

Some Results On Non-Expansive Type Mappings*

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Abstract

The purpose of this paper is to present some fixed and coincidence point theorems for single-valued and multi-valued mappings satisfying the non-expansive type conditions in the context of asymptotically regular mappings and orbital completeness of the space.

1 Introduction

Let (X, d) be a metric space, and $T : X \rightarrow X$. Then T is said to be contraction (resp. non-expansive) mapping, if there exists a nonnegative real number $k < 1$ (resp. $k = 1$) such that the inequality $d(Tx, Ty) \leq kd(x, y)$ holds for any $x, y \in X$. Moreover, if X is a complete space, by Banach contraction principle [2], T has a unique fixed point in X . However, non-expansive mappings may not have any fixed point or have more than one fixed point on complete metric spaces.

From last few decades, there exists very abundant literature about contractive and non-expansive type mappings, where the contractive and nonexpansive conditions are replaced with more general conditions, and many fixed point theorems have been obtained for non-expansive type mappings in metric spaces with their applications in various areas (see [4, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19], and references therein).

In this paper, \mathbb{N} and \mathbb{R} stand for set of positive integers and set of reals, respectively.

2 Main Results

In this section, we have proved three important results of fixed point theory, which are for a single valued mapping, a pair of single valued mappings and a hybrid pair of mappings, by using non-expansive type conditions in the context of asymptotically regular mappings and orbitally complete spaces.

2.1 Fixed Point Theorem for a Single Valued Mapping

Throughout this part, for all $x, y \in X$, we consider the following non-expansive type condition:

$$\begin{aligned} d(Tx, Ty) \leq & \alpha d(x, y) + \beta [(x, Tx) + d(y, Ty)] + \gamma [(x, Ty) + d(y, Tx)] \\ & + \delta [M(x, y) + hm(x, y)] \end{aligned} \quad (1)$$

where

$$\alpha \geq 0, \beta, \gamma, \delta > 0, \quad 0 < h < 1, \quad (2)$$

$$\alpha + 2\beta + 2\gamma + 2\delta = 1, \quad (3)$$

$$M(x, y) = \max\{d(x, Ty), d(y, Tx)\},$$

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and

$$m(x, y) = \min\{d(x, Ty), d(y, Tx)\}.$$

Browder and Petryshyn [5] introduced the concept of asymptotically regular mapping at a point in the Hilbert spaces. A mapping T on a metric space (X, d) is said to asymptotically regular at $x \in X$, if $\lim_{n \rightarrow \infty} d(T^n x, T^{n+1} x) = 0$.

Proposition 1 *Let T be a self-mapping on X satisfying the condition (1) with (2) and (3). Then T is asymptotically regular at each point in X .*

Proof. Let x_0 be any point in X , we define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n = T^n x_0$, where $n \in \mathbb{N} \cup \{0\}$. From (1),

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha d(x_{n-1}, x_n) + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \gamma[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &\quad + \delta[M(x_{n-1}, x_n) + hm(x_{n-1}, x_n)]. \end{aligned} \tag{4}$$

Here, $m(x_{n-1}, x_n) = 0$ and $M(x_{n-1}, x_n) = d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$.

Suppose that $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$ for some n . Then by (4) and (3), we have

$$\begin{aligned} d(x_n, x_{n+1}) &< \alpha d(x_n, x_{n+1}) + 2\beta d(x_n, x_{n+1}) + 2\gamma d(x_n, x_{n+1}) + 2\delta d(x_n, x_{n+1}) \\ &< (\alpha + 2\beta + 2\gamma + 2\delta)d(x_n, x_{n+1}) \\ &= d(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. Therefore $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all n . Hence,

$$d(x_{n-1}, Tx_{n-1}) \leq d(x_0, Tx_0) \quad (n = 1, 2, 3, \dots). \tag{5}$$

Now using (1), we have

$$\begin{aligned} d(x_1, Tx_2) &= d(Tx_0, Tx_2) \\ &\leq \alpha d(x_0, x_1) + \beta[d(x_0, Tx_0) + d(x_2, Tx_2)] + \gamma[(x_0, Tx_2) + d(x_2, Tx_0)] \\ &\quad + \delta[M(x_0, x_2) + hm(x_0, x_2)], \end{aligned}$$

from (5) and the triangle inequality, we get

$$d(x_1, Tx_2) \leq 2\alpha d(x_0, x_1) + 2\beta d(x_0, x_1) + 4\gamma d(x_0, x_1) + \delta(3 + h)d(x_0, x_1).$$

Then

$$d(x_1, Tx_2) \leq [2(1 - \beta) - \delta(1 - h)]d(x_0, x_1). \tag{6}$$

By (1), we have

$$\begin{aligned} d(Tx_1, Tx_2) &\leq \alpha' d(x_1, x_2) + \beta' [d(x_1, Tx_1) + d(x_2, Tx_2)] \\ &\quad + \gamma' [(x_1, Tx_2) + d(x_2, Tx_1)] + \delta' [M(x_1, x_2) + hm(x_1, x_2)]. \end{aligned}$$

Here, $M(x_1, x_2) = d(x_1, Tx_2)$, $m(x_1, x_2) = 0$, and $\alpha', \beta', \gamma', \delta'$ are nothing but $\alpha, \beta, \gamma, \delta$ in such a way that $\alpha' + 2\beta' + 2\gamma' + 2\delta' = 1$. Then by using (5) and (6), we have

$$\begin{aligned} d(Tx_1, Tx_2) &\leq \alpha' d(x_0, x_1) + 2\beta' d(x_0, x_1) + (\gamma' + \delta')d(x_1, Tx_2) \\ &\leq (\alpha' + 2\beta')d(x_0, x_1) + (\gamma' + \delta')(2(1 - \beta) - \delta(1 - h))d(x_0, x_1) \\ &\leq (\alpha' + 2\beta' + 2\gamma' + 2\delta' - 2\gamma'\beta - \gamma'\delta(1 - h) - 2\delta'\beta - \delta'\delta(1 - h))d(x_0, x_1) \\ \Rightarrow d(Tx_1, Tx_2) &\leq (1 - s^2(1 - h))d(x_0, x_1), \text{ where } s^2 = \delta\delta'. \end{aligned} \tag{7}$$

Again from (5) and (7) we have,

$$d(Tx_2, Tx_3) \leq d(Tx_1, Tx_2) \leq (1 - s^2(1 - h))d(x_0, x_1).$$

And from (1) and (5) we obtain,

$$\begin{aligned} d(Tx_2, Tx_4) &\leq \alpha d(x_2, x_4) + \beta[d(x_2, Tx_2) + d(x_4, Tx_4)] + \gamma[(x_2, Tx_4) + d(x_4, Tx_2)] \\ &\quad + \delta[M(x_2, x_4) + hm(x_2, x_4)] \\ &\leq 2\alpha d(x_2, x_3) + 2\beta d(x_2, x_3) + 4\gamma d(x_2, x_3) + \delta(3 + h)d(x_2, x_3) \\ &\leq (2(1 - \beta) - \delta(1 - h))d(x_0, Tx_0). \end{aligned} \tag{8}$$

From (5), (8), we obtain

$$\begin{aligned} d(Tx_3, Tx_4) &\leq \alpha' d(x_3, x_4) + \beta'[d(x_3, Tx_3) + d(x_4, Tx_4)] + \gamma'[(x_3, Tx_4) + d(x_4, Tx_3)] \\ &\quad + \delta[M(x_3, x_4) + hm(x_3, x_4)] \\ &\leq (\alpha' + 2\beta')d(x_3, x_4) + (\gamma' + \delta')d(x_3, Tx_4) \\ &\leq (\alpha' + 2\beta')d(x_3, x_4) + (\gamma' + \delta')(2(1 - \beta) - \delta(1 - h))d(x_2, x_3) \\ &\leq (1 - s^2(1 - h))^2 d(x_0, x_1). \end{aligned}$$

Moreover,

$$d(Tx_4, Tx_5) \leq d(Tx_3, Tx_4) \leq (1 - s^2(1 - h))^2 d(x_0, x_1),$$

where $\alpha' \geq 0, \beta', \gamma', \delta' > 0, \alpha' + 2\beta' + 2\gamma' + 2\delta' = 1$ and $s^2 = \delta\delta'$.

Analogously, $d(Tx_5, Tx_6) \leq (1 - s^2(1 - h))^3 d(x_0, x_1)$, and continuing this process, we have

$$d(T^n x_0, T^{n+1} x_0) \leq (1 - s^2(1 - h))^{\lfloor \frac{n}{2} \rfloor} d(x_0, Tx_0), \tag{9}$$

for all $n \in \mathbb{N}$, where $\lfloor \frac{n}{2} \rfloor$ denotes the greatest integer not exceeding $\frac{n}{2}$. Since $0 < s \leq \frac{1}{2}$ and $h < 1$, we have $\lim_{n \rightarrow \infty} d(T^n x_0, T^{n+1} x_0) = 0$, i.e., T is asymptotically regular at x_0 . ■

Proposition 2 *If T is a self-mapping on X satisfying the condition (1) with (2) and (3). If T has a fixed point (say p), then T is continuous at p .*

Proof. Let $x_n \rightarrow p = Tp$. Then, by using (3) and triangular inequality in the condition (1), we get

$$\begin{aligned} d(Tx_n, Tp) &\leq \alpha d(x_n, p) + \beta[d(x_n, Tx_n) + d(p, Tp)] + \gamma[(x_n, Tp) + d(p, Tx_n)] \\ &\quad + \delta[M(x_n, p) + hm(x_n, p)], \\ &\leq \alpha d(x_n, p) + \beta[d(x_n, p) + d(p, Tx_n)] + \gamma[d(x_n, p) + d(p, Tx_n)] \\ &\quad + \delta[\max\{d(x_n, p), d(p, Tx_n)\} + h \min\{d(x_n, p), d(p, Tx_n)\}] \\ &< (\alpha + 2\beta + 2\gamma + 2\delta)d(x_n, p) = d(x_n, p). \end{aligned}$$

Now, $Tx_n \rightarrow Tp$ whenever $n \rightarrow \infty$. Therefore, T is continuous at p . ■

Theorem 1 *Let (X, d) be a non-empty complete metric space, and let T be a self-mapping on X satisfying the condition (1) with (2) and (3). Then T has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = T^n x_0$. Then, from (9), we obtain that $\{T^n x_0\}$ is a Cauchy sequence. Since X is complete, there is a $p \in X$ such that

$$\lim_{n \rightarrow \infty} T^n x = p.$$

Now, from (1), we have

$$\begin{aligned} d(T^n x, Tp) &\leq \alpha d(T^{n-1} x, p) + \beta[d(T^{n-1} x, T^n x) + d(p, Tp)] + \gamma[(T^{n-1} x, Tp) + d(p, T^n x)] \\ &\quad + \delta[M(T^{n-1} x, p) + hm(T^{n-1} x, p)]. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$,

$$d(p, Tp) \leq (\beta + \gamma + \delta)d(p, Tp).$$

Then $p = Tp$. Now, suppose that p and q are two fixed points of T . Then by (1), we get

$$\begin{aligned} d(p, q) &= d(Tp, Tq) \\ &\leq \alpha d(p, q) + \beta[d(p, Tp) + d(q, Tq)] + \gamma[d(p, Tq) + d(q, Tp)] \\ &\quad + \delta[M(p, q) + hm(p, q)] \\ &= (\alpha + 2\gamma + \delta(1 + h))d(p, q). \end{aligned}$$

Using (3), we get

$$d(p, q) \leq (1 - (2\beta + \delta(1 - h)))d(p, q)$$

which implies $p = q$. Hence, T has a unique fixed point. ■

Now, we take the following examples for vindication of Theorem 1.

Example 1 Let $X = [0, 10] \cup \{12\}$ be a usual metric space. Define $T : X \rightarrow X$ as:

$$T(x) = \begin{cases} 10 & \text{if } 0 \leq x \leq 10, \\ 9 & \text{if } x = 12. \end{cases}$$

Then, it can be seen easily, T is satisfying the condition (1) for $\alpha = \frac{2}{10}$, $\beta = \frac{2}{10}$, $\gamma = \frac{1}{10}$, $\delta = \frac{1}{10}$, and 10 is only fixed point of T .

Example 2 Let $X = \{1, 2, 4\}$ be a usual metric space. Define $T : X \rightarrow X$ as: $T1 = 2$, $T2 = 2$, $T4 = 1$. Then, it can be seen easily, T is satisfying the condition (1) for $\alpha = \frac{2}{10}$, $\beta = \frac{2}{10}$, $\gamma = \frac{1}{10}$, $\delta = \frac{1}{10}$, and 2 is only fixed point of T .

Here, we also give an another example in support of Theorem 1.

Example 3 Let $X = \{0, 1, 2\}$. Define $T : X \rightarrow X$ as: $T0 = 0$, $T1 = 0$, $T2 = 1$. The metric d on X is defined by $d(0, 0) = d(1, 1) = d(2, 2) = 0$, $d(0, 1) = d(1, 0) = 2$, $d(0, 2) = d(2, 0) = 1$ and $d(1, 2) = d(2, 1) = 1$. Then T satisfying all the condition (1) for $\alpha = \frac{2}{10}$, $\beta = \frac{2}{10}$, $\gamma = \frac{1}{10}$, $\delta = \frac{1}{10}$, and 0 is only fixed point of T . However, T does not satisfy the condition of result of [12] for $x = 0$, $y = 2$ ([12, Theorem 1.2]).

Remark 1 Theorem 1 is generalization of results of [4, 6, 10, 11, 12], and others.

2.2 Coincidence Point Theorem for a Pair of Single Valued Mappings

First we recall some definitions which are very important to our work in this part.

Definition 1 ([13]) Let f and g be two self-mapping of a metric space X . Then f and g are said to be compatible if $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t \in X$.

Let T and f be self-mappings on a metric space (X, d) such that $T(X) \subset f(X)$. Then, for any $x_0 \in X$, we choose a point $x_1 \in X$ such that $fx_1 = Tx_0$. Continuing this process, we can choose a sequence $\{x_k\}$ in X such that $fx_{k+1} = Tx_k$, $k = 0, 1, 2, \dots$, and the sequence $\{fx_n\}$ is called a T -sequence with initial point x_0 .

Definition 2 ([1]) Let T and f be self-mappings on a metric space (X, d) such that $T(X) \subset f(X)$. Then the mapping T is said to be asymptotically f -regular at point $x_0 \in X$, if $\lim_{n \rightarrow \infty} d(fx_n, fx_{n+1}) = 0$, where $\{fx_n\}$ is a T -sequence with initial point x_0 .

In the same paper, Abbas et al. [1] obtained the following result on metric space under the notion of asymptotically f -regular mappings.

Theorem 2 *Let T and f be self-mappings on a metric space (X, d) such that $T(X) \subset f(X)$. Assume that, for all $x, y \in X$, the following condition holds:*

$$d(Tx, Ty) \leq a_1F_1[\min\{d(fx, Tx), d(fy, Ty)\}] + a_2F_2[d(fx, Tx)d(fy, Ty)] \\ + a_3d(fx, fy) + a_4[d(fx, Tx) + d(fy, Ty)] + a_5[d(fx, Ty) + d(fy, Tx)],$$

where for $i = 1, 2, 3, 4, 5$, $a_i \geq 0$ such that for every arbitrary fixed $k > 0$, $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, we have $a_4 + a_5 \leq \lambda_1$, $a_3 + 2a_5 \leq \lambda_2$ and $a_1, a_2 \leq k$. If $f(X)$ or $T(X)$ is a complete subspace of X and T is asymptotically f -regular at some point $x_0 \in X$, then T and f have a coincidence point.

Now, in the direction of asymptotically f -regular mappings, we establish the following result.

Theorem 3 *Let T and f be self-mappings on a metric space (X, d) satisfying, for all $x, y \in X$, the following condition:*

$$d(Tx, Ty) \leq \alpha d(fx, fy) + \beta[d(fx, Tx) + d(fy, Ty)] \\ + \gamma[d(fx, Ty) + d(fy, Tx)] + \delta[M_f(x, y) + hm_f(x, y)], \tag{10}$$

where $\alpha \geq 0$, $\beta, \gamma, \delta > 0$ and $0 < h < 1$ with $\alpha + 2\beta + 2\gamma + 2\delta = 1$, $M_f(x, y) = \max\{d(fx, Ty), d(fy, Tx)\}$ and $m_f(x, y) = \min\{d(fx, Ty), d(fy, Tx)\}$. If $T(X) \subset f(X)$ and T is asymptotically f -regular at some point x_0 in X , and one of the following holds:

- (a) X is complete and f is surjective;
- (b) X is complete, f is continuous, and T and f are compatible;
- (c) $f(X)$ is complete subspace of X ;
- (d) $T(X)$ is complete subspace of X .

Then f and T have a coincidence point in X . Moreover, the coincidence value is unique, i.e., $fp = fq$, whenever $fp = Tp$ and $fq = Tq$.

Proof. Let $x_0 \in X$ be an arbitrary. Since $T(X) \subset f(X)$, we may construct a sequence $fx_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Now, from (10), we get (with $h = 1$)

$$d(fx_{n+1}, fx_{n+2}) \leq \alpha d(fx_n, fx_{n+1}) + \beta[d(fx_n, Tx_n) + d(fx_{n+1}, Tx_{n+1})] \\ + \gamma[d(fx_n, Tx_{n+1}) + d(fx_{n+1}, Tx_n)] + \delta[M_f(x_n, x_{n+1}) + m_f(x_n, x_{n+1})].$$

If for some n , $d(fx_{n+1}, fx_{n+2}) > d(fx_n, fx_{n+1})$, then in the above inequality results as

$$d(fx_{n+1}, fx_{n+2}) < (\alpha + 2\beta + 2\gamma + 2\delta)d(fx_{n+1}, fx_{n+2}) = d(fx_{n+1}, fx_{n+2}),$$

which is a contradiction. Therefore,

$$d(fx_{n+1}, fx_{n+2}) \leq d(fx_n, fx_{n+1}) \text{ for all } n. \tag{11}$$

Without loss of generality, for $m > n$, we have

$$d(fx_n, fx_m) \leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{m+1}) + d(fx_{m+1}, fx_m) \\ = d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m) + d(Tx_n, Tx_m).$$

Now from (10), (11) and triangular inequality, we obtain

$$\begin{aligned}
d(fx_n, fx_m) &\leq d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m) \\
&\quad + \alpha d(fx_n, fx_m) + \beta [d(fx_n, Tx_n), d(fx_m, Tx_m)] \\
&\quad + \gamma [d(fx_n, Tx_m) + d(fx_m, Tx_n)] + \delta [M_f(x_n, x_m) + m_f(x_n, x_m)] \\
&\leq d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m) + \alpha d(fx_n, fx_m) \\
&\quad + \gamma [d(fx_n, fx_m) + d(fx_m, fx_{m+1}) + d(fx_m, fx_n) + d(fx_n, fx_{n+1})] \\
&\quad + \beta [d(fx_n, fx_{n+1}) + d(fx_m, fx_{m+1})] \\
&\quad + \delta [(d(fx_n, fx_m) + d(fx_m, fx_{m+1})) + (d(fx_m, fx_n) + d(fx_n, fx_{n+1}))] \\
&= (\alpha + 2\gamma + 2\delta)d(fx_n, fx_m) + \beta [d(fx_n, fx_{n+1}) + d(fx_m, fx_{m+1})] \\
&\quad + (1 + \gamma + \delta)[d(fx_m, fx_{m+1}) + d(fx_n, fx_{n+1})].
\end{aligned}$$

Then

$$d(fx_n, fx_m) = \left(\frac{1 + \gamma + \delta}{2\beta} \right) [d(fx_m, fx_{m+1}) + d(fx_n, fx_{n+1})].$$

Since T is asymptotically f -regular, then the right hand side of the above inequality tends to zero, as $m, n \rightarrow \infty$. Thus, $\lim_{m, n \rightarrow \infty} d(fx_n, fx_m) = 0$. It follows that $\{fx_n\}$ is a Cauchy sequence in X . Now, we consider the following cases:

- (a) Suppose X is complete and f is surjective. Then there exists a point $p \in X$ with $\lim_{n \rightarrow \infty} fx_n = p$ and a point $z \in X$ such that $p = fz$, and from (10), we get (with $h = 1$)

$$\begin{aligned}
d(fz, Tz) &\leq d(fz, fx_{n+1}) + d(Tx_n, Tz) \\
&\leq d(fz, fx_{n+1}) + \alpha d(fx_n, fz) + \beta [d(fx_n, Tx_n) + d(fz, Tz)] \\
&\quad + \gamma [d(fx_n, Tz) + d(fz, Tx_n)] + \delta [M_f(x_{n+1}, z) + m_f(x_{n+1}, z)]. \\
&\leq d(fz, fx_{n+1}) + \alpha d(fx_n, fz) + \beta [d(fx_n, fx_{n+1}) + d(fz, Tz)] \\
&\quad + \gamma [d(fx_n, Tz) + d(fz, fx_{n+1})] \\
&\quad + \delta [\max\{d(fx_n, fx_{n+1}), d(fz, Tz)\} + \min\{d(fx_n, fx_{n+1}), d(fz, Tz)\}],
\end{aligned}$$

making $n \rightarrow \infty$, we get $d(fz, Tz) \leq (\beta + \gamma + \delta)(fz, Tz)$ which implies $p = fz = Tz$. Hence, f and T have a coincidence point.

- (b) Suppose that X is complete, f is continuous, and f and T are compatible. Then, $\lim_{n \rightarrow \infty} fx_n = p$ implies $\lim_{n \rightarrow \infty} ffx_n = fp$, and from (10), we get (with $h = 1$)

$$\begin{aligned}
d(fp, Tp) &\leq d(fp, ffx_{n+1}) + d(fTx_n, Tp) \\
&\leq d(fp, ffx_{n+1}) + d(fTx_n, Tfx_n) + d(Tfx_n, Tp) \\
&\leq d(fp, ffx_{n+1}) + d(fTx_n, Tfx_n) + \alpha d(ffx_n, fp) \\
&\quad + \beta [d(ffx_n, Tfx_n) + d(fp, Tp)] + \gamma [d(ffx_n, Tp) + d(fp, Tfx_n)] \\
&\quad + \delta [M_f(fx_n, p) + m_f(fx_n, p)].
\end{aligned}$$

Note that, since $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Tx_n = p$, and f and T are compatible, $\lim_{n \rightarrow \infty} d(fTx_n, Tfx_n) = 0$. Taking the limit as $n \rightarrow \infty$, above inequality yields

$$d(fp, Tp) \leq (2\beta + 2\gamma + 2\delta)d(fp, Tp)$$

Then

$$d(fp, Tp) \leq (1 - \alpha)d(fp, Tp).$$

It implies that $fp = Tp$.

(c) If $f(X)$ is complete subspace of X , then $p \in f(X)$. Suppose $z \in f^{-1}p$, then $p = fz$ and the proof is completed by case (a).

(d) If $T(X)$ is complete subspace of X , then $p \in T(X) \subset f(X)$, and the proof is completed by case (c).

To establish uniqueness, suppose that q is another coincidence point of f and T , i.e., $fp = Tp$, $fq = Tq$. Then, from (10), we have

$$\begin{aligned} d(Tp, Tq) &\leq \alpha d(fp, fq) + \beta[d(fp, Tp) + d(fq, Tq)] + \gamma[d(fp, Tq) + d(fq, Tp)] \\ &\quad + \delta[M_f(p, q) + m_f(p, q)] \\ &\leq (\alpha + 2\gamma + 2\delta)d(Tp, Tq). \end{aligned}$$

Then

$$d(Tp, Tq) \leq (1 - 2\beta)d(Tp, Tq)$$

which implies that $Tp = Tq$, and hence $fp = fq$. ■

Remark 2 Taking $f = I$ (identity mapping on X) in Theorem 3, we find an another version of Theorem 1 for $h = 1$.

2.3 Coincidence Point Theorem for a Hybrid Pair of Mappings

Let (X, d) be a metric space. A mapping $T : X \rightarrow 2^X$ is said to be multi-valued mapping, where 2^X is collection of all non-empty subsets of X . We denote by $CB(X)(C(X)$ respectively) the family of all non-empty closed bounded (compact respectively) subsets of X . Then a function $H : CB(X) \rightarrow \mathbb{R}$, defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},$$

is called the Hausdorff metric for $A, B \in CB(X)(C(X)$ respectively), where $d(x, B) = \inf_{y \in B} d(x, y)$. Now, we recall some definitions which are the following.

Definition 3 ([9]) An orbit of the multi-valued mapping T at a point $x_0 \in X$ is a sequence $\{x_n : x_n \in Tx_{n-1}\}$. A space X is T -orbitally complete if every Cauchy sequence of the form $\{y_n : y_n \in Ty_{n-1}\}$ converges in X .

Definition 4 ([18]) Let (X, d) be a metric space, $T : X \rightarrow C(X)$, and let S be a self-mapping of X . If, for a point $x_0 \in X$, there exists a sequence $\{x_n\} \subset X$ such that $Sx_{n+1} \in Tx_n$, $n \in \mathbb{N} \cup \{0\}$, then $O_S(x_0) = \{Sx_n : n = 1, 2, 3, \dots\}$ is an orbit of (T, S) at x_0 . A space X is called (T, S) -orbitally complete if every Cauchy sequence of the form $\{Sx_n : Sx_n \in Tx_{n-1}\}$ converges in X .

Now, we prove the following result by using above mentioned concepts.

Theorem 4 Let (X, d) be a metric space, and $T : X \rightarrow C(X)$. Let S be a self-mapping of X such that for all $x, y \in X$,

$$\begin{aligned} H(Tx, Ty) &\leq \alpha d(Sx, Sy) + \beta[(Sx, Tx) + d(Sy, Ty)] \\ &\quad + \gamma[(Sx, Ty) + d(Sy, Tx)] + \delta[M_s(x, y) + hm_s(x, y)], \end{aligned} \tag{12}$$

where $\alpha \geq 0, \beta, \gamma, \delta > 0$ and $0 < h < 1$ with $\alpha + 2\beta + 2\gamma + 2\delta = 1$ and $M_s(x, y) = \max\{(Sx, Ty), d(Sy, Tx)\}$, $m_s(x, y) = \min\{(Sx, Ty), d(Sy, Tx)\}$. If $T(X) \subset S(X)$, and one of the following holds:

- (a) X is (T, S) -orbitally complete and S is surjective,
- (b) $S(X)$ is (T, S) -orbitally complete,

(c) $T(X)$ is (T, S) -orbitally complete.

Then S and T have a coincidence point in X , i.e., there exists $z \in X$ such that $Sz \in Tz$.

Proof. Let $x_0 \in X$ and $y_0 = Sx_0$. Then, we construct sequences $\{x_n\}$ and $\{y_n\}$ as follows. Since $T(X) \subset S(X)$, we can choose $y_1 = Sx_1 \in Tx_0$. If $Tx_0 = Tx_1$, choose $y_2 = Sx_2 \in Tx_1$ such that $y_1 = y_2$. If $Tx_0 \neq Tx_1$, choose $y_2 = Sx_2 \in Tx_1$ such that $d(y_1, y_2) \leq H(Tx_0, Tx_1)$ (such a choice is possible because Tx is compact for each $x \in X$). In general, choose $y_{n+1} = Sx_{n+1} \in Tx_n$, for each $n \in \mathbb{N} \cup \{0\}$, and $y_{n+1} = y_{n+2}$ if $Tx_n = Tx_{n+1}$ and $d(y_{n+1}, y_{n+2}) \leq H(Tx_n, Tx_{n+1})$, otherwise. Now, from (12) implies (with $h = 1$)

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\leq H(Tx_n, Tx_{n+1}) \\ &\leq \alpha d(Sx_n, Sx_{n+1}) + \beta [d(Sx_n, Tx_n) + d(Sx_{n+1}, Tx_{n+1})] \\ &\quad + \gamma [d(Sx_n, Tx_{n+1}) + d(Sx_{n+1}, Tx_n)] + \delta [M_s(x_n, x_{n+1}) + m_s(x_n, x_{n+1})] \\ &\leq \alpha d(y_n, y_{n+1}) + \beta [d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})] \\ &\quad + \gamma [d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})] \\ &\quad + \delta [\max\{d(y_n, y_{n+2}), d(y_{n+1}, y_{n+1})\} + \min\{d(y_n, y_{n+2}), d(y_{n+1}, y_{n+1})\}]. \end{aligned}$$

Suppose that $d(y_{n+1}, y_{n+2}) > d(y_n, y_{n+1})$ for some n . Then, substituting in the above inequality, we have

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\leq \alpha d(y_n, y_{n+1}) + \beta [d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})] \\ &\quad + \gamma [d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})] + \delta [d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})]. \end{aligned}$$

It implies that $d(y_{n+1}, y_{n+2}) < d(y_{n+1}, y_{n+2})$, which is a contradiction. Therefore,

$$d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1}) \text{ for all } n. \quad (13)$$

Now, if $d(Sx_{n-1}, Sx_{n+1}) \geq d(Sx_{n-1}, Tx_{n-1})$, then the condition (12) implies (with $h = 1$),

$$\begin{aligned} d(y_n, y_{n+2}) &\leq H(Tx_{n-1}, Tx_{n+1}) \\ &\leq \alpha d(Sx_{n-1}, Sx_{n+1}) + \beta [d(Sx_{n-1}, Tx_{n-1}) + d(Sx_{n+1}, Tx_{n+1})] \\ &\quad + \gamma [d(Sx_{n-1}, Tx_{n+1}) + d(Sx_{n+1}, Tx_{n-1})] \\ &\quad + \delta [M_s(x_{n-1}, x_{n+1}) + m_s(x_{n-1}, x_{n+1})] \\ &\leq \alpha d(y_{n-1}, y_{n+1}) + \beta [d(y_{n-1}, y_n) + d(y_{n+1}, y_{n+2})] \\ &\quad + \gamma [d(y_{n-1}, y_{n+2}) + d(y_{n+1}, y_n)] + \delta [\max\{d(y_{n-1}, y_{n+2}), d(y_{n+1}, y_n)\} \\ &\quad + \min\{d(y_{n-1}, y_{n+2}), d(y_{n+1}, y_n)\}]. \end{aligned}$$

Using (13) and triangular, we obtain

$$\begin{aligned} d(y_n, y_{n+2}) &\leq (2\alpha + 2\beta + 2\gamma + 2\delta)d(y_{n-1}, y_n) + (\gamma + \delta)d(y_n, y_{n+2}) \\ \text{it implies } d(y_n, y_{n+2}) &\leq \left(\frac{1 + \alpha}{1 - \gamma - \delta} \right) d(y_{n-1}, y_n). \end{aligned}$$

Thus, $d(y_n, y_{n+2}) \leq k_1 d(y_{n-1}, y_n)$, where $k_1 = \frac{(1+\alpha)}{(1-\gamma-\delta)} < 2$.

Now, if $d(Sx_{n-1}, Sx_{n+1}) < d(Sx_{n-1}, Tx_{n-1})$, then the condition (12) implies (with $h = 1$),

$$\begin{aligned} d(y_n, y_{n+1}) &\leq H(Tx_{n-1}, Tx_n) \\ &\leq \alpha d(Sx_{n-1}, Sx_n) + \beta [d(Sx_{n-1}, Tx_{n-1}) + d(Sx_n, Tx_n)] \\ &\quad + \gamma [d(Sx_{n-1}, Tx_n) + d(Sx_n, Tx_{n-1})] \\ &\quad + \delta [M_s(x_{n-1}, x_n) + m_s(x_{n-1}, x_n)] \\ &\leq \alpha d(y_{n-1}, y_n) + \beta [d(y_{n-1}, y_n) + d(y_n, y_{n+1})] \\ &\quad + \gamma [d(y_{n-1}, y_{n+1}) + d(y_n, y_n)] + \delta [\max\{d(y_{n-1}, y_{n+1}), d(y_n, y_n)\} \\ &\quad + \min\{d(y_{n-1}, y_{n+1}), d(y_n, y_n)\}]. \end{aligned}$$

It implies that

$$d(y_n, y_{n+1}) < \left(\frac{\alpha + \beta + \gamma + \delta}{1 - \beta} \right) d(y_{n-1}, y_n).$$

So

$$\begin{aligned} d(y_n, y_{n+2}) &\leq d(y_n, y_{n+1}) + (y_{n+1}, y_{n+2}) \leq 2d(y_n, y_{n+1}) \\ &< \frac{2(\alpha + \beta + \gamma + \delta)}{(1 - \beta)} d(y_{n-1}, y_n) = \frac{(1 + \alpha)}{(1 - \beta)} d(y_{n-1}, y_n). \end{aligned}$$

Thus $d(y_n, y_{n+2}) \leq k_2 d(y_{n-1}, y_n)$, where $k_2 = \frac{(1+\alpha)}{(1-\beta)} < 2$. Taking $k = \max\{k_1, k_2\}$, we get $0 < k < 2$, and

$$d(y_n, y_{n+2}) \leq k d(y_{n-1}, y_n), \text{ for all } n \in \mathbb{N}. \tag{14}$$

From inequalities (12), (13) and (14), we get (with $h = 1$)

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\leq \alpha d(y_n, y_{n+1}) + \beta [d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})] \\ &\quad + \gamma [d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})] + \delta [\max\{d(y_n, y_{n+2}), d(y_{n+1}, y_{n+1})\} \\ &\quad + \min\{d(y_n, y_{n+2}), d(y_{n+1}, y_{n+1})\}] \\ &\leq (\alpha + 2\beta + k(\gamma + \delta)) d(y_{n-1}, y_n). \end{aligned}$$

Thus,

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_{n-1}, y_n), \forall n \in \mathbb{N}, \tag{15}$$

where $\lambda = (\alpha + 2\beta + k(\gamma + \delta)) < 1$.

Using induction and (13) and (15), we obtain $d(y_n, y_{n+1}) \leq \lambda^{\lfloor \frac{n}{2} \rfloor} d(y_0, y_1)$, for all $n \in \mathbb{N}$, where $\lfloor \frac{n}{2} \rfloor$ denotes the greatest integer value. Hence, $\{y_n\}$ is a Cauchy sequence in X , and it is convergent to a point $p \in X$ in cases (a)–(c).

Now, if S is surjective, then there exists a point $z \in X$ such that $p = Sz$. This is obviously true in cases (b) and (c). From (12),

$$\begin{aligned} d(Sz, Tz) &\leq d(Sz, Sx_{n+1}) + H(Tx_n, Tz) \\ &\leq d(Sz, Sx_{n+1}) + \alpha d(Sx_n, Sz) + \beta [d(Sx_n, Tx_n) + d(Sz, Tz)] \\ &\quad + \gamma [d(Sx_n, Tz) + d(Sz, Tx_n)] + \delta [M_s(x_n, z) + m_s(x_n, z)] \end{aligned}$$

Then

$$\begin{aligned} d(p, Tz) &\leq d(p, y_{n+1}) + \alpha d(y_n, p) + \beta [d(y_n, y_{n+1}) + d(p, Tz)] \\ &\quad + \gamma [d(y_n, Tz) + d(p, y_{n+1})] + [\max\{d(y_n, Tz), d(p, y_{n+1})\} \\ &\quad + \min\{d(y_n, Tz), d(p, y_{n+1})\}]. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we have $d(p, Tz) \leq (\beta + \gamma + \delta)d(p, Tz)$ which implies that $Sz \in Tz$. Hence, S and T have a coincidence point. ■

3 An Application to Dynamic Programming

In this section, we consider the functional equation given by Bellman and Lee [3] and find solution using our condition.

Let U and V be Banach spaces and $W \subset U, D \subset V$. Let $B(W)$ denote the set of all bounded real valued functions on W . It is well known that $B(W)$ endowed with the metric,

$$d_B(h, k) = \sup_{x \in W} |h(x) - k(x)|, \quad h, k \in B(W) \tag{16}$$

is a complete metric space. Bellman and Lee [3] introduced the following basic form of the functional equation of dynamic programming:

$$p(x) = \sup_y H(x, y, p(\tau(x, y))), \quad (17)$$

where x and y represent the state and decision vectors respectively, and $\tau : W \times D \rightarrow W$ represents the transformation of the process and $p(x)$ represents the optimal return function with initial state x .

Now, we will study the existence and uniqueness of the solution of the following functional equation:

$$p(x) = \sup_y [g(x, y) + G(x, y, p(\tau(x, y)))], \quad x \in W \quad (18)$$

where $g : W \times D \rightarrow \mathbb{R}$ and $G : W \times D \times \mathbb{R} \rightarrow \mathbb{R}$ are bounded functions. Let $T : B(W) \rightarrow B(W)$ be a mapping defined by

$$T(h(x)) = \sup_y \{g(x, y) + G(x, y, h(\tau(x, y)))\}, \quad (19)$$

where $h \in B(W)$ and $x \in W$.

Theorem 5 *If there exist $\alpha \geq 0$, $\beta, \gamma, \delta > 0$ and $0 < h < 1$ such that $\alpha + 2\beta + 2\gamma + 2\delta = 1$, and*

$$|G(x, y, h(t)) - G(x, y, k(t))| \leq M(h(t), k(t)), \quad (20)$$

for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$, where

$$\begin{aligned} M(h(t), k(t)) &= \alpha |h(t) - k(t)| + \beta \left\{ |h(t) - T(h(t))| + |k(t) - T(k(t))| \right\} \\ &+ \gamma \left\{ |h(t) - T(k(t))| + |k(t) - T(h(t))| \right\} \\ &+ \delta \left\{ \max\{|h(t) - T(k(t))|, |k(t) - T(h(t))|\} \right\} \\ &+ h \min\{|h(t) - T(k(t))|, |k(t) - T(h(t))|\} \end{aligned}$$

then the functional equation (18) has a unique bounded solution in $B(W)$.

Proof. Let $\epsilon > 0$ be an arbitrary, and $h, k \in B(W)$. Then for $x \in W$, we can choose $y_1, y_2 \in D$ so that

$$T(h(x)) < g(x, y_1) + G(x, y_1, h(\tau(x, y_1))) + \epsilon, \quad (21)$$

$$T(k(x)) < g(x, y_2) + G(x, y_2, k(\tau(x, y_2))) + \epsilon. \quad (22)$$

Also from (19),

$$T(h(x)) \geq g(x, y_2) + G(x, y_2, h(\tau(x, y_2))), \quad (23)$$

$$T(k(x)) \geq g(x, y_1) + G(x, y_1, k(\tau(x, y_1))). \quad (24)$$

If the inequality (20) holds, from inequalities (21) and (24) we have

$$\begin{aligned} T(h(x)) - T(k(x)) &< G(x, y_1, h(\tau(x, y_1))) - G(x, y_1, k(\tau(x, y_1))) + \epsilon \\ &\leq |G(x, y_1, h(\tau(x, y_1))) - G(x, y_1, k(\tau(x, y_1)))| + \epsilon \\ &\leq M(h(x), k(x)) + \epsilon. \end{aligned} \quad (25)$$

Similarly from (22) and (23), we obtain

$$T(k(x)) - T(h(x)) \leq M(h(x), k(x)) + \epsilon. \quad (26)$$

From (25) and (26), we establish

$$|T(h(x)) - T(k(x))| \leq M(h(x), k(x)) + \epsilon,$$

for each $x \in W$, and for arbitrary $\epsilon > 0$.

Thus, $d_B(T(h), T(k)) \leq M(h, k)$, where $\alpha, \delta > 0, \beta, \gamma \geq 0$ such that $\alpha + 2\beta + 2\gamma + 2\delta = 1$. Moreover, the conditions of Theorem 1 are satisfied for the mapping T . Hence, the functional equation (18) has a unique bounded solution. ■

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