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A Further Improvement Of The Ostrowski-Taussky Inequality For Real Matrices*

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Abstract

Let $A \in \mathbb{C}^{n\times n}$, $A = H + iK$, $H = \frac{1}{2}(A + A^*)$, $iK = \frac{1}{2}(A - A^*)$. It is well known that if H is positive definite, then

$$\det H + |\det K| \leq |\det A|.$$

We improve this inequality assuming that $A \in \mathbb{R}^{n\times n}$.

1 Introduction

If $A \in \mathbb{C}^{n\times n}$, then

$$A = H + iK, \quad H = \frac{A + A^*}{2}, \quad iK = \frac{A - A^*}{2}. \tag{1}$$

If $A \in \mathbb{R}^{n\times n}$, then

$$S = iK \tag{2}$$

is real, and

$$A = H + S, \quad H = \frac{A + A^T}{2}, \quad S = \frac{A - A^T}{2}. \tag{3}$$

These decompositions hold also for $n = 1$. Then A is a scalar a, (1) reads

$$a = h + ik, \quad h = \Re a, \quad k = \Im a,$$

and (3) reads

$$a = h + s, \quad h = a, \quad s = 0.$$

Let $M > O$ denote that $M \in \mathbb{C}^{n\times n}$ is positive definite.

Theorem 1 ([3, Theorem 7.8.24]) *Let $A \in \mathbb{C}^{n\times n}$ be as in (1) with $n \geq 2$. If $H > O$, then*

$$\det H + |\det K| \leq |\det A|. \tag{4}$$

Equality is attained for $n = 2$ if and only if there exists $c \in \mathbb{R}$ such that $K = cH$, and for $n \geq 3$ if and only if $K = O$ (i.e., $A > O$).

For $n = 1$, (4) is equivalent to $(h+|k|)^2 \leq h^2+k^2$, which (with $h > 0$) holds if and only if $k = 0$. Omitting $|\det K|$ from (4), we get the Ostrowski-Taussky inequality [3, Theorem 7.8.19]. So (4) is its improvement.

Corollary 1 *Let $A \in \mathbb{R}^{n\times n}$ be as in (3). If $H > O$, then*

$$\det H + \det S \leq \det A. \tag{5}$$

Equality is attained if and only if $n = 1$ or $S = O$ (i.e., $A > O$).

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Proof. If $n \geq 2$, then by (4) and (2),

$$\det H + |\det S| \leq |\det A|.$$

By Corollary 2 below, $\det S \geq 0$ and $\det A > 0$, verifying (5). If $n = 1$, then (5) holds trivially (and is actually an equation).

Studying equality for $n \geq 2$ remains. If $n = 2$, the equality condition given in Theorem 1 is $K = cH$, i.e., $S = icH$. If $c \neq 0$, this does not hold, because S is real but all nonzero entries of icH are pure imaginary. Thus equality is attained for $n \geq 2$ if and only if $K = O$ or, equivalently, $S = O$. ■

We improve Corollary 1.

2 Preliminaries

We let $\operatorname{spec} M$ denote the (multi)set of the eigenvalues (not necessarily distinct) of $M \in \mathbb{C}^{n\times n}$.

Lemma 1 *Let $A \in \mathbb{R}^{n\times n}$ be as in (3). If $H > O$, then*

$$\operatorname{spec} H^{-1}S = \{it_1, -it_1, \ldots, it_m, -it_m, 0, \ldots, 0\}, \tag{6}$$

where $0 \leq m \leq \frac{n}{2}$, $t_1, \ldots, t_m \in \mathbb{R}_+$. (If $m = 0$, then omit the $\pm it_k$s. If $m = n/2$, then omit the zeros.)

Proof. Let $T = H^{-\frac{1}{2}} S H^{-\frac{1}{2}}$. Since T is skew-symmetric, $\operatorname{spec} T$ is of the form (6). Since $H^{-1}S = H^{-\frac{1}{2}} T H^{\frac{1}{2}}$, it follows that $\operatorname{spec} H^{-1}S = \operatorname{spec} T$. ■

Corollary 2 *Under the assumptions of Lemma 1*

$$\det H^{-1}S = 0 \text{ for } n > 2m, \quad \det H^{-1}S = t_1^2 \cdots t_m^2 \text{ for } n = 2m,$$
$$\det(I + H^{-1}S) = (1 + t_1^2) \cdots (1 + t_m^2), \quad \det S \geq 0, \ \det A > 0.$$

Also the converse of Lemma 1 is true: if $\operatorname{spec} H^{-1}S$ is of type (6), then $H > O$. Because we do not need it for our purpose, we did not present the proof. The corresponding lemma (with converse) for complex matrices is well known [1, 4, 6, 7].

3 Improving Corollary 1

Hartfiel [2, Corollary] proved that if $A, B \in \mathbb{C}^{n\times n}$ and $A, B > O$, then

$$\det(A + B) \geq \det A + \det B + (2^n - 2)(\det AB)^{\frac{1}{2}}. \tag{7}$$

Our $H + S$ is partly like $A + B$ (H vs. A) but partly unlike (S vs. B). Can we find such an inequality $\det(H + S) \geq \ldots$ that is in some sense a reminiscent of (7)? The answer will appear to be positive.

Lemma 2 *If $t_1, \ldots, t_n \in \mathbb{R}$, then*

$$\prod_{k=1}^{n}(1 + t_k^2) \geq 1 + (2^n - 2)\prod_{k=1}^{n}|t_k| + \Big(\prod_{k=1}^{n}|t_k|\Big)^2 \tag{8}$$

with equality if and only if $n = 1$ or $t_1 = \cdots = t_n = 0$.

Proof. We proceed by induction. If $n = 1$, then (8) is trivially true (and is actually an equation). If (8) is true for n, then it is true for $n+1$, since

$$\begin{aligned}
\prod_{k=1}^{n+1}(1+t_k^2) &= (1+t_{n+1}^2)\prod_{k=1}^{n}(1+t_k^2) \\
&\geq (1+t_{n+1}^2)\Big[1+(2^n-2)\prod_{k=1}^{n}|t_k| + \Big(\prod_{k=1}^{n}|t_k|\Big)^2\Big] \\
&= (1+t_{n+1}^2)\Big[1+\Big(\prod_{k=1}^{n}|t_k|\Big)^2\Big] + (1+t_{n+1}^2)(2^n-2)\prod_{k=1}^{n}|t_k| \\
&\geq (1+t_{n+1}^2)\Big[1+\Big(\prod_{k=1}^{n}|t_k|\Big)^2\Big] + 2|t_{n+1}|(2^n-2)\prod_{k=1}^{n}|t_k| \\
&= 1+\Big(\prod_{k=1}^{n}|t_k|\Big)^2 + t_{n+1}^2 + t_{n+1}^2\Big(\prod_{k=1}^{n}|t_k|\Big)^2 + (2^{n+1}-4)\prod_{k=1}^{n+1}|t_k| \\
&= 1+\Big(\prod_{k=1}^{n}|t_k|\Big)^2 + t_{n+1}^2 + \Big(\prod_{k=1}^{n+1}|t_k|\Big)^2 + (2^{n+1}-4)\prod_{k=1}^{n+1}|t_k| \\
&\geq 1+2|t_{n+1}|\prod_{k=1}^{n}|t_k| + \Big(\prod_{k=1}^{n+1}|t_k|\Big)^2 + (2^{n+1}-4)\prod_{k=1}^{n+1}|t_k| \\
&= 1+2\prod_{k=1}^{n+1}|t_k| + \Big(\prod_{k=1}^{n+1}|t_k|\Big)^2 + (2^{n+1}-4)\prod_{k=1}^{n+1}|t_k| \\
&= 1+(2^{n+1}-2)\prod_{k=1}^{n+1}|t_k| + \Big(\prod_{k=1}^{n+1}|t_k|\Big)^2.
\end{aligned}$$

The first inequality follows from the induction hypothesis. The inequality $a^2+b^2 \geq 2|ab|$ with appropriate a and b verifies the second and third. Equality is attained in the first inequality if and only if $t_{n+1} = 0$ and (by the induction hypothesis concerning equality) $t_1 = \cdots = t_n = 0$. Clearly, it is then attained in the second and third, too. ■

Theorem 2 *Let $A \in \mathbb{R}^{n\times n}$, $n \geq 2$, be as in (3) with $H > O$, and let m be as in Lemma 1. Then*

$$\det A \geq \det H + \det S + (2^m - 2)(\det HS)^{\frac{1}{2}}. \tag{9}$$

Equality is attained if and only if $S = O$ (i.e., $A > O$). If S is invertible (equivalently, if $n = 2m$), then

$$\det A \geq \det H + \det S + (2^{\frac{n}{2}} - 2)(\det HS)^{\frac{1}{2}}.$$

Equality is attained if and only if $n = 2$.

Proof. If $n > 2m$, then $\det S = 0$ by Corollary 2, so (9) is nothing but (5) with $\det S = 0$, i.e., the Ostrowski-Taussky inequality.

If $n = 2m$, then, by Corollary 2 and Lemma 2,

$$\begin{aligned}
\det(I + H^{-1}S) = \prod_{k=1}^{m}(1+t_k^2) &\geq 1 + (2^m-2)\prod_{k=1}^{m}t_k + \Big(\prod_{k=1}^{m}t_k\Big)^2 \\
&= 1 + (2^m-2)(\det H^{-1}S)^{\frac{1}{2}} + \det(H^{-1}S).
\end{aligned}$$

Equality is attained if and only if $m = 1$ or $t_1 = \cdots = t_m = 0$. Consequently,

$$\begin{aligned}\det A &= \det(H+S) = \det H \det(I + H^{-1}S) \\ &\geq \det H\big[1 + (2^m - 2)(\det H^{-1}S)^{\frac{1}{2}} + \det(H^{-1}S)\big] \\ &= \det H + (2^m - 2)(\det HS)^{\frac{1}{2}} + \det S.\end{aligned}$$

If $t_1 = \cdots = t_m = 0$, then $S = O$, which is impossible, since S is invertible. Therefore the equality condition is $m = 1$, i.e., $n = 2$. ■

Inequality (7) is a corollary of the inequality [2, Theorem]

$$\det(A+B) \geq \Big(1 + \sum_{i=1}^{n-1} \frac{\det B_i}{\det A_i}\Big) \det A + \Big(1 + \sum_{i=1}^{n-1} \frac{\det A_i}{\det B_i}\Big) \det B + (2^n - 2n)(\det AB)^{\frac{1}{2}}. \tag{10}$$

Here $A, B \in \mathbb{C}^{n\times n}$ are positive definite, and A_i and B_i are the $i \times i$ principal submatrices in the upper left corner of A and B, respectively. We address to (10) (instead of (7)) and to its recent extensions (e.g., [5]) the question asked in the beginning of this section. These questions remain for further study.

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