alpha t)}}dt=\frac{1}{\sqrt{2}}\int_0^1 \frac{\eta}{\sqrt{-F(\eta t)}}dt \\ &= \frac{1}{\sqrt{2}}\int_0^1 \sqrt{\frac{3}{-at\left(1-t\right)\left(\eta t-\bar{\eta}\right)}}dt=\sigma \\ &< \sqrt{\frac{3}{2a\bar{\eta}}}\int_0^1 \frac{1}{\sqrt{t\left(1-t\right)}}dt=\sqrt{\frac{3}{2a\bar{\eta}}}\pi.\end{aligned}$$

where σ is defined by (7). It follows that $\lim_{\alpha\to\eta^+} T(\alpha) = \sigma$ exists. By similar argument in the proof of Theorem 1(4), we obtain $\lim_{\alpha\to\infty} T(\alpha) = 0$. So by (9), the proof of Theorem 1(5) is complete.

3.6 Proof of Theorem 1(6)

Assume that (C6) holds. Recall r_1 defined by (4). By (C6), we observe that $r_0 < 0 < r_1$. By (10), then

$$F'(u) = f(u) \begin{cases} > 0 & \text{for } 0 < u < r_1, \\ = 0 & \text{for } u = r_1, \\ < 0 & \text{for } u > r_1. \end{cases} \tag{27}$$

Since $F(0) = 0$, and by (27), we obtain $F(\alpha) - F(u) > 0$ for $0 < u < \alpha < r_1$. It implies that the domain of $T(\alpha)$ is $(0, r_1)$. Since $a > 0$ and $c > 0$, we see that

$$\theta(\alpha) - \theta(u) = \frac{a\left(\alpha^3 - u^3\right) + 3c\left(\alpha - u\right)}{3} > 0 \quad \text{for } 0 < u < \alpha,$$

from which it follows that by (12), $T'(\alpha) > 0$ on $(0, r_1)$. In addition, by (10), we compute

$$0 < \lim_{u\to r_1^-} \frac{f(u)}{r_1 - u} = a\left(r_0 - r_1\right) < \infty \quad \text{and} \quad \lim_{u\to 0^+} \frac{f(u)}{u} = \infty.$$

So by [1, Theorems 2.6 and 2.9], we obtain

$$\lim_{\alpha\to 0^+} T(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha\to r_1^-} T(\alpha) = \infty.$$

So by (9), the proof of Theorem 1(6) is complete.

3.7 Proof of Theorem 1(7)

Assume that (C7) holds. Then

$$F'(u) = f(u) = u\left(-au + b\right) \begin{cases} > 0 & \text{for } 0 < u < \frac{b}{a}, \\ = 0 & \text{for } u = \frac{b}{a}, \\ < 0 & \text{for } u > \frac{b}{a}. \end{cases} \tag{28}$$

Since $F(0) = 0$, and by (28), we obtain $F(\alpha) - F(u) > 0$ for $0 < u < \alpha < \frac{b}{a}$. It implies that the domain of T is $(0, \frac{b}{a})$. Since $a > 0$, we see that

$$\theta(\alpha) - \theta(u) = \frac{a\left(\alpha^3 - u^3\right)}{3} > 0 \quad \text{for } 0 < u < \alpha,$$

from which it follows that by (12), $T'(\alpha) > 0$ on $(0, \frac{b}{a})$. In addition, we compute

$$0 < \lim_{u\to \frac{b}{a}^-} \frac{f(u)}{\frac{b}{a} - u} = b < \infty \quad \text{and } f(u) - uf'(0) = -au^2 < 0 \text{ for } u > 0.$$

So by [1, Theorems 2.6 and 2.10], we obtain

$$\lim_{\alpha\to 0^+} T(\alpha) = \frac{\pi}{\sqrt{f'(0)}} = \frac{\pi}{\sqrt{b}} \quad \text{and} \quad \lim_{\alpha\to \frac{b}{a}^-} T(\alpha) = \infty.$$

By (9), the proof of Theorem 1(7) is complete.

3.8 Proof of Theorem 1(8): $a > 0$, $b > 0$, $c < 0$ and $\Delta > 0$

Before we prove Theorem 1(8), we need the following Lemmas 2–4.

Lemma 2 *Consider (1). Assume that (C8) holds. Then the following statements (i)–(ii) hold:*

(i) If $b \le \frac{4}{\sqrt{3}}\sqrt{-ac}$, then the domain of $T(\alpha)$ is empty; and if $b > \frac{4}{\sqrt{3}}\sqrt{-ac}$, then the domain of $T(\alpha)$ is (η, r_1) where η and r_1 are defined by (5) and (4), respectively.

(ii) $\partial\eta/\partial b < 0$ for $b > \frac{4}{\sqrt{3}}\sqrt{-ac}$. Moreover,

$$\begin{cases} \sqrt{\frac{-c}{a}} < \eta < \sqrt{\frac{-3c}{a}} & \text{for } \frac{4}{\sqrt{3}}\sqrt{-ac} < b < \frac{8}{3}\sqrt{-ac}, \\ \eta \le \sqrt{\frac{-c}{a}} & \text{for } b \ge \frac{8}{3}\sqrt{-ac}. \end{cases} \tag{29}$$

Proof. (I) By (C8) and (10), then

$$F'(u) = f(u) \begin{cases} < 0 & \text{on } (0, r_0) \cup (r_1, \infty), \\ = 0 & \text{for } u = r_0 \text{ and } u = r_1, \\ > 0 & \text{on } (r_0, r_1). \end{cases} \tag{30}$$

We compute

$$F(r_1) = r_1 \frac{b^2 + 8ac + b\sqrt{b^2 + 4ac}}{12a}. \tag{31}$$

Then we consider two cases.

Case 1. Assume that $b \le \frac{4}{\sqrt{3}}\sqrt{-ac}$. Since $a > 0$, and by (31), we observe that

$$F(r_1) \le \frac{r_1}{12a}\left[\left(\frac{4}{\sqrt{3}}\sqrt{-ac}\right)^2 + 8ac + \frac{4}{\sqrt{3}}\sqrt{-ac}\sqrt{\left(\frac{4}{\sqrt{3}}\sqrt{-ac}\right)^2 + 4ac}\right] = 0. \tag{32}$$

Since $F(0) = 0$, and by (32), we see that, for any $\alpha > 0$, there exists $\bar{u} \in (0, \alpha)$ such that $F(\alpha) - F(\bar{u}) \le 0$. Thus the domain of $T(\alpha)$ is empty.

Case 2. Assume that $b > \frac{4}{\sqrt{3}}\sqrt{-ac}$. Since $a > 0$, and by (31), we see that

$$F(r_1) > \frac{r_1}{12a}\left[\left(\frac{4}{\sqrt{3}}\sqrt{-ac}\right)^2 + 8ac + \frac{4}{\sqrt{3}}\sqrt{-ac}\sqrt{\left(\frac{4}{\sqrt{3}}\sqrt{-ac}\right)^2 + 4ac}\right] = 0. \tag{33}$$

Since $F(0) = 0$, and by (33), we see that $0 < \eta < r_1 < \bar{\eta}$. Moreover, $F(\alpha) - F(u) > 0$ for $0 < u < \alpha$ and $\eta < \alpha \le r_1$. Since $f(r_1) = 0$, the domain of $T(\alpha)$ contains in (η, r_1).

By Cases 1–2, the statement (i) holds.

(II) Since $a > 0$ and $ac < 0$, we see that

$$\frac{\partial}{\partial b}\eta = \frac{3}{4}\frac{\sqrt{3b^2 + 16ac} - \sqrt{3b^2}}{a\sqrt{3b^2 + 16ac}} < 0. \tag{34}$$

We compute

$$\eta = \begin{cases} \sqrt{\frac{-3c}{a}} & \text{if } b = \frac{4}{\sqrt{3}}\sqrt{-ac}, \\ \sqrt{\frac{-c}{a}} & \text{if } b = \frac{8}{3}\sqrt{-ac}. \end{cases}$$

So (29) holds by (34). Then the statement (ii) holds.

The proof is complete. ■

Lemma 3 *Consider (1). Assume that (C8) holds and* $b > \frac{4}{\sqrt{3}}\sqrt{-ac}$*. Then the following statements (i)–(ii) hold:*

(i) $\theta(\alpha) - \theta(u) < 0$ *for* $0 < u < \alpha \leq \sqrt{\frac{-c}{a}}$.

(ii) $g(\alpha, u) > 0$ *for* $0 < u < \alpha$ *and* $\alpha > \sqrt{\frac{-c}{a}}$ *where*

$$g(\alpha, u) \equiv abu^3 + 2a\,(b\alpha + 4c)\,u^2 + \left(2ab\alpha^2 + 8ac\alpha - 3bc\right)u + ab\alpha^3 + 8ac\alpha^2 - 3bc\alpha.$$

Proof. Since $a > 0$ and $c < 0$, we have

$$\theta'(u) = au^2 + c \begin{cases} < 0 & \text{for } 0 < u < \sqrt{\frac{-c}{a}}, \\ = 0 & \text{for } u = \sqrt{\frac{-c}{a}}, \\ > 0 & \text{for } u > \sqrt{\frac{-c}{a}}. \end{cases} \tag{35}$$

It follows that the statement (i) holds. We find that

$$\frac{\partial}{\partial u} g(\alpha, u) = 3abu^2 + 4a\,(b\alpha + 4c)\,u + 2ab\alpha^2 + 8ac\alpha - 3bc$$

is a quadratic polynomial with variable u. For $\alpha > \sqrt{-\frac{c}{a}}$, its discriminant

$$\begin{aligned} & [4a\,(b\alpha + 4c)]^2 - 4\,[3ab]\left[2ab\alpha^2 + 8ac\alpha - 3bc\right] \\ = \;& a\left(-2ab^2\alpha^2 + 8abc\alpha + 64ac^2 + 9b^2c\right) \\ < \;& a\left[-2ab^2\left(\sqrt{\frac{-c}{a}}\right)^2 + 8abc\sqrt{\frac{-c}{a}} + 64ac^2 + 9b^2c\right] \quad (\text{because } \alpha > \sqrt{-\frac{c}{a}}) \\ = \;& ac\left(11b^2 + 8\sqrt{-ac}\,b + 64ac\right) \\ < \;& ac\left[11\left(\frac{4}{\sqrt{3}}\sqrt{-ac}\right)^2 + 8\sqrt{-ac}\left(\frac{4}{\sqrt{3}}\sqrt{-ac}\right) + 64ac\right] \quad (\text{because } b > \frac{4}{\sqrt{3}}\sqrt{-ac}) \\ = \;& -\frac{16}{3}\left(2\sqrt{3} - 1\right)a^2c^2 < 0. \end{aligned}$$

It follows that

$$\partial g(\alpha, u)/\partial u > 0 \;\; \text{for } 0 < u < \alpha \text{ and } \alpha > \sqrt{\frac{-c}{a}}. \tag{36}$$

Since $b > \frac{4}{\sqrt{3}}\sqrt{-ac}$, and by (36), we observe that, for $0 < u < \alpha$ and $\alpha > \sqrt{\frac{-c}{a}}$,

$$\begin{aligned} g(\alpha, u) \;&>\; g(\alpha, 0) = \alpha\left(ab\alpha^2 + 8ac\alpha - 3bc\right) \\ &=\; \alpha\left[a\left(\frac{4}{\sqrt{3}}\sqrt{-ac}\right)\left(\sqrt{-\frac{c}{a}}\right)^2 + 8ac\sqrt{-\frac{c}{a}} - 3\left(\frac{4}{\sqrt{3}}\sqrt{-ac}\right)c\right] \\ &=\; \alpha\left(8 - \frac{16}{\sqrt{3}}\right)\sqrt{-ac}\,c > 0. \end{aligned}$$

Then the statement (ii) holds. The proof is complete. ■

Lemma 4 *Consider (1). Assume that (C8) holds and* $b > \frac{4}{\sqrt{3}}\sqrt{-ac}$*. Then*

$$\lim_{\alpha \to \eta^+} T(\alpha) \; \textit{exists}, \;\; \lim_{\alpha \to r_1^-} T(\alpha) = \infty \;\; \textit{and} \;\; \lim_{\alpha \to \eta^+} T'(\alpha) = -\infty.$$

Proof. By (8) and (10), we see that

$$\begin{aligned}\lim_{\alpha\to\eta^+} T(\alpha) &= \lim_{\alpha\to\eta^+}\frac{1}{\sqrt{2}}\int_0^1 \frac{\alpha}{\sqrt{F(\alpha)-F(\alpha t)}}dt = \frac{1}{\sqrt{2}}\int_0^1 \frac{\eta}{\sqrt{-F(\eta t)}}dt \\ &= \sqrt{\frac{3}{2a}}\int_0^1 \frac{1}{\sqrt{t\,(1-t)\,(\bar{\eta}-\eta t)}}dt < \sqrt{\frac{3}{a(\bar{\eta}-\eta)}}\int_0^1 \frac{1}{\sqrt{t\,(1-t)}}dt \\ &= \sqrt{\frac{3}{a(\bar{\eta}-\eta)}}\pi.\end{aligned}$$

So $\lim_{\alpha\to\eta^+} T(\alpha)$ exists. Since $a>0$ and $r_1>r_0$, we see that

$$0<\lim_{u\to r_1^-}\frac{f(u)}{r_1-u}=\lim_{u\to r_1^-}\frac{-a\,(u-r_0)\,(u-r_1)}{r_1-u}=a\,(r_1-r_0)<\infty.$$

So by [1, Theorem 2.6], we obtain $\lim_{\alpha\to r_1^-} T(\alpha)=\infty$.

In order to prove $\lim_{\alpha\to\eta^+} T'(\alpha)=-\infty$, we consider two cases.

Case 1. Assume that $b\geq\frac{8}{3}\sqrt{-ac}$. By Lemma 2(ii), we have $\eta\leq\sqrt{\frac{-c}{a}}$. Then by (35), we see that

$$\theta(\eta)-\theta(\eta t)<0 \ \text{ for } 0<t<1$$

and

$$\theta(\eta)-\theta(\eta t)<\theta(\eta)-\theta(\tfrac{\eta}{2})<0 \ \text{ for } 0<t<\frac{1}{2}.$$

So by (12) and (10), we observe that

$$\begin{aligned}\lim_{\alpha\to\eta^+} T'(\alpha) &= \frac{1}{2\sqrt{2}}\int_0^1 \frac{\theta(\eta)-\theta(\eta t)}{[F(\eta)-F(\eta t)]^{3/2}}dt < \frac{1}{2\sqrt{2}}\int_0^{1/2} \frac{\theta(\eta)-\theta(\eta t)}{[-F(\eta t)]^{3/2}}dt \\ &< \frac{1}{2\sqrt{2}}\int_0^{1/2} \frac{\theta(\eta)-\theta(\frac{\eta}{2})}{[-F(\eta t)]^{3/2}}dt \\ &= \frac{\theta(\eta)-\theta(\frac{\eta}{2})}{2\sqrt{2}(\frac{a}{3}\eta^2)^{3/2}}\int_0^{1/2} \frac{1}{[t\,(1-t)\,(\bar{\eta}-\eta t)]^{3/2}}dt \\ &< \frac{\theta(\eta)-\theta(\frac{\eta}{2})}{2\sqrt{2}\left(\frac{a}{3}\eta^2\bar{\eta}\right)^{3/2}}\int_0^{1/2}\frac{1}{t^{3/2}}dt=-\infty.\end{aligned}$$

Case 2. Assume that $\frac{4}{\sqrt{3}}\sqrt{-ac}<b<\frac{8}{3}\sqrt{-ac}$. By Lemma 2(ii), we have $\sqrt{\frac{-c}{a}}<\eta<\sqrt{\frac{-3c}{a}}$. Then by (35), there exists $t^*\in(0,1)$ such that

$$0<\eta t^*<\sqrt{\frac{-c}{a}} \ \text{ and } \ \theta(\eta)-\theta(\eta t)\begin{cases} <0 & \text{for } 0<t<t^*, \\ =0 & \text{for } t=t^*, \\ >0 & \text{for } t^*<t<1. \end{cases} \tag{37}$$

Since $\theta(u) = \left(au^2 + 3c\right) u/3$, and by (10), (35) and (37), we compute

$$\begin{aligned}
\int_{t^*}^{1} \frac{\theta(\eta) - \theta(\eta t)}{\left[F(\eta) - F(\eta t)\right]^{3/2}} dt &= \int_{t^*}^{1} \frac{\theta(\eta) - \theta(\eta t)}{\left[-F(\eta t)\right]^{3/2}} dt = \int_{t^*}^{1} \frac{\theta(\eta) - \theta(\eta t)}{\left[\frac{a}{3}\eta^2 t \left(1-t\right)\left(\bar{\eta} - \eta t\right)\right]^{3/2}} dt \\
&< \frac{1}{\left[\frac{a}{3}\eta^2 t^* \left(\bar{\eta} - \eta\right)\right]^{3/2}} \int_{t^*}^{1} \frac{\theta(\eta) - \theta(\eta t)}{\left(1-t\right)^{3/2}} dt \\
&= \frac{1}{\left[\frac{a}{3}\eta^2 t^* \left(\bar{\eta} - \eta\right)\right]^{3/2}} \int_{t^*}^{1} \frac{a\eta^3 \left(t + t^2 + 1\right) + 3c\eta}{\sqrt{1-t}} dt \\
&< \frac{3a\eta^3 + 3c\eta}{\left[\frac{a}{3}\eta^2 t^* \left(\bar{\eta} - \eta\right)\right]^{3/2}} \int_{t^*}^{1} \frac{1}{\sqrt{1-t}} dt < \infty
\end{aligned}$$

and

$$\begin{aligned}
\int_{0}^{t^*} \frac{\theta(\eta) - \theta(\eta t)}{\left[F(\eta) - F(\eta t)\right]^{3/2}} dt &< \int_{0}^{t^*} \frac{\theta(\eta) - \theta(\eta t^*)}{\left[-F(\eta t)\right]^{3/2}} dt = \int_{0}^{t^*} \frac{\theta(\eta) - \theta(\eta t^*)}{\left[\frac{a}{3}\eta^2 t \left(1-t\right)\left(\bar{\eta} - \eta t\right)\right]^{3/2}} dt \\
&= \frac{\theta(\eta) - \theta(\eta t^*)}{\left[\frac{a}{3}\eta^2 \bar{\eta}\right]^{3/2}} \int_{0}^{t^*} \frac{1}{t^{3/2}} dt = -\infty.
\end{aligned}$$

So by (12), we obtain

$$\lim_{\alpha \to \eta^+} T'(\alpha) = \int_{0}^{t^*} \frac{\theta(\eta) - \theta(\eta t)}{\left[F(\eta) - F(\eta t)\right]^{3/2}} dt + \int_{t^*}^{1} \frac{\theta(\eta) - \theta(\eta t)}{\left[F(\eta) - F(\eta t)\right]^{3/2}} dt = -\infty.$$

The proof is complete. ■

Proof of Theorem 1(8). By Lemma 2(i) and (9), the bifurcation curve S does not exist for $b \leq \frac{4}{\sqrt{3}}\sqrt{-ac}$. Thus we assume that $b > \frac{4}{\sqrt{3}}\sqrt{-ac}$. We assert that

$$T(\alpha) \text{ is strictly decreasing and then strictly increasing on } \left(\eta, r_1\right). \tag{38}$$

So by (38) and Lemma 4, the proof is complete.

Next, we prove (38). We consider two cases:

Case 1. Assume that $b > \frac{8}{3}\sqrt{-ac}$. By Lemma 2(ii), we have $\eta < \sqrt{\frac{-c}{a}}$. Then by (35), we see that

$$\theta(\alpha) - \theta(u) < 0 \quad \text{for } 0 < u < \alpha \text{ and } \eta < \alpha \leq \sqrt{\frac{-c}{a}}.$$

So by (12), $T'(\alpha) < 0$ for $\eta < \alpha \leq \sqrt{\frac{-c}{a}}$. Since $\lim_{\alpha \to r_1^-} T(\alpha) = \infty$, we see that $r_1 > \sqrt{\frac{-c}{a}}$. Then by Lemma 3(ii), we observe that, for $\sqrt{-c/a} < \alpha < r_1$,

$$\begin{aligned}
T''(\alpha) + \frac{1}{\alpha} T'(\alpha) &= \frac{1}{4\sqrt{2}\alpha} \int_{0}^{\alpha} \frac{-6AB - 2BC + 3A^2 + 4B^2}{B^{5/2}} du \\
&= \frac{1}{4\sqrt{2}\alpha} \int_{0}^{\alpha} \frac{\frac{1}{9}\left[\left[a\left(\alpha^3 - u^3\right) + 3c\left(\alpha - u\right)\right]^2 + 3\left(\alpha - u\right)^2 g(\alpha, u)\right]}{B^{5/2}} du > 0.
\end{aligned}$$

It implies that $T(\alpha)$ has exactly one critical point on (η, r_1). So (38) holds by Lemma 4

Case 2. Assume that $\frac{4}{\sqrt{3}}\sqrt{-ac} < b < \frac{8}{3}\sqrt{-ac}$. By Lemma 2(ii), we have $\eta > \sqrt{\frac{-c}{a}}$. Similarly, by Lemma 3(ii), we obtain

$$T''(\alpha) + \frac{1}{\alpha}T'(\alpha) > 0 \text{ for } \eta < \alpha < r_1.$$

It implies that $T(\alpha)$ has exactly one critical point on (η, r_1). So (38) holds by Lemma 4. ■

References

[1] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, Indiana Univ. Math. J., 20 (1970), 1–13.

[2] S.-Y. Huang, K.-C. Hung and S.-H. Wang, A global bifurcation theorem for a multiparameter positone problem and its application to the one-dimensional perturbed Gelfand problem. Electron. J. Qual. Theory Differ. Equ. 2019, Paper No. 99, 25 pp.

[3] S.-Y. Huang and S.-H. Wang, A variational property on the evolutionary bifurcation curves for the one-dimensional perturbed Gelfand problem from combustion theory. Electron. J. Qual. Theory Differ. Equ. 2016, Paper No. 94, 21 pp.

[4] S.-Y. Huang and S.-H. Wang, Proof of a conjecture for the one-dimensional perturbed Gelfand problem from combustion theory. Arch. Ration. Mech. Anal. 222 (2016), no. 2, 769–825.

[5] S.-Y. Huang and S.-H. Wang, An evolutionary property of the bifurcation curves for a positone problem with cubic nonlinearity. Taiwanese J. Math. 20 (2016), no. 3, 639–661.

[6] S.-Y. Huang and S.-H. Wang, On S-shaped bifurcation curves for a two-point boundary value problem arising in a theory of thermal explosion. Discrete Contin. Dyn. Syst. 35 (2015), no. 10, 4839–4858.

[7] K.-C. Hung and S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity and their applications, Trans. Amer. Math. Soc., 365 (2013), 1933–1956.