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On A Generalized Class Of Analytic Functions With Bounded Radius Rotation*

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Abstract

In this paper, we use Salagean and Ruscheweyh operator to introduce certain new classes of functions with bounded radius rotation. Some interesting results including inclusion relation and geometric properties of linear combinations of these functions are studied. Relevant connections to various known results are also pointed out.

1 Introduction

Let A denote the class of functions f which are analytic in the open unit disc $E = \{z :| z |< 1\}$ and are of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \qquad (z \in E). \tag{1}$$

Let $S^*(\alpha)$ and $C(\alpha)$ denote the subclasses of A consisting of functions which are starlike and convex of order α, $0 \le \alpha < 1$, respectively. Let p be a function analytic in E with $p(0) = 1$. Then p is said to be in the class $P(\alpha)$, $0 \le \alpha < 1$, if and only if, $\Re p(z) > \alpha$ for $z \in E$. For $\alpha = 0$, we obtain the class P of Caratheodory functions of positive real part. It can easily be seen that, for $p \in P(\alpha)$, we can write

$$p(z) = (1-\alpha)p_1(z) + \alpha, \qquad p_1 \in P.$$

Also $P(\alpha) \subset P$, $0 \le \alpha < 1$. The class $P(\alpha)$ is generalized in [5, 6] as follows: Let $p : p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j$ be analytic in E. Then p is said to belong to the class $P_m(\alpha)$, if it satisfies the condition

$$\int_0^{2\pi} \left| \frac{\Re p(z) - \alpha}{1-\alpha} \right| \mathrm{d}\theta \le m\pi, \qquad m \ge 2, \quad z = re^{i\theta}.$$

The class $P_m(0) = P_m$ has been introduced by Pinchuk in [7]. Also, for $m = 2$, we obtain $P(\alpha)$. For $p \in P_m(\alpha)$, we can write

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z), \qquad p_1, p_2 \in P(\alpha).$$

In [11], a linear operator $D_*^n :: A \to A$ is introduced as:

$$D_*^0 f(z) = f(z), \qquad D'_* f(z) = z f'(z)$$

and

$$D_*^{n+1} f(z) = z(D_*^n f(z))', \quad \text{for} \quad z \in E, \quad f \in A, \quad n \in \mathbb{N}_0\{0, 1, 2, 3, ...\}. \tag{2}$$

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We note that, if $f(z)$ is given by (1), then

$$D_*^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j, \qquad z \in E.$$

The operator D_*^n is called Salagean operator of order n. Also, Ruscheweyh differential operator D^n of order n, $n \in \mathbb{N}_\circ$, is defined as:

$$D^n : A \to A, \qquad f \in A, \qquad D^0 f(z) = f(z), \qquad D' f(z) = z f'(z)$$

and

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \qquad \text{see [9]}.$$

The following identity can easily be obtained for the operator D^n :

$$(n+1) D^{n+1} f(z) = z(D^n f(z))'^n f(z), \qquad z \in E. \tag{3}$$

Using these operators, we define:

Definition 1 *Let $f \in A$. Then $f \in R_m(n, \alpha)$, $m \geq 2$, $n \in \mathbb{N}_0$, $\alpha \in [0, 1)$, if and only if,*

$$\frac{z(D^n f(z))'}{D^n f(z)} \in P_m(\alpha), \qquad z \in E.$$

We note that $R_2(0, \alpha) = S^*(\alpha)$ and $R_2(0, 0) = S^*$. The class $R_m(0, \alpha)$ will be denoted as $R_m(\alpha)$. The class $R_m^*(n, \alpha)$ is defined in similar way as:

Definition 2 *Let $f \in A$. Then f is said to belong to the class $R_m^*(n, \alpha)$, if and only if,*

$$\frac{z(D_*^n f(z))'}{D_*^n f(z)} \in P_m(\alpha), \qquad z \in E.$$

We note

(i) $R_m^*(0, 0) = R_m$ is class of functions of bounded radius rotation, see [2].

(ii) $R_2^*(1, 0) = C$ the class of convex univalent functions and $R_2^*(0, \alpha) = S^*(\alpha)$.

2 Preliminary Results

Lemma 1 ([12]) *Let a, d, k, ρ be reals with $a > d \geq 0$, $k > 0$ and $\rho \in (0, \pi)$. Suppose $|u - a| \leq d$ and $|v - a| \leq d$ and set*

$$w = \frac{u}{1 + k e^{i\rho}} + \frac{v}{1 + k^{-1} e^{-i\rho}}.$$

Then

$$\Re\{w\} \geq a - d\left(\sec \frac{\rho}{2}\right).$$

Lemma 2 *Let $f \in R_m$, $m \geq 2$. Then, for $z \in E$,*

(i) $|arg f(z)| \leq m \sin^{-1} r$, *see [1].*

(ii) $\left|\frac{zf'(z)}{f(z)} - \frac{1+r^2}{1-r^2}\right| \leq \frac{mr}{1-r^2}$, *see [8].*

Let $*$ denote the convolution (Hadamard product). Then, using convolution techniques given in [10], we have the following.

Lemma 3 *Let $\beta > 0$, $\gamma \geq 0$ and an analytic function $p(z)$ satisfying $p(0) = 1$. Then*

$$p(z) * \frac{\phi_{\beta,\gamma}(z)}{z} = p(z) + \frac{zp'(z)}{\beta p(z) + \gamma},$$

where

$$\phi_{\beta,\gamma}(z) = \sum_{j=1}^{\infty}(\frac{\beta+\gamma}{\beta_j+\gamma})z^j.$$

Lemma 4 ([3]) *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and let $\psi(u, v)$ be a complex-valued function satisfying the conditions:*

(i) $\psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,

(ii) $(1, 0) \in D$ and $\psi(1, 0) > 0$

(iii) $\Re\{\psi(iu_2, v_1)\} \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.

If $h(z) = 1+c_1z+c_2z^2+\ldots$ is an analytic function in E such that $(h(z), zh'(z)) \in D$ and $\Re\{\psi[h(z), zh'(z)]\} > 0$ for $z \in E$, then $\Re\{h(z)\} > 0$ in E.

3 Main Results

In this section, we obtain the main results.

Theorem 5 *For $m \geq 2$, $n \in \mathbb{N}_0$,*

$$R_m(n+1, \beta_{n+1}) \subset R_m(n, \beta_n) \subset \ldots \subset R_m,$$

where

$$\beta_n = \frac{2}{\lambda_{n+1} + \sqrt{\lambda_{n+1}^2 + \delta}}, \qquad \lambda_{n+1} = (2n - 2\beta_{n+1} + 1). \tag{4}$$

Proof. Let $f \in R_m(n+1, \beta_{n+1})$. Then

$$\frac{z(D^{n+1}f)'}{D^{n+1}f} \in P_m(\beta_{n+1}), \quad z \in E.$$

Set

$$\frac{z(D^n f(z))'}{D^n f(z)} = H(z) = (1-\beta_n)\left\{\left(\frac{m}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)h_2(z)\right\} + \beta_n, \tag{5}$$

We note that $h_i(z)$ is analytic in E with $h_i(0) = 1$, $i = 1, 2$. Using identity (3), Lemma 3 and from (5) together with some simple computations (see [4]), we obtain

$$\begin{aligned}
\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} &= H(z) + \frac{zH'(z)}{H(z)+\delta_n} \\
&= \left(\frac{m}{4}+\frac{1}{2}\right)\left\{(1-\beta_n)h_1(z) + \frac{zh_1'(z)}{h_1(z)+\delta_n} + \beta_n\right\} \\
&\quad -\left(\frac{m}{4}-\frac{1}{2}\right)\left\{(1-\beta_n)h_2(z) + \frac{zh_2'(z)}{h_2(z)+\delta_n} + \beta_n\right\},
\end{aligned} \tag{6}$$

where

$$\delta_n = \frac{n+\beta_n}{1-\beta_n}. \tag{7}$$

Since $f \in R_m(n+1, \beta_{n+1})$, it follows from (6) that, for $i = 1, 2$

$$\Re\left\{(1-\beta_n)h_i(z) + \frac{zh_i(z)}{h_i(z)+\delta_n} + (\beta_n - \beta_{n+1})\right\} > 0, \quad z \in E. \tag{8}$$

We construct the functional $\psi(u, v)$ with $u = h_i$, $v = zh_i'$ in (8) and have

$$\psi(u, v) = (1-\beta_n)u + \frac{v}{u+\delta_n} + \beta_n - \beta_{n+1}.$$

The first two conditions of Lemma 4 can easily be verified. For condition (3), we proceed as follows:

$$\begin{aligned}
\Re\{\psi(iu_2, v_1)\} &= (\beta_n - \beta_{n+1}) + \Re\left(\frac{v_1}{iu_2+\delta_n}\right) \\
&\leq \frac{2(\beta_n - \beta_{n+1})[(n+\beta_n)^2 + (1-\beta_n)^2u_2^2] - (n+\beta_n)(1-\beta_n)(1+u_2^2)}{2[(1-\beta_n)^2u_2^2 + (n+\beta_n)^2]} \\
&= \frac{A+Bu_2^2}{2C}, \qquad \text{with} \quad v_1 \leq -\frac{(1+u_2^2)}{2}, \quad \delta_n \quad \text{given in (7)} \\
&\leq 0,
\end{aligned}$$

if $A \leq 0$, $B \leq 0$ where as C is obviously positive. From $A \leq 0$, we compute the value of β_n, which is as in (4) and $B \leq 0$ ensures that $\beta_n \in [0, 1)$. Thus condition (iii) of Lemma 4 also holds and we apply it to have $\Re h_i(z) > 0$, $i = 1, 2$, $z \in E$.

Consequently, it follows from (5), that $H \in P_m(\beta_n)$, where β_n is defined by (4). This completes the proof. ∎

Special case:

Let $\beta_{n+1} = 0$. Then, from (4), we have

$$\beta_n = \frac{2}{(2n+1) + \sqrt{4n^2+4n+9}}$$

and $\beta_1 = 0$ gives us $\beta_0 = \frac{1}{2}$, for $n = 0$. Therefore

$$\frac{(zf'(z))'}{f'(z)} \in P_m \quad \text{implies} \quad \frac{zf'(z)}{f(z)} \in P_m\left(\frac{1}{2}\right).$$

Next we prove an inclusion result for $R_m^*(n, \alpha)$.

Theorem 6 *For $n \in \mathbb{N}_\circ$, $m \geq 2$ and $z \in E$,*

$$R_m^*(n+1, \gamma_{n+1}) \subset R_m^*(n, \gamma_n) \subset ... \subset R_m^*(1, \gamma_1) \subset R_m,$$

where

$$\gamma_n = \frac{2}{(1-2\gamma_{n+1}) + \sqrt{(1-2\gamma_{n+1})^2 + 8}}. \tag{9}$$

Proof. We follow the similar procedure of Theorem 5 and let

$$\begin{aligned}
\frac{z(D_*^n f(z))'}{D_*^n f(z)} = p(z) &= \left(\frac{m}{4}+\frac{1}{2}\right)\{(1-\gamma_n)p_1(z) + \gamma_n\} \\
&\quad -\left(\frac{m}{4}-\frac{1}{2}\right)\{(1-\gamma_n)p_2(z) + \gamma_n\}.
\end{aligned} \tag{10}$$

Here $p(z)$ is analytic in E with $p(0) = 1$. From (10), we have

$$z(D_*^n f(z))'^n_* f(z))p(z),$$

and this leads to

$$\frac{z(D_*^{n+1} f(z))'}{D_*^{n+1} f(z)} = p(z) + \frac{zp'(z)}{p(z)}. \tag{11}$$

Following the same technique, we have from (10) and (11)

$$\Re\left\{(1-\gamma_n)p_i(z) + \frac{zp_i'(z)}{p_i(z) + \frac{\gamma_n}{1-\gamma_n}} + (\gamma_n - \gamma_{n+1})\right\} > 0, \quad \text{for} \quad i = 1, 2., \quad z \in E.$$

Now, constructing the functional $\psi(u, v)$ as

$$\psi(u, v) = (1-\gamma_n)u + \frac{v}{u + \frac{\gamma_n}{1-\gamma_n}} + (\gamma_n - \gamma_{n+1}).$$

We verify the three conditions of Lemma 4 and apply it to get $\Re p_i(z) > 0$, $i = 1, 2$. While verifying condition (iii), we also obtain the value of γ_n given by (9). Since $f \in R_m^*(\gamma_{m+1})$, it implies from (11) that

$$\left\{p(z) + \frac{zp'(z)}{p(z)}\right\} \in P_m(\gamma_{n+1})$$

and hence $p \in P_m(\gamma_n)$, where γ_n is given by (9). This completes the proof. ■

As a special case, we have

$$R_m^*(n+1, 0) \subset R_m^*\left(n, \frac{1}{2}\right), \qquad z \in E.$$

Remark 1 *From Definition 1, Definition 2, Theorem 5 and Theorem 6, we can easily deduce that $R_2(n, \alpha)$ and $R_2^*(0, \alpha)$ are subclasses of S^* of starlike functions.*

We have

Theorem 7 *Let*

$$F(z) = \lambda f(z) + (1-\lambda)g(z), \tag{12}$$

where $0 \leq \arg \frac{\lambda}{1-\lambda} \leq \sigma < \pi$, $f \in R_m(n, \alpha)$, $g \in R_m^*(0, \alpha)$ in E. Then $F \in S^*$ in $|z| < r_m$, where r_m is the smallest positive value of r satisfying the equation

$$T(r) = A_m(1+r^2) - mr = 0, \qquad A_m = \cos\left(\frac{\sigma}{2} + m\sin^{-1} r\right)$$

Proof. Differentiating (12), we get

$$\begin{aligned} \frac{zF'(z)}{F(z)} &= \frac{\lambda z f'(z) + (1-\lambda)zg'(z)}{\lambda f(z) + (1-\lambda)g(z)} \\ &= \frac{zf'(z)}{f(z)}\left[1 + \left(\frac{\lambda}{1-\lambda}\frac{f(z)}{g(z)}\right)^{-1}\right]^{-1} + \frac{zg'(z)}{g(z)}\left[1 + \left(\frac{\lambda}{1-\lambda}\frac{f(z)}{g(z)}\right)\right]^{-1}. \end{aligned} \tag{13}$$

Let

$$u = \frac{zg'(z)}{g(z)}, \quad v = \frac{zf'(z)}{f(z)}, \quad k = \left|\frac{\lambda}{1-\lambda}\frac{f(z)}{g(z)}\right|. \tag{14}$$

Then, from (13) and (14), we have

$$w(z) = \frac{zF'(z)}{F(z)} = \frac{u}{1+ke^{i\rho}} + \frac{v}{1+k^{-1}e^{-i\rho}}.$$

We now apply Lemma 1 and Lemma 2 to obtain

$$\Re \frac{zF'(z)}{F(z)} \geq \frac{1+r^2}{1-r^2} - \frac{mr}{1-r^2} \sec\left(\frac{\rho}{2}\right), \tag{15}$$

where

$$\rho = \arg \frac{\lambda}{1-\lambda} \frac{f(z)}{g(z)} = 2n\pi + \arg \frac{\lambda}{1-\lambda} + \arg f(z) - \arg g(z).$$

This gives us $|\rho| \leq \sigma + 2m\sin^{-1} r$. Therefore

$$\Re\left\{\frac{zF'(z)}{F(z)}\right\} > 0,$$

if $T(r) = (1+r^2)\cos\left(\frac{\sigma}{2} + m\sin^{-1} r\right) - mr > 0$. We note that

$$T(r) = \cos\frac{\sigma}{2}, \quad \text{for} \quad r = 0$$

and

$$T(r) = -m\sin(\frac{\pi-\sigma}{2m}) < 0, \quad \text{when} \quad r = \sin(\frac{\pi-\sigma}{2m}).$$

This implies $T(r) = 0$ has a root in the interval $(0, \sin(\frac{\pi-\sigma}{2m}))$ and right hand side of (15) is positive in the dics $|z| < r_m$, where r_m is the least positive value of r satisfying $T(r) = 0$. This gives

$$r_m = \frac{m + \sqrt{m^2 - 4A_m^2}}{2A_m}, \quad A_m = \cos(\frac{\sigma}{2} + m\sin^{-1} r),$$

and the proof is complete. ■

We have the following special cases

Corollary 8 *Let $m = 2$. Then f and g are starlike in E and*

$$A = A_2 = \cos\left(\frac{\sigma}{2} + 2\sin^{-1} r\right).$$

From Theorem 7, it follows that the linear combination of two starlike functions is starlike in the disc $|z| < r_2 = \frac{1-\sqrt{1-A_2^2}}{A_2}$.

Corollary 9 *Let, for $m = 2$, F be defined as in Theorem 7. Then, F maps the disc $|z| < r_\sigma$ onto a convex domain, where r_σ is the least positive root of the equation*

$$T_\sigma(r) = Dr^2 - 2rr_1 + Dr_1^2, \quad r_1 = 2 - \sqrt{3},$$

and

$$D = \cos\left(\frac{\sigma}{2} + 2\sin^{-1}\left(\frac{r}{r_1}\right)\right).$$

It is well known that every starlike function is convex in the disc $|z| < r_1 = 2 - \sqrt{3}$. Therefore we proceed with similar technique as follows.

We can write

$$\begin{aligned}\frac{(zF'(z))'}{F'(z)} &= \frac{(zf'(z))'}{f'(z)}\left[1 + \left(\frac{\lambda}{1-\lambda}\frac{f'(z)}{g'(z)}\right)^{-1}\right]^{-1} \\ &+ \frac{(zg'(z))'}{g'(z)}\left[1 + \left(\frac{\lambda}{1-\lambda}\frac{f'(z)}{g'(z)}\right)^{-1}\right]^{-1}.\end{aligned}$$

With
$$u = \frac{(zg'(z))'}{g'(z)}, \quad v = \frac{(zf'(z))'}{f'(z)}, \quad r_1 = 2 - \sqrt{3}$$
and
$$k = |\frac{\lambda}{1-\lambda}\frac{f'(z)}{g'(z)}|, \quad \rho = \arg(\frac{\lambda}{1-\lambda}\frac{f'(z)}{g'(z)}),$$
we have
$$\left| u - \frac{r_1^2 + r^2}{r_1^2 - r^2} \right| \le \frac{2rr_1}{r_1^2 - r^2},$$
and
$$\left| v - \frac{r_1^2 + r^2}{r_1^2 - r^2} \right| \le \frac{2rr_1}{r_1^2 - r^2}.$$
We construct
$$w(z) = \frac{(zF'(z))'}{F'(z)} = \frac{u}{1 + ke^{i\rho}} + \frac{v}{1 + k^{-1}e^{-i\rho}}.$$
Then, as in Theorem 7,
$$\rho = \arg\left(\frac{\lambda}{1-\lambda}\frac{f'(z)}{g'(z)}\right) = 2n\pi \arg\left(\frac{\lambda}{1-\lambda}\right) + \arg f'(z) - \arg g'(z)$$
and this gives us
$$|\rho| \le \sigma + 4\sin^{-1}(\frac{r}{r_1}),$$
since f and g are convex in $|z| < r_1$. Combining these facts together, it follows that
$$\Re\left[\frac{(zF'(z))'}{F'(z)}\right] > 0,$$
if
$$T_\sigma(r) = Dr^2 - 2r_1 r + Dr_1^2, \qquad D = \cos\left(\frac{\sigma}{2} + 2\sin^{-1}\left(\frac{r}{r_1}\right)\right).$$
This gives us
$$r_\sigma = \frac{r_1 - \sqrt{r_1^2 - D^2 r_1^2}}{D}.$$
It can easily be checked that
$$r_\sigma \in \left(0, r_1 \sin\left(\frac{\pi - \sigma}{4}\right)\right).$$
Hence F maps the disc $|z| < r_\sigma$ onto a convex domain.

Conclusion

In this paper, certain new classes of functions with bounded radius rotation using Salagean and Ruscheweyh operators are introduced. Several interesting results including inclusion relation and geometric properties of linear combinations of these functions are studied. Several special cases are considered as applications of these new results. The ideas and techniques of this paper may be starting point for further research in Geometric Function Theory and related areas.

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References

[1] D. A. Brannan, On functions of bounded boundary rotation, Proc. Edin. Math. Soc., 2(1968-1969), 337–347.

[2] A. W. Goodman, Univalent Functions, Vols. I & II, Polygonal Publishing House, Washington, USA, (1993).

[3] S. Miller, Differential inequalities and Carathéodory functions, Bull. Amer. Math. Soc., 81(1975), 79–81.

[4] K. I. Noor, W. Ul-Haq, M. Arif and S. Mustafa, On bounded boundary and bounded radius rotations, J. Inequal. Appl., 2009, Art. ID 813687, 12 pp.

[5] K. I. Noor, B. Malik and S. Mustafa, A survey on functions of bounded boundary and bounded radius rotation, Appl. Math. E-Notes, 12(2012), 136–152.

[6] K. S. Padmanabham and R. Parvathan, Properties of a class of functions with bounded boundary rotation, Ann. Polon. Math., 31(1975), 311–323.

[7] B. Pinchuk, Functions with bounded boundary rotation, Israel J. Math., 10(1971), 7–16.

[8] M. S. Robertson, Coefficients of functions with bounded boundary rotation, Canad. J. Math., 21(1969), 77–82.

[9] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49(1975), 109–115.

[10] St. Ruscheweyh, Ein invarianz eign shaft der Basilevic functionen, Math. Z., 134(1973), 215–219.

[11] G. S. Salagean, Subclasses of univalent functions. Complex analysis-fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), 362–372, Lecture Notes in Math., 1013, Springer, Berlin, 1983.

[12] R. K. Stump, Linear conmbinations of univalent functions with complex coefficients, Canad. J. Math., 23(1971), 712–717.