

A Study Of The Evolution Of Space Curves With Modified Orthogonal Frame In Euclidean 3-Space*

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Abstract

In this study, we investigate the evolution of a moving space curve with a modified orthogonal frame. Firstly, the relations between the Frenet frame and the modified orthogonal frame is given. Later, the time evolution equation and curvatures of a moving space curve according to the modified orthogonal frame are obtained. Finally, some theorems and results are given according to the special cases of the evolution of the moving space curve.

1 Introduction

In recent years, the connection between the motion of space curves and differential equations is a subject of research attention. At the same time, the applications of this subject in differential geometry and physics have attracted attention. The subject of how space curves change depending on time is of great interest and has been explored by many researchers. The evolution of the Serret-Frenet equations and the evolution of the curvatures of this moving frame have been studied in Euclidean 3-space and at the same time, nonlinear differential equations have been obtained by using the evolution of the curvature and torsion under suitable conditions [1, 2, 3]. The geometry of evolving plane curves, the evolution of inelastic plane curves, the evolution of a helix curve and evolution of space curves in R^n are given in the studies [4, 5, 6, 7]. Also, Richardson and King investigated the evolution of space curves by means of curvature and torsion [8]. Hasimoto showed that the evolution of the vortex filament, which is considered to be the motion of the space curve, can be transformed into the nonlinear Schrodinger equation [9]. In addition, the Hasimoto surfaces are obtained by evolving the regular space curve. The equation of the Hasimoto surface is given as follows:

$$r_t(s, t) = r_s \wedge r_{ss} \quad (1)$$

where $r(s, t)$ is a position vector of a moving space curve, s is the arc-length parameter, t denotes time and the subscript denote the partial differential equation [10]. All the above studies have been examined according to the Serret-Frenet frame and alternatively, the evolution of the space curve and characterization of the inelastic curve according to Bishop frame are given in [11, 12]. Serret-Frenet frame is insufficient at points where the curvature of a space curve is zero. Because at points where the curvature is zero, the principal normal and binormal vector of a space curve becomes discontinuous. To find a solution to this problem, Sasai defined the modified orthogonal frame as an alternative to the Frenet frame in his study [13]. Then, the modified orthogonal frame was defined by Bükcü and Karaca for the curvature and torsion non-zero of a space curve in Minkowski 3-space [14]. Some studies on spherical and some special curves have been made using this modified orthogonal frame [15, 16, 17]. This study aims to investigate the geometric properties of the evolution of the space curve according to the modified orthogonal frame. Primarily for this, the time evolution equation and curvatures of a space curve according to the modified orthogonal frame are obtained. Later, some theorems and results are given according to the special cases of the evolution of the space curve. In addition, another purpose of this study is to contribute to the scientific world by examining the evolution of the space curve from a different perspective.

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2 Preliminaries

Let E^3 denote the Euclidean 3-space. The norm of a vector $x \in E^3$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. Also, a space curve r is called unit speed curve, if $\|r'(s)\| = 1$. Assume that r is a moving space curve with respect to the arc-length s in Euclidean 3-space E^3 and t , n , and b denote the tangent, principal normal and binormal unit vectors at any point $r(s)$ of the curve r , respectively. Then, there exists an orthonormal frame $\{t, n, b\}$ which satisfies the Frenet-Serret equation

$$\begin{aligned} t' &= \kappa n, \\ n' &= -\kappa t + \tau b, \\ b' &= -\tau n \end{aligned}$$

where κ and τ are the curvature and torsion of the curve r , respectively.

For the reason that the principal normal and binormal vectors of the Frenet frame of a space curve become discontinuous at the points where the curvature is zero, the modified orthogonal frame was introduced by Sasai as an alternative to the Frenet frame. In this sense, one assumes that the curvature $\kappa(s)$ of the space curve r is not zero and then the modified orthogonal frame $\{T, N, B\}$ can be defined as follows:

$$T = \frac{dr}{ds}, \quad N = \frac{dT}{ds}, \quad B = T \wedge N$$

where $T \wedge N$ is the vector product of T and N . The relations between the modified orthogonal frame $\{T, N, B\}$ and Serret-Frenet frame $\{t, n, b\}$ at non-zero points of κ are

$$T = t, \quad N = \kappa n, \quad B = \kappa b.$$

From these equations, it is known that the differentiation of the elements of the modified orthogonal frame $\{T, N, B\}$ satisfies

$$\begin{aligned} T'(s) &= N(s), \\ N'(s) &= -\kappa^2 T(s) + \frac{\kappa_s}{\kappa} N(s) + \tau B(s), \\ B'(s) &= -\tau N(s) + \frac{\kappa_s}{\kappa} B(s) \end{aligned}$$

and the matrix form is given by

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_s = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa_s}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa_s}{\kappa} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where κ_s denotes the differentiation with respect to the arc-length parameter s and $\tau = \frac{\det(r', r'', r''')}{\kappa^2}$ is the torsion of the space curve r . Moreover, the modified orthogonal frame $\{T, N, B\}$ satisfies

$$\langle T, T \rangle = 1, \langle N, N \rangle = \langle B, B \rangle = \kappa^2, \langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0.$$

3 Evolution of Space Curves with Modified Orthogonal Frame

In this section, we study the evolution of a moving space curve using a modified orthogonal frame. We derive the time evolution equation and curvatures of a space curve for the modified orthogonal frame.

Let $r(s, t)$ be a position vector of a moving space curve r which a unit speed curve for all t parameters in Euclidean 3-spaces. Then, the time evolution of the modified orthogonal frame $\{T, N, B\}$ can be written in matrix form as follows:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_t = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & \kappa \kappa_t & \gamma \\ -\beta & -\gamma & \kappa \kappa_t \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where α, β and γ are smooth functions for s and t . These functions define the evolution according to the parameter t on the space curve r . Under compatibility conditions $T_{st} = T_{ts}, N_{st} = N_{ts}, B_{st} = B_{ts}$, we obtain

$$\begin{aligned} \alpha_s &= \tau\beta - \frac{\kappa_s}{\kappa}\alpha + \kappa\kappa_t, \\ \beta_s &= \gamma - \tau\alpha - \frac{\kappa_s}{\kappa}\beta, \\ \gamma_s &= \tau_t - \beta. \end{aligned} \tag{2}$$

We suppose that the velocity vector $r_t = \frac{dr}{dt}$ of a moving space curve r is given by

$$r_t = \frac{dr}{dt} = aT + bN + cB. \tag{3}$$

From the imposition of condition $r_{st} = r_{ts}$, we find the following equations

$$\begin{aligned} 0 &= a_s - b\kappa^2, \\ \alpha &= a + b_s + \frac{\kappa_s}{\kappa}b - \tau c, \\ \beta &= \tau b + c_s + \frac{\kappa_s}{\kappa}c \end{aligned} \tag{4}$$

where a, b and c as functions of the parameters s and t correspond to the tangent, normal and binormal vectors of the velocity, respectively. Substituting the equations (4) into the second equation of (2), we get

$$\gamma = \left(\tau b + c_s + \frac{\kappa_s}{\kappa}c\right)_s + \tau \left(a + b_s + \frac{\kappa_s}{\kappa}b - \tau c\right) + \frac{\kappa_s}{\kappa} \left(\tau b + c_s + \frac{\kappa_s}{\kappa}c\right). \tag{5}$$

Also, considering the equation (2), we find the time evolution equations of the curvature and torsion of the curve r as

$$\begin{aligned} \kappa_t &= \frac{1}{\kappa} \left(\alpha_s + \frac{\kappa_s}{\kappa}\alpha - \tau\beta\right), \\ \tau_t &= \gamma_s + \beta. \end{aligned} \tag{6}$$

This equation is the main result of this paper. The motion of the space curve via the modified orthogonal frame is determined from these equations for $\{\alpha, \beta, \gamma\}$. Now, let's express the special cases with respect to the coefficients a, b, c of the velocity of a curve r indicated in equation (3).

Theorem 1 *Let $r(s, t)$ be a position vector of a moving space curve r which is a unit speed curve for all t parameters in Euclidean 3-space and the velocity vector of $r(s, t)$ be $r_t = B$, then with respect to the modified orthogonal frame the evolution equation of the curvature and the torsion of the curve r are $\kappa_t = -\frac{1}{\kappa} (2\tau \frac{\kappa_s}{\kappa} + \tau_s)$ and $\tau_t = (\frac{\kappa_{ss}}{\kappa} - \tau^2)_s + \frac{\kappa_s}{\kappa}$, respectively.*

Proof. If we assume that the velocity vector of $r(s, t)$ is $r_t = B$, then from the equation (3), we get $a = 0, b = 0$ and $c = 1$. Then the time evolution of the modified orthogonal frame $\{T, N, B\}$ is found as follows:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_t = \begin{bmatrix} 0 & -\tau & \frac{\kappa_s}{\kappa} \\ \tau & \kappa\kappa_t & \frac{\kappa_{ss}}{\kappa} - \tau^2 \\ -\frac{\kappa_s}{\kappa} & -\frac{\kappa_{ss}}{\kappa} + \tau^2 & \kappa\kappa_t \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

such that $\alpha = -\tau, \beta = \frac{\kappa_s}{\kappa}, \gamma = \frac{\kappa_{ss}}{\kappa} - \tau^2$. Finally, considering the equation (6) completes the proof. ■

If we recall the equation of the hasimoto surface that $r_t(s, t) = r_s \wedge r_{ss} = B$ is satisfied with respect to the modified orthogonal frame, the following corollary is given immediately.

Corollary 2 *If a moving space curve $r(s, t)$ generates a Hasimoto surface, then with respect to the modified orthogonal frame the evolution equation of the curvature and the torsion of the curve r are $\kappa_t = -\frac{1}{\kappa} (2\tau \frac{\kappa_s}{\kappa} + \tau_s)$ and $\tau_t = (\frac{\kappa_{ss}}{\kappa} - \tau^2)_s + \frac{\kappa_s}{\kappa}$, respectively, see Figure 1.*

Theorem 3 *Let $r(s, t)$ be a position vector of a moving space curve r which is a unit speed curve for all t parameters in Euclidean 3-space and the velocity vector of $r(s, t)$ be $r_t = T$, then with respect to the modified orthogonal frame the evolution equation of the curvature and the torsion of the curve r are $\kappa_t = \frac{\kappa_s}{\kappa^2}$ and $\tau_t = \tau_s$, respectively.*

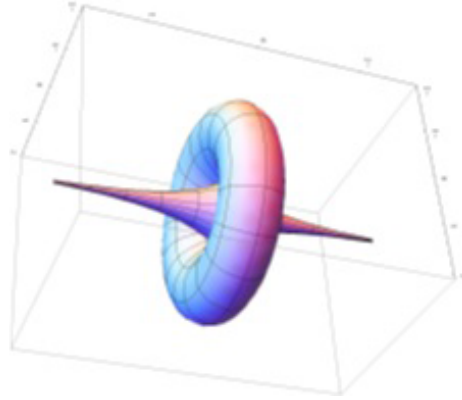


Figure 1: A Hasimoto Surface.

Proof. Assume that the velocity vector of a moving space curve r is $r_t = T$. $a = 1, b = 0$ and $c = 0$ have to be satisfied in the equation (3). Thus the time evolution of the modified orthogonal frame $\{T, N, B\}$ is found as follows:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_t = \begin{bmatrix} 0 & -1 & 0 \\ 1 & \kappa\kappa_t & \tau \\ 0 & -\tau & \kappa\kappa_t \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

such that $\alpha = 1, \beta = 0, \gamma = \tau$. From the equation (6), the evolution equation of the curvature and the torsion of the curve r are $\kappa_t = \frac{\kappa_s}{\kappa^2}$ and $\tau_t = \tau_s$. ■

Theorem 4 Let $r(s, t)$ be a position vector of a moving space curve r which is a unit speed curve for all t parameters in Euclidean 3-space and the velocity vector of $r(s, t)$ be $r_t = N$, then with respect to the modified orthogonal frame the evolution equation of the curvature and the torsion of the curve r are $\kappa_t = \frac{1}{\kappa} \left(\frac{\kappa_{ss}}{\kappa} - \tau^2 \right)$ and $\tau_t = \left(\tau_s + \tau \frac{\kappa_s}{\kappa} \right)_s + \tau$, respectively.

Proof. If $r_t = N$ is assumed then $a = 0, b = 1$ and $c = 0$, then, the time evolution of the modified orthogonal frame $\{T, N, B\}$ is found as follows:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_t = \begin{bmatrix} 0 & \frac{\kappa_s}{\kappa} & \tau \\ -\frac{\kappa_s}{\kappa} & \kappa\kappa_t & \tau_s + \tau \frac{\kappa_s}{\kappa} \\ -\tau & -\tau_s - \tau \frac{\kappa_s}{\kappa} & \kappa\kappa_t \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

such that $\alpha = \frac{\kappa_s}{\kappa}, \beta = \tau, \gamma = \tau_s + \tau \frac{\kappa_s}{\kappa}$. From the equation (6), the evolution equation of the curvature and the torsion of the space curve r are $\kappa_t = \frac{1}{\kappa} \left(\frac{\kappa_{ss}}{\kappa} - \tau^2 \right)$ and $\tau_t = \left(\tau_s + \tau \frac{\kappa_s}{\kappa} \right)_s + \tau$. ■

Theorem 5 Let $r(s, t)$ be a position vector of a moving space curve r which is a unit speed curve for all t parameters in Euclidean 3-space and the velocity vector of $r(s, t)$ be $r_t = \lambda T + \frac{B}{\lambda}$ where λ is a non-zero constant, then with respect to the modified orthogonal frame the evolution equation of the curvature and the torsion of the curve r are $\kappa_t = \frac{1}{\lambda\kappa^2} \left(-\kappa\tau_s + \kappa_s (\lambda^2 - 2\tau) \right)$ and $\tau_t = \frac{1}{\lambda} \left(\left(\frac{\kappa_{ss}}{\kappa} + \tau (\lambda^2 - \tau) \right)_s + \frac{\kappa_s}{\kappa} \right)$, respectively.

Proof. Let $r(s, t)$ be the position vector of a moving space curve r and $a = \text{constant} = \lambda \neq 0, b = 0$ and $c = \frac{1}{\lambda}$, then the time evolution of the modified orthogonal frame $\{T, N, B\}$ is found as follows:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_t = \begin{bmatrix} 0 & \frac{\lambda^2 - \tau}{\lambda} & \frac{\kappa_s}{\lambda\kappa} \\ \frac{\tau - \lambda^2}{\lambda} & \kappa\kappa_t & \left(\frac{\kappa_{ss} + \tau\kappa(\lambda^2 - \tau)}{\lambda\kappa} \right) \\ -\frac{\kappa_s}{\lambda\kappa} & -\left(\frac{\kappa_{ss} + \tau\kappa(\lambda^2 - \tau)}{\lambda\kappa} \right) & \kappa\kappa_t \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

such that $\alpha = \frac{\lambda^2 - \tau}{\lambda}, \beta = \frac{\kappa_s}{\lambda\kappa}, \gamma = \frac{\kappa_{ss} + \kappa\tau(\lambda^2 - \tau)}{\lambda\kappa}$. From the equation (6), the evolution equation of the curvature and the torsion of the curve r are $\kappa_t = \frac{1}{\lambda\kappa^2} (-\kappa\tau_s + \kappa_s(\lambda^2 - 2\tau))$ and $\tau_t = \frac{1}{\lambda} \left(\left(\frac{\kappa_{ss}}{\kappa} + \tau(\lambda^2 - \tau) \right)_s + \frac{\kappa_s}{\kappa} \right)$. ■

Corollary 6 *If a moving space curve $r(s, t)$ generates a Dini surface, then with respect to the modified orthogonal frame the evolution equation of the curvature and the torsion of the curve r are $\kappa_t = \frac{1}{\lambda\kappa^2} (-\kappa\tau_s + \kappa_s(\lambda^2 - 2\tau))$ and $\tau_t = \frac{1}{\lambda} \left(\left(\frac{\kappa_{ss}}{\kappa} + \tau(\lambda^2 - \tau) \right)_s + \frac{\kappa_s}{\kappa} \right)$, respectively, see Figure 2.*

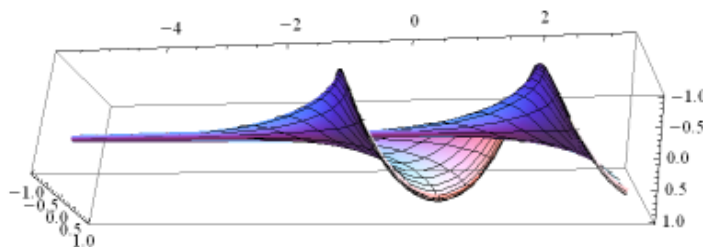


Figure 2: A Dini Surface.

Theorem 7 *Let $r(s, t)$ be a position vector of a moving space curve r which is a unit speed curve for all t parameters in Euclidean 3-space and the velocity vector of $r(s, t)$ be $r_t = T + N$, then with respect to the modified orthogonal frame the evolution equation of the curvature is $\kappa_t = \frac{\kappa_{ss} + \kappa_s}{\kappa^2}$ and the torsion is $\tau_t = 0$.*

Proof. If the velocity vector of the moving curve r is $r_t = T + N$, then $a = 1, b = 1$ and $c = 0$. So, $\tau = 0, \beta = 0$ and $\gamma = 0$ are obtained, then the time evolution of the modified orthogonal frame is found as follows:

$$\begin{bmatrix} T \\ N \end{bmatrix}_t \begin{bmatrix} 0 & \alpha \\ -\alpha & \kappa\kappa_t \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix}$$

such that $\alpha = a + b_s + \frac{\kappa_s}{\kappa} b$. From the equation (6), the evolution equation of the curvature and the torsion of the curve r are $\kappa_t = \frac{1}{\kappa} \left(\alpha_s + \frac{\kappa_s}{\kappa} \alpha \right)$ and $\tau_t = 0$. Here the curvature of the curve r is $\kappa_t = \frac{\kappa_{ss} + \kappa_s}{\kappa^2}$. ■

Corollary 8 *Let $r(s, t)$ be the position vector of a moving planar curve, then concerning the modified orthogonal frame, the evolution equation of the curvature is $\kappa_t = \frac{\kappa_{ss} + \kappa_s}{\kappa^2}$.*

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