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General Estimates For Coupled System Of Damped Hyperbolic Equations With Power External Forces*

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Abstract

In this article, we study the decay rate for system of coupled semi-linear wave equations with power external forces in $\mathbb{R}^n$, including damping term of memory type which is very meaningful. We use the weighted spaces to deal with unbounded domain. Owing to the Faedo-Galerkin method combined with the stable set, we prove the existence of global solution. With the help of some special estimates and generalized Poincaré's inequality, we obtain a non classical decay rate for the energy function to generalize a similar result in literature.

1 Introduction and Preliminaries

Some natural materials have viscoelastic structures. The structure of viscoelasticity manifests in different types. It is very important to study the differential and integro-differential equations with viscoelasticity in unbounded domains, which are models appearing in many applications: theory of viscoelasticity, thermal physics, dynamics of multi-phase media. At present, the qualitative properties of global solutions of systems with memory terms have been investigated.

We consider, for $x \in \mathbb{R}^n,\ t > 0$, the following system

$$\begin{cases} u_{tt} + \alpha u_t = \theta(x)\Delta\Big(u - \int_0^t \varpi_1(t-s)u(s)\,ds\Big) + h_1(u,v) \\ v_{tt} + \alpha v_t = \theta(x)\Delta\Big(v - \int_0^t \varpi_2(t-s)v(s)\,ds\Big) + h_2(u,v), \end{cases} \tag{1}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x), v(x,0) = v_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), \end{cases} \tag{2}$$

where $n \geq 3, \alpha > 0$, the functions $h_i(.,.) \in (\mathbb{R}^2, \mathbb{R}), i = 1,2$ are given by

$$h_1(y,z) = (q+1)\Big[d|y+z|^{(q-1)}(y+z) + e|y|^{(q-3)/2}y|z|^{(q+1)/2}\Big]$$

$$h_2(y,z) = (q+1)\Big[d|y+z|^{(q-1)}(y+z) + e|z|^{(q-3)/2}z|y|^{(q+1)/2}\Big],$$

with $d, e > 0, q > 3$. The function $\frac{1}{\theta(x)} \sim \vartheta(x) > 0$, for all $x \in \mathbb{R}^n$, is a density such that

$$\vartheta(x) \in L^\tau(\mathbb{R}^n) \quad \text{with} \quad \tau = \frac{2n}{2n - rn + 2r} \quad \text{for} \quad 2 \leq r \leq \frac{2n}{n-2}. \tag{3}$$

As in [17], here exists a function $\mathcal{G} \in C^1(\mathbb{R}^3, \mathbb{R})$ such that

$$uh_1(u,v) + vh_2(u,v) = (q+1)\mathcal{G}(u,v),\ \forall (u,v) \in \mathbb{R}^2, \tag{4}$$

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satisfies

$$(q+1)\mathcal{G}(u,v) = |u+v|^{q+1} + 2|uv|^{(q+1)/2}. \tag{5}$$

As in [4, 19], we introduce the function spaces $\mathcal{H}$ as the closure of $C_0^\infty(\mathbb{R}^n)$ as follows

$$\mathcal{H} = \{v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla v \in L^2(\mathbb{R}^n)^n\},$$

defined with the norm $\|v\|_{\mathcal{H}} = (v,v)_{\mathcal{H}}^{1/2}$ for the inner product

$$(v,w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx,$$

and $L^2_\vartheta(\mathbb{R}^n)$ with the norm $\|v\|_{L^2_\vartheta} = (v,v)_{L^2_\vartheta}^{1/2}$ for

$$(v,w)_{L^2_\vartheta} = \int_{\mathbb{R}^n} \vartheta v w \, dx.$$

For general $r \in [1,+\infty)$

$$\|v\|_{L^r_\vartheta} = \left(\int_{\mathbb{R}^n} \vartheta \, |v|^r \, dx \right)^{\frac{1}{r}},$$

is the norm of the weighted space $L^r_\vartheta(\mathbb{R}^n)$.

The main aim of this work is to consider important properties for growth of the relaxation function depending on a convex function, which make our contribution very interesting. We use a classical methods to solve a new model with a nontrivial result related to the existence of global solution in $\mathbb{R}^n$ and obtained an unusual decay rate for the energy function. The following references are related to our system for a single equation [7] and [8]. The paper [7] is one of the pioneers in the literature for the single equation, which is the source of inspiration of several researches, while the work [8] is a recent generalization of [7] by introducing less dissipative effects.

We review the related papers regarding the semi-linear wave system, from a qualitative and quantitative study. For a single wave equation, we beginning with the work treated in [13], for $(x,t) \in \Omega \times (0,\infty)$ where the goal was mainely on the system

$$u_{tt} + \mu u_t - \Delta u - \omega \Delta u_t = u \ln |u|, \tag{6}$$

with initial and boundary conditions

$$u(x,t) = 0, x \in \partial\Omega, u(x,0) = u_0(x), u_t(x,0) = u_1(x),$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. The author constructed, firstly, a local existence of weak solution by using the contraction mapping principle and of course showed the global existence, decay rate and infinite time blow up of the solution with conditions on initial energy.

Next, a nonexistence of global solutions for system of three semi-linear hyperbolic equations was introduced in [3]. A coupled system for semi-linear hyperbolic equations was investigated by many authors and a different results were obtained with the nonlinearities in the form $f_1 = |u|^{q-1}|v|^{q+1}u$, $f_2 = |v|^{q-1}|u|^{q+1}v$. (Please, see [2, 15, 24, 30])

In the non-bounded domain $\mathbb{R}^n$, we refer to the article recently published by T. Miyasita and Kh. Zennir in [18], where the considered problem is as follows

$$u_{tt} + a u_t - \phi(x)\Delta \left(u + \omega u_t - \int_0^t g(t-s)u(s)\, ds \right) = u|u|^{q-1}, \tag{7}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x) \\ u_t(x,0) = u_1(x). \end{cases} \tag{8}$$

The authors established the existence of unique local solution and they continued to extend it to be global in time. The rate of the decay for solution was the main result by considering the relaxation function which is strictly convex. For more results related to decay rate of solution of this type of problems, please see [14, 25, 26, 27, 29, 31].

Regarding the study of the coupled system of two nonlinear wave equations, it is worth recalling some of the work recently published. Baowei et *al.* developed in [11], a coupled system for viscoelastic wave equations with nonlinear sources in bounded domain ($(x,t) \in \Omega \times (0,\infty)$) with smooth boundary as follows

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + u_t = f_1(u,v) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + v_t = f_2(u,v). \end{cases} \tag{9}$$

Here, the authors are concerned with a system in $\mathbb{R}^n (n = 1, 2, 3)$. Under appropriate hypotheses, the authors showed a very general decay estimate by multiplied techniques to extend some existing results for a single equation to the case of a coupled system.

It is worth noting here that there are several studies in this field and we particularly refer to the generalization that Shun et *al.* made in studying a complicate non-linear case with degenerate damping term in [22]. The IBVP for a system of nonlinear wave equations in viscoelasticity in a bounded domain was considered in the system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + (|u|^k + |v|^q)|u_t|^{m-1}u_t = f_1(u,v) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + (|v|^\kappa + |u|^\rho)|v_t|^{r-1}v_t = f_2(u,v) \\ u(x,t) = v(x,t) = 0, x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), v(x,0) = v_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), \end{cases} \tag{10}$$

where Ω is bounded domain with a smooth boundary. Given some conditions on the memory terms, nonlinear source terms and degenerate damping, they got a new decay estimate of associated energy functional with certain initial conditions.

The lack of existence (Blow up) is considered one of the most important qualitative studies that must be spoken of, given its importance in terms of application in various applied sciences. Regarding the global non-existence for solutions of more degenerate case for coupled system of damped wave equations with different damping, we mention the articles [5, 6, 9, 20, 21, 23, 28]. The next Sobolev embedding and generalized Poincaré inequalities will be very useful.

Lemma 1 ([18]) *Let ϑ satisfy (3). For a positive constants $C_\tau > 0$ and $C_P > 0$ depending only on ϑ and n, we have*

$$\|v\|_{\frac{2n}{n-2}} \leq C_\tau \|v\|_{\mathcal{H}} \quad \text{and} \quad \|v\|_{L^2_\vartheta} \leq C_P \|v\|_{\mathcal{H}}$$

for $v \in \mathcal{H}$.

Lemma 2 ([12]) *Let ϑ satisfy (3). Then the estimates*

$$\|v\|_{L^r_\vartheta} \leq C_r \|v\|_{\mathcal{H}} \quad \text{and} \quad C_r = C_\tau \|\vartheta\|_\tau^{\frac{1}{r}}$$

hold for $v \in \mathcal{H}$. Here $\tau = 2n/(2n - rn + 2r)$ for $1 \leq r \leq 2n/(n-2)$.

We assume that the kernel functions $\varpi_1, \varpi_2 \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfy

$$1 - \overline{\varpi_1} = l > 0 \quad \text{for} \quad \overline{\varpi_1} = \int_0^{+\infty} \varpi_1(s)\,ds, \;\; \varpi_1'(t) \leq 0 \tag{11}$$

and

$$1 - \overline{\varpi_2} = m > 0 \quad \text{for} \quad \overline{\varpi_2} = \int_0^{+\infty} \varpi_2(s)\,ds, \;\; \varpi_2'(t) \leq 0. \tag{12}$$

Noting by

$$\varpi(t)=\max_{t\geq 0}\Big\{\varpi_1(t),\varpi_2(t)\Big\}, \tag{13}$$

and

$$\varpi_0(t)=\min_{t\geq 0}\Big\{\int_0^t\varpi_1(s)ds,\int_0^t\varpi_2(s)ds\Big\}. \tag{14}$$

We assume that there is a function $\chi\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ such that

$$\varpi_i'(t)+\chi(\varpi_i(t))\leq 0,\quad \chi(0)=0,\quad \chi'(0)>0,\ i=1,2, \tag{15}$$

for any $\xi\geq 0$.

Hölder and Young inequalities give

$$\|uv\|^{(q+1)/2}_{L^{(q+1)/2}_{\vartheta}}\leq\Big(\|u\|^2_{L^{(q+1)}_{\vartheta}}+\|v\|^2_{L^{(q+1)}_{\vartheta}}\Big)^{(q+1)/2}\leq\left(l\|u\|^2_{\mathcal{H}}+m\|v\|^2_{\mathcal{H}}\right)^{(q+1)/2}.$$

Thanks to Minkowski's inequality, we have

$$\|u+v\|^{(q+1)}_{L^{(q+1)}_{\vartheta}}\leq c\Big(\|u\|^2_{L^{(q+1)}_{\vartheta}}+\|v\|^2_{L^{(q+1)}_{\vartheta}}\Big)^{(q+1)/2}\leq c\left(\|u\|^2_{\mathcal{H}}+\|v\|^2_{\mathcal{H}}\right)^{(q+1)/2}.$$

Then, there exist $\eta>0$ such that

$$\|u+v\|^{(q+1)}_{L^{(q+1)}_{\vartheta}}+2\,\|uv\|^{(q+1)/2}_{L^{(q+1)/2}_{\vartheta}}\leq\eta\left(l\|u\|^2_{\mathcal{H}}+m\|v\|^2_{\mathcal{H}}\right)^{(q+1)/2}. \tag{16}$$

We need to define positive constants λ_0 and $\mathcal{E}_0$ by

$$\lambda_0\equiv\eta^{-1/(q-1)}\quad\text{and}\quad\mathcal{E}_0=\Big(\frac{1}{2}-\frac{1}{q+1}\Big)\eta^{-2/(q-1)}. \tag{17}$$

The maine aim of the present paper is to obtain a new decay estimate of solution by the convexity property of the function χ given in Theorem 3.

We denote an eigenpair $\{(\lambda_i,e_i)\}_{i\in\mathbb{N}}\subset\mathbb{R}\times\mathcal{H}$ of

$$-\theta(x)\Delta e_i=\lambda_i e_i\quad x\in\mathbb{R}^n,$$

for any $i\in\mathbb{N}$, $\frac{1}{\theta(x)}\sim\vartheta(x)$. Then

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_i\leq\cdots\uparrow+\infty,$$

holds and $\{e_i\}$ is a complete orthonormal system in $\mathcal{H}$.

Definition 1 *The pair (u,v) is said to be a weak solution to (1)-(2) on $[0,T]$ if it satisfies for $x\in\mathbb{R}^n$,*

$$\begin{cases}\int_{\mathbb{R}^n}\vartheta(x)(u_{tt}++\alpha u_t)\varphi dx+\int_{\mathbb{R}^n}\nabla u\nabla\varphi dx-\int_0^t\varpi_1(t-s)\nabla u(s)\,ds\nabla\varphi dx=\int_{\mathbb{R}^n}\vartheta(x)h_1(u,v)\varphi dx,\\ \int_{\mathbb{R}^n}\vartheta(x)(v_{tt}+\alpha v_t)\psi dx+\int_{\mathbb{R}^n}\nabla v\nabla\psi dx-\int_0^t\varpi_2(t-s)\nabla v(s)\,ds\nabla\psi dx,\\ \int_{\mathbb{R}^n}\vartheta(x)(v_{tt}+\alpha v_t)\psi dx+\int_{\mathbb{R}^n}\nabla v\nabla\psi dx-\int_0^t\varpi_2(t-s)\nabla v(s)\,ds\nabla\psi dx=\int_{\mathbb{R}^n}\vartheta(x)h_2(u,v)\psi dx,\end{cases} \tag{18}$$

for all test functions $\varphi,\psi\in\mathcal{H}$ for almost all $t\in[0,T]$.

2 Statement of Main Results

The next Theorem is concerned with the local solution (in time $[0,T]$).

Theorem 1 (Local existence) *Assume that*

$$1 < q \leq \frac{n+2}{n-2} \quad \textit{and that} \quad n \geq 3. \tag{19}$$

Let $(u_0, v_0) \in \mathcal{H}^3$ *and* $(u_1, v_1) \in L^2_\vartheta(\mathbb{R}^n) \times L^2_\vartheta(\mathbb{R}^n)$. *Under the assumptions (3)–(5) and (11)–(15), we have (1)–(2) admits a unique local solution* (u, v) *such that*

$$(u,v) \in \mathcal{X}_T^2,\ \ \mathcal{X}_T \equiv C\big([0,T];\mathcal{H}\big) \cap C^1\big([0,T]; L^2_\vartheta(\mathbb{R}^n)\big),$$

for sufficiently small $T > 0$.

We prove the existence of global solution in time. Let us introduce the potential energy $J : \mathcal{H}^3 \to \mathbb{R}$ defined by

$$\mathcal{J}(u,v) = \left(1 - \int_0^t \varpi_1(s)\,ds\right)\|u\|^2_{\mathcal{H}} + (\varpi_1 \circ u) + \left(1 - \int_0^t \varpi_2(s)\,ds\right)\|v\|^2_{\mathcal{H}} + (\varpi_2 \circ v)\,. \tag{20}$$

where

$$(\varpi_j \circ w)(t) = \int_0^t \varpi_j(t-s)\,\|w(t) - w(s)\|^2_{\mathcal{H}}\,ds,$$

for any $w \in L^2(\mathbb{R}^n), j = 1, 2$. The modified energy function is defined by

$$\mathcal{E}(t) = \frac{1}{2}\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta}\Big) + \frac{1}{2}\mathcal{J}(u,v) - \int_{\mathbb{R}^n} \vartheta(x)\mathcal{G}(u,v)dx, \tag{21}$$

Theorem 2 (Global existence) *Let (3)–(5) and (11)–(15) hold. Under (19) and for sufficiently small* $(u_0, u_1), (v_0, v_1) \in \mathcal{H} \times L^2_\vartheta(\mathbb{R}^n)$, *problem (1)–(2) admits a unique global solution* (u, v) *such that*

$$(u,v) \in \mathcal{X}^2,\ \ \mathcal{X} \equiv C\big([0,+\infty);\mathcal{H}\big) \cap C^1\big([0,+\infty); L^2_\vartheta(\mathbb{R}^n)\big). \tag{22}$$

The decay rate for solution is given in the next Theorem.

Theorem 3 (Decay of solution) *Let (3)–(5) and (11)–(15) hold. Under condition (19) and*

$$\gamma = \eta\Big(\frac{2(q+1)}{q-1}\mathcal{E}(0)\Big)^{(q-1)/2} < 1, \tag{23}$$

there exists $t_0 > 0$ *depending only on* ϖ_1, ϖ_2, λ_1 *and* $\chi'(0)$ *such that*

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0)\exp\left(-\int_{t_0}^t \frac{\varpi(s)}{1 - \varpi_0(t)}\right), \tag{24}$$

holds for all $t \geq t_0$.

Next Lemma will be very useful and play an important role.

Lemma 3 *For* $(u, v) \in \mathcal{X}_T^2$, *the functional* $\mathcal{E}(t)$ *associated with problem (1)–(2) is decreasing.*

Proof. For $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
\mathcal{E}(t_2) - \mathcal{E}(t_1) &= \int_{t_1}^{t_2} \frac{d}{dt}\mathcal{E}(t)\,dt \\
&= -\frac{1}{2}\int_{t_1}^{t_2}\Big(\varpi_1(t)\,\|u\|^2_{\mathcal{H}} - (\varpi_1' \circ u)\Big)\,dt - \frac{1}{2}\int_{t_1}^{t_2}\Big(\varpi_2(t)\,\|v\|^2_{\mathcal{H}} - (\varpi_2' \circ v)\Big)\,dt \\
&\quad -\alpha\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta}\Big) \\
&\leq 0,
\end{aligned}$$

owing to (11)–(15). ■

3 Proofs of Main Results

We sketch here the outline of the proof for local solution by a standard procedure (See [10, 14, 31]).

Proof of Theorem 1. Let $(u_0, u_1), (v_0, v_1) \in \mathcal{H} \times L^2_\vartheta(\mathbb{R}^n)$. For any $(u, v) \in \mathcal{X}^2_T$, we can obtain a weak solution of the related system

$$\begin{cases} \vartheta(x)(z_{tt} + \alpha z_t) - \Delta z = -\int_0^t \varpi_1(t-s)\Delta u(s)\, ds + \vartheta(x)h_1(u, v) \\ \vartheta(x)(y_{tt} + \alpha y_t) - \Delta y = -\int_0^t \varpi_2(t-s)\Delta v(s)\, ds + \vartheta(x)h_2(u, v) \\ z(x, 0) = u_0(x), y(x, 0) = v_0(x) \\ z_t(x, 0) = u_1(x), y_t(x, 0) = v_1(x). \end{cases} \tag{25}$$

We reduce problem (25) to a related Cauchy problem for system of ODE and then, by the Faedo-Galerkin approximation, we find weak solution of (25). We then find a solution map $\top : (u, v) \mapsto (z, y)$ from $\mathcal{X}^2_T$ to $\mathcal{X}^2_T$. We are now ready to show that $\top$ is a contraction mapping in an appropriate subset of $\mathcal{X}^2_T$ for a small $T > 0$. Hence $\top$ has a fixed point $\top(u, v) = (u, v)$, which gives a unique solution in $\mathcal{X}^2_T$. ■

We will show the global solution. By using conditions on functions ϖ_1, ϖ_2, we have

$$\begin{aligned} \mathcal{E}(t) &\geq \frac{1}{2}\mathcal{J}(u, v) - \int_{\mathbb{R}^n} \vartheta(x)\mathcal{G}(u, v)dx \\ &\geq \frac{1}{2}\mathcal{J}(u, v) - \frac{1}{q+1}\|u + v\|^{(q+1)}_{L^{(q+1)}_\vartheta} - \frac{2}{q+1}\|uv\|^{(q+1)/2}_{L^{(q+1)/2}_\vartheta} \\ &\geq \frac{1}{2}\mathcal{J}(u, v) - \frac{\eta}{q+1}\left[l\,\|u\|^2_{\mathcal{H}} + m\,\|v\|^2_{\mathcal{H}}\right]^{(q+1)/2} \\ &\geq \frac{1}{2}\mathcal{J}(u, v) - \frac{\eta}{q+1}\Big(\mathcal{J}(u, v)\Big)^{(q+1)/2} \\ &= G(\varsigma), \end{aligned} \tag{26}$$

here $\varsigma^2 = \mathcal{J}(u, v)$, for $t \in [0, T)$, where

$$G(\xi) = \frac{1}{2}\xi^2 - \frac{\eta}{q+1}\xi^{(q+1)}.$$

Noting that $\mathcal{E}_0 = G(\lambda_0)$, given in (17). Then

$$\begin{cases} G(\xi) \geq 0 & \text{in } \ \xi \in [0, \lambda_0], \\ G(\xi) < 0 & \text{in } \ \xi > \lambda_0. \end{cases}$$

Moreover, $\lim\limits_{\xi \to +\infty} G(\xi) \to -\infty$. Then, we have the following Lemma.

Lemma 4 *Let* $0 \leq \mathcal{E}(0) < \mathcal{E}_0$.

(i) If $\|u_0\|^2_{\mathcal{H}} + \|v_0\|^2_{\mathcal{H}} < \lambda_0^2$, *then local solution of (1)–(2) satisfies*

$$\mathcal{J}(u, v) < \lambda_0^2, \ \forall t \in [0, T).$$

(ii) If $\|u_0\|^2_{\mathcal{H}} + \|v_0\|^2_{\mathcal{H}} > \lambda_0^2$, *then local solution of (1)–(2) satisfies*

$$\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}} > \lambda_1^2, \ \forall t \in [0, T), \lambda_1 > \lambda_0.$$

Proof. Since $0 \leq \mathcal{E}(0) < \mathcal{E}_0 = G(\lambda_0)$, there exist ξ_1 and ξ_2 such that $G(\xi_1) = G(\xi_2) = \mathcal{E}(0)$ with $0 < \xi_1 < \lambda_0 < \xi_2$.

The case (i) By (26), we have

$$G(\mathcal{J}(u_0,v_0)) \leq \mathcal{E}(0) = G(\xi_1),$$

which implies that $\mathcal{J}(u_0,v_0) \leq \xi_1^2$. Then we claim that $\mathcal{J}(u,v) \leq \xi_1^2,\ \forall t \in [0,T)$. Then, there exists $t_0 \in (0,T)$ such that

$$\xi_1^2 < \mathcal{J}(u(t_0),v(t_0)) < \xi_2^2.$$

Then

$$G(\mathcal{J}(u(t_0),v(t_0))) > \mathcal{E}(0) \geq \mathcal{E}(t_0),$$

by Lemma 3, which contradicts (26). Hence we have

$$\mathcal{J}(u,v) \leq \xi_1^2 < \lambda_0^2,\ \forall t \in [0,T).$$

The case (ii) We could prove that $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 \geq \xi_2^2$ and that $\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 \geq \xi_2^2 > \lambda_0^2$ in the same way as (i).

■

Proof of Theorem 2.

$(u_0,u_1),(v_0,v_1) \in \mathcal{H} \times L_\vartheta^2(\mathbb{R}^n)$ satisfy both $0 \leq \mathcal{E}(0) < \mathcal{E}_0$ and $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 < \lambda_0^2$. By Lemma 3 and Lemma 4, we have

$$\begin{aligned}
&\frac{1}{2}\Big(\|u_t\|_{L_\vartheta^2}^2 + \|v_t\|_{L_\vartheta^2}^2\Big) + l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 \\
\leq\ & \frac{1}{2}\Big(\|u_t\|_{L_\vartheta^2}^2 + \|v_t\|_{L_\vartheta^2}^2\Big) + \left(1 - \int_0^t \varpi_1(s)\,ds\right)\|u\|_{\mathcal{H}}^2 + (\varpi_1 \circ u) \\
& + \left(1 - \int_0^t \varpi_2(s)\,ds\right)\|u\|_{\mathcal{H}}^2 + (\varpi_2 \circ v) \\
\leq\ & 2\mathcal{E}(t) + \frac{2\eta}{q+1}\Big[l\,\|u\|_{\mathcal{H}}^2 + m\,\|u\|_{\mathcal{H}}^2\Big]^{(q+1)/2} \\
\leq\ & 2\mathcal{E}(0) + \frac{2\eta}{q+1}\Big(\mathcal{J}(u,v)\Big)^{(q+1)/2} \\
\leq\ & 2\mathcal{E}_0 + \frac{2\eta}{q+1}\lambda_0^{q+1} \\
=\ & \eta^{-2/(q-1)}.
\end{aligned} \tag{27}$$

This completes the proof. ■

Let

$$\begin{aligned}
\Lambda(u,v) =\ & \frac{1}{2}\left(1 - \int_0^t \varpi_1(s)\,ds\right)\|u\|_{\mathcal{H}}^2 + \frac{1}{2}(\varpi_1 \circ u) + \frac{1}{2}\left(1 - \int_0^t \varpi_2(s)\,ds\right)\|v\|_{\mathcal{H}}^2 + \frac{1}{2}(\varpi_2 \circ v) \\
& - \int_{\mathbb{R}^n} \vartheta(x)\mathcal{G}(u,v)dx
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
\Pi(u,v) =\ & \left(1 - \int_0^t \varpi_1(s)\,ds\right)\|u\|_{\mathcal{H}}^2 + (\varpi_1 \circ u) + \left(1 - \int_0^t \varpi_2(s)\,ds\right)\|v\|_{\mathcal{H}}^2 + (\varpi_2 \circ v) \\
& -(q+1)\int_{\mathbb{R}^n} \vartheta(x)\mathcal{G}(u,v)dx.
\end{aligned} \tag{29}$$

Lemma 5 *Let (u,v) be the solution of problem (1)–(2). If*

$$\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 - (q+1)\int_{\mathbb{R}^n} \vartheta(x)\mathcal{G}(u_0,v_0)dx > 0. \tag{30}$$

Then under condition (23), the functional $\Pi(u,v) > 0,\ \forall t > 0$.

Proof. By (30) and continuity, there exists a time $t_1 > 0$ such that

$$\Pi(u,v) \geq 0, \forall t < t_1.$$

Let

$$Y = \{(u,v) \mid \Pi(u(t_0),v(t_0)) = 0,\ \Pi(u,v) > 0, \forall t \in [0,t_0)\}.$$

Then, by (28), (29), we have for all $(u,v) \in Y$,

$$\begin{aligned}
\Lambda(u,v) &= \frac{q-1}{2(q+1)}\Big[\Big(1-\int_0^t \varpi_1(s)\,ds\Big)\|u\|_{\mathcal{H}}^2 + \Big(1-\int_0^t \varpi_2(s)\,ds\Big)\|v\|_{\mathcal{H}}^2\Big] \\
&\quad + \frac{q-1}{2(q+1)}\Big[(\varpi_1 \circ u) + (\varpi_2 \circ v)\Big] + \frac{1}{q+1}\Pi(u,v) \\
&\geq \frac{q-1}{2(q+1)}\Big[l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 + (\varpi_1 \circ u) + (\varpi_2 \circ v)\Big].
\end{aligned}$$

Owing to (21), it follows for $(u,v) \in Y$

$$l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 \leq \frac{2(q+1)}{q-1}\Lambda(u,v) \leq \frac{2(q+1)}{q-1}\mathcal{E}(t) \leq \frac{2(q+1)}{q-1}\mathcal{E}(0). \tag{31}$$

By (16), (23) we have

$$\begin{aligned}
(q+1)\int_{\mathbb{R}^n} \mathcal{G}(u(t_0),v(t_0)) &\leq \eta\left(l\|u(t_0)\|_{\mathcal{H}}^2 + m\|v(t_0)\|_{\mathcal{H}}^2\right)^{(q+1)/2} \\
&\leq \eta\Big(\frac{2(q+1)}{q-1}E(0)\Big)^{(q-1)/2}(l\,\|u(t_0)\|_{\mathcal{H}}^2 + m\,\|v(t_0)\|_{\mathcal{H}}^2) \\
&\leq \gamma(l\,\|u(t_0)\|_{\mathcal{H}}^2 + m\,\|v(t_0)\|_{\mathcal{H}}^2) \\
&< \Big(1-\int_0^{t_0}\varpi_1(s)ds\Big)\|u(t_0)\|_{\mathcal{H}}^2 + \Big(1-\int_0^{t_0}\varpi_2(s)ds\Big)\|v(t_0)\|_{\mathcal{H}}^2 \\
&< \Big(1-\int_0^{t_0}\varpi_1(s)ds\Big)\|u(t_0)\|_{\mathcal{H}}^2 + \Big(1-\int_0^{t_0}\varpi_2(s)ds\Big)\|v(t_0)\|_{\mathcal{H}}^2 \\
&\quad +(\varpi_1 \circ u) + (\varpi_2 \circ v),
\end{aligned}$$

hence $\Pi(u(t_0),v(t_0)) > 0$ on Y, which contradicts the definition of Y since $\Pi(u(t_0),v(t_0)) = 0$. Thus $\Pi(u,v) > 0,\ \forall t > 0$. ∎

We are now ready to show the decay estimate.

Proof of Theorem *3*. By (16) and (31), we have for $t \geq 0$

$$0 < l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 \leq \frac{2(q+1)}{q-1}\mathcal{E}(t).$$

Let

$$\mathcal{I}(t) = \frac{\varpi(t)}{1-\varpi_0(t)}, \tag{32}$$

where ϖ and ϖ_0 defined in (13) and (14). Noting that $\lim_{t\to+\infty}\varpi(t) = 0$ by (11)-(14), we have

$$\lim_{t\to+\infty}\mathcal{I}(t) = 0,\ \ \mathcal{I}(t) > 0,\ \forall t \geq 0.$$

Then we take $t_0 > 0$ such that

$$0 < \frac{1}{2}\mathcal{I}(t) < \chi'(0),$$

with (15) for all $t > t_0$. Due to (21), we have

$$\begin{aligned}\mathcal{E}(t) &\leq \frac{1}{2}\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta}\Big) + \frac{1}{2}[(\varpi_1 \circ u) + (\varpi_2 \circ v)] \\ &+ \frac{1}{2}\left(1 - \int_0^t \varpi_1(s)\,ds\right)\|u\|^2_{\mathcal{H}} + \frac{1}{2}\left(1 - \int_0^t \varpi_2(s)\,ds\right)\|v\|^2_{\mathcal{H}} \\ &\leq \frac{1}{2}\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta}\Big) + \frac{1}{2}[(\varpi_1 \circ u) + (\varpi_2 \circ v)] \\ &+ \frac{1}{2}(1 - \varpi_0(t))[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}}].\end{aligned}$$

Then by definition of $\mathcal{I}(t)$, we have

$$\mathcal{I}(t)\mathcal{E}(t) \leq \frac{1}{2}\mathcal{I}(t)\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta}\Big) + \frac{1}{2}\varpi(t)[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}}] + \frac{1}{2}\mathcal{I}(t)[(\varpi_1 \circ u) + (\varpi_2 \circ v)],$$

and Lemma 3, we have for all $t_1, t_2 \geq 0$,

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) \leq -\frac{1}{2}\int_{t_1}^{t_2}\Big(\varpi(t)[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}}]\Big)\,dt + \frac{1}{2}\int_{t_1}^{t_2}((\varpi_1' \circ u) + (\varpi_2' \circ v))\,dt - \alpha\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta}\Big),$$

then,

$$\mathcal{E}'(t) \leq -\frac{1}{2}\varpi(t)[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}}] + \frac{1}{2}[(\varpi_1' \circ u) + (\varpi_2' \circ v)] - \alpha\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta}\Big).$$

Finally, $\forall t \geq t_0$, we have

$$\begin{aligned}\mathcal{E}'(t) + \mathcal{I}(t)\mathcal{E}(t) &\leq \Big(\frac{1}{2}\mathcal{I}(t) - \alpha\Big)\Big(\|u_t\|^2_{L^2_\vartheta} + \|v_t\|^2_{L^2_\vartheta}\Big) + \frac{1}{2}[(\varpi_1' \circ u) + (\varpi_2' \circ v)] \\ &+ \frac{1}{2}\mathcal{I}(t)((\varpi_1 \circ u) + (\varpi_2 \circ v)),\end{aligned}$$

and we can choose $t_0 > 0$ large enough such that

$$\frac{1}{2}\mathcal{I}(t) < \alpha,$$

then

$$\begin{aligned}\mathcal{E}'(t) + \mathcal{I}(t)\mathcal{E}(t) &\leq \frac{1}{2}\int_0^t \{\varpi_1'(t-\tau) + \mathcal{I}(t)\varpi_2(t-\tau)\}\,\|u(t) - u(\tau)\|^2_{\mathcal{H}}\,d\tau \\ &+ \frac{1}{2}\int_0^t \{\varpi_2'(t-\tau) + \mathcal{I}(t)\varpi_2(t-\tau)\}\,\|v(t) - v(\tau)\|^2_{\mathcal{H}}\,d\tau \\ &\leq \frac{1}{2}\int_0^t \{\varpi_1'(\tau) + \mathcal{I}(t)\varpi_1(\tau)\}\,\|u(t) - u(t-\tau)\|^2_{\mathcal{H}}\,d\tau \\ &+ \frac{1}{2}\int_0^t \{\varpi_2'(\tau) + \mathcal{I}(t)\varpi_2(\tau)\}\,\|v(t) - v(t-\tau)\|^2_{\mathcal{H}}\,d\tau \\ &\leq \frac{1}{2}\int_0^t \Big\{-\chi\Big(\varpi_1(\tau)\Big) + \chi'(0)\varpi_1(\tau)\Big\}\,\|u(t) - u(t-\tau)\|^2_{\mathcal{H}}\,d\tau \\ &+ \frac{1}{2}\int_0^t \Big\{-\chi\Big(\varpi_2(\tau)\Big) + \chi'(0)\varpi_2(\tau)\Big\}\,\|v(t) - v(t-\tau)\|^2_{\mathcal{H}}\,d\tau \\ &\leq 0.\end{aligned}$$

By the convexity of χ and (15), we have

$$\chi(\xi) \geq \chi(0) + \chi'(0)\xi = \chi'(0)\xi.$$

Then

$$\mathcal{E}(t) \leq \mathcal{E}(t_0) \exp \left(- \int_{t_0}^{t} \mathcal{I}(s) ds \right),$$

which completes the proof. ■

Conclusion

Our novelty lies mainly in the study of the effect of terms to develop the quality of growth of the unique global solution. This is based on the following:

1. The use of weighted spaces constructed by the function $\vartheta(x)$, is to compensate the role of Poincare's inequality which considered as a key of the proofs.

2. We have found that the solution decays in general way depends on a convex function χ, which represents the development of relaxation function.

3. The main contribution is the rate of obtained solution, in which it is expressed with the functional (32). This rate was developed firstly in [18].

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