Applied Mathematics E-Notes, 22(2022), 299-309 © ISSN 1607-2510
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Modified Oscillation Results For Second-Order Nonlinear Differential Equations With Sublinear Neutral Terms*

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Received 25 January 2021

Abstract

The authors present some new criteria for the oscillation of all solutions to a class of second-order differential equations with sublinear neutral terms. The results established are new and extend those reported in the literature. Examples are included to demonstrate the importance and novelty of the presented results.

1 Introduction

In this article, we deal with the oscillatory properties of second-order differential equations with sublinear neutral terms of the form

$$(b(t)(w'(t))^{\delta})' + f(t)u^{\beta}(\theta(t)) = 0, \tag{1}$$

where $t \geq t_0 > 0$, $w(t) = u(t) + \sum_{i=1}^{m} g_i u^{\alpha_i}(t-\eta_i)$, δ and β are the ratios of odd positive integers. Throughout this paper, we assume that:

(B_1) $\theta \in C([t_0,\infty),\mathbb{R})$, $\theta(t) \leq t$ with $\lim_{t\to\infty}\theta(t) = \infty$;

(B_2) g_i and η_i are positive constants for $i = 1, 2, ..., m$;

(B_3) $b \in C([t_0,\infty),\mathbb{R}^+)$, $f \in C([t_0,\infty),\mathbb{R})$, $f(t) \geq 0$ for all $t \geq t_0 > 0$ and $f(t)$ is not identically zero in any interval $[d,\infty)$;

(B_4) $\lim_{t\to\infty} B(t) = \infty$, where $B(t) = \int_{t_0}^{t} b^{-1/\delta}(s)ds$;

(B_5) α_i are the quotients of odd positive integers with $0 < \alpha_i < 1$ for $i = 1, 2, ..., m$, and

$$\sum_{i=1}^{m}\left(\alpha_i + (1-\alpha_i)g_i^{\frac{1}{1-\alpha_i}}\right) < 1.$$

By a *solution* of equation (1), we mean a function $u \in C([T_u,\infty),\mathbb{R})$ for some $T_u \geq t_0$ such that $b(w')^{\delta} \in C^1([T_u,\infty),\mathbb{R})$ and u satisfies equation (1) on $[T_u,\infty)$. We consider only those solutions u of equation (1) which satisfy $\sup\{|u(t)| : t \geq T\} > 0$ for any $T \geq T_u$, and assume that equation (1) possesses such solutions. A solution of (1) is said to be *oscillatory* if it has infinitely many zeros on $[T_u,\infty)$ and otherwise, it is said to be *nonoscillatory*. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Neutral differential equations are differential equations in which the highest-order derivative of the unknown function is evaluated both at the present state t and at one or more past or future states. Besides

*Mathematics Subject Classifications: 34C10, 34K11, 34K40.

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their theoretical interest, such equations have numerous applications in natural sciences and technology; for example, see the monographs [13, 14]. Therefore, there has been great interest in obtaining conditions for the oscillation and other asymptotic properties of such equations. So in the past several years, many oscillatory results have been established for second-order differential equations of neutral type; for example, see [1, 4, 5, 8, 10, 16, 17, 18, 21, 24].

In recent years the authors studied the oscillatory behavior of the following equation

$$(b(t)(w'(t))^{\delta})' + f(t)u^{\beta}(\theta(t)) = 0, \quad t \geq t_0, \tag{2}$$

where $w(t) = u(t) + \sum_{i=1}^{m} g_i(t)u^{\alpha_i}(\eta(t))$ in [3, 6, 20, 25] for $i = 1, 2, ..., m$, and in [2, 7, 9, 11, 12, 19, 22, 23] in the case $m = 1$. In all these results it is required implicitly or explicitly that $\lim_{t\to\infty} g_i(t) = 0$ for $i = 1, 2, ..., m$, and thus the results obtained in these papers are not applicable when $g_i(t)$ for $i = 1, 2, ..., m$ is a constant. This observation motivated us to find new criteria for the oscillation of equation (1) where we have constants $g_i \in (0, 1)$ for $i = 1, 2, ..., m$ instead of functions $g_i(t) \to 0$ as $t \to \infty$ for $i = 1, 2, ..., m$. Thus, the results obtained in this paper are new and applicable to new classes of differential equations with sublinear neutral terms. Examples are provided to show the importance and novelty of our main results.

2 Main Results

In this section, we obtain sufficient conditions for the oscillation of all solutions of (1). Without loss of generality, we deal only with positive solutions of (1); since if $u(t)$ is a solution of (1), then $-u(t)$ is also a solution. We start with the following lemmas.

Lemma 1 (See [3, Lemma 2.1]) *If a and b are positive and $0 < \alpha \leq 1$, then*

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1 - \alpha)b, \tag{3}$$

where equality holds if and only if $a = b$.

Lemma 2 (See [13, Lemma 1.5.1]) *Let $h, g \in C([t_0, \infty), \mathbb{R})$ and $h(t) = g(t)+pg(t-c)$, $t \geq t_0+\max\{0, c\}$, where $p \neq 1$ and c are constants. Assume that there exists a constant $l \in \mathbb{R}$ such that $\lim_{t\to\infty} h(t) = l$.*

(S_1) If $\liminf_{t\to\infty} g(t) = g_* \in \mathbb{R}$, then $g_* = \frac{l}{(1+p)}$;

(S_2) If $\limsup_{t\to\infty} g(t) = g^* \in \mathbb{R}$, then $g^* = \frac{l}{(1+p)}$.

Lemma 3 *Let $u(t)$ be a positive solution of (1). If*

$$\int_{t_0}^{\infty} \left(\frac{1}{b(t)}\int_{t}^{\infty} f(s)ds\right)^{1/\delta} dt = \infty, \tag{4}$$

then the corresponding function w satisfies

(i) $w(t) > 0$, $w'(t) > 0$ and $(b(t)(w'(t))^{\delta})' \leq 0$ for $t \geq t_1$ for some $t_1 \geq t_0$;

(ii) $\frac{w(t)}{B(t)}$ is decreasing for $t \geq t_1$;

(iii) $w(t) \geq B(t)b^{1/\delta}(t)w'(t)$ for $t \geq t_1$;

(iv) $w(t) \to \infty$ as $t \to \infty$.

Proof. Let $u(t)$ be a positive solution of (1), say $u(t) > 0$, $u(\theta(t)) > 0$, and $u(t-\eta_i) > 0$ for $i = 1, 2, ..., m$ for all $t \geq t_1$ for some $t_1 \geq t_0$. From (1) and (B_4), it follows that case (i) holds true, which implies

$$w(t) \geq B(t) b^{1/\delta}(t) w'(t) \quad \text{for } t \geq t_1,$$

and hence $w(t)/B(t)$ is a decreasing function for $t \geq t_1$. Next, we claim that (4) implies that $w(t) \to \infty$ as $t \to \infty$. Since $w(t)$ is a positive increasing function, there exists a constant $M > 0$ such that

$$\lim_{t\to\infty} w(t) = M > 0. \tag{5}$$

Let $\liminf_{t\to\infty} u(t) = c$. Then using Lemma 2, we obtain

$$M = c + \sum_{i=1}^{m} g_i c^{\alpha_i}. \tag{6}$$

Since $g_i > 0$ for $i = 1, 2, ..., m$, we see that $c > 0$. If not, $c = 0$, then (6) implies $M = 0$, which contradicts (5). Hence there exists $t_2 \geq t_1$ such that

$$u(\theta(t)) \geq \frac{c}{2} \quad \text{for } t \geq t_2.$$

Using this in (1), we obtain

$$(b(t)(w'(t))^{\delta})' + \left(\frac{c}{2}\right)^{\beta} f(t) \leq 0.$$

Integrating the last inequality from t to ∞ gives

$$w'(t) \geq \left(\frac{c}{2}\right)^{\beta/\delta} \left(\frac{1}{b(t)} \int_t^{\infty} f(s) ds\right)^{1/\delta}.$$

Integrating the above inequality from t_2 to t yields

$$w(t) \geq w(t_2) + \left(\frac{c}{2}\right)^{\beta/\delta} \int_{t_2}^{t} \left(\frac{1}{b(s)} \int_s^{\infty} f(x) dx\right)^{1/\delta} ds,$$

which in view of (4) implies that $w(t) \to \infty$ as $t \to \infty$. The proof is now completed. ■

Lemma 4 *Let $u(t)$ be a positive solution of (1) and (4) holds. Then*

$$u(t) \geq g w(t) \tag{7}$$

for $t \geq t_1$ for some $t_1 \geq t_0$, where

$$g = \left[1 - \sum_{i=1}^{m} \left(\alpha_i + (1-\alpha_i) g_i^{1/(1-\alpha_i)}\right)\right] > 0.$$

Proof. Let $u(t)$ be a positive solution of (1), say $u(t) > 0$, $u(\theta(t)) > 0$, and $u(t-\eta_i) > 0$ for $i = 1, 2, ..., m$ for all $t \geq t_1$ for some $t_1 \geq t_0$. Then $w(t) > 0$ for $t \geq t_1$ and from Lemma 3(iv), we see that $w(t) \to \infty$ as $t \to \infty$. Then there exists $t_2 \geq t_1$ such that

$$w(t) \geq 1 \quad \text{for } t \geq t_2. \tag{8}$$

From the definition of $w(t)$ and the fact that $w(t)$ is increasing, we see that

$$u(t) = w(t) - \sum_{i=1}^{m} g_i u^{\alpha_i}(t-\eta_i) \geq w(t) - \sum_{i=1}^{m} w^{\alpha_i}(t) \left(g_i^{1/(1-\alpha_i)}\right)^{1-\alpha_i}.$$

Now using (3) and (8), we obtain

$$\begin{aligned} u(t) &\geq w(t) - \sum_{i=1}^{m} \left(\alpha_i w(t) + (1-\alpha_i) g_i^{1/(1-\alpha_i)} w(t) \right) \\ &= \left[1 - \sum_{i=1}^{m} \left(\alpha_i + (1-\alpha_i) g_i^{1/(1-\alpha_i)} \right) \right] w(t). \end{aligned}$$

Hence,

$$u(t) \geq g w(t) \quad \text{for } t \geq t_2. \tag{9}$$

This completes the proof. ■

Lemma 5 *Let $u(t)$ be a positive solution of (1). If (4) and*

$$\int_{t_0}^{\infty} f(t) B^{\beta}(\theta(t)) dt = \infty \tag{10}$$

hold, then

$$\lim_{t\to\infty} \frac{w(t)}{B(t)} = 0. \tag{11}$$

Proof. Proceeding as in the proof of Lemma 4, we again arrive at (9) for $t \geq t_2$. Using (9) in (1), we obtain

$$(b(t)(w'(t))^{\delta})' + g^{\beta} f(t) w^{\beta}(\theta(t)) \leq 0, \quad t \geq t_2. \tag{12}$$

Since $w(t)/B(t)$ is positive and decreasing (see Lemma 3(ii)), there exists a constant l such that $\lim_{t\to\infty} \frac{w(t)}{B(t)} = l \geq 0$. Assume on the contrary that $l > 0$. Then $w(t)/B(t) \geq l$ for $t \geq t_2$. Using this in (12) and then integrating the resulting inequality from t_2 to t, we obtain

$$b(t_2)(w'(t_2))^{\delta} \geq l^{\beta} g^{\beta} \int_{t_2}^{t} f(s) B^{\beta}(\theta(s)) ds,$$

which contradicts (10) for $t \to \infty$, and so $\lim_{t\to\infty} \frac{w(t)}{B(t)} = 0$. The proof is completed. ■

Theorem 1 *Let (B_1)–(B_5) and (4) hold and let $\beta = \delta$. If*

$$\liminf_{t\to\infty} \int_{\theta(t)}^{t} f(s) B^{\beta}(\theta(s)) ds > \frac{1}{g^{\beta} e}, \tag{13}$$

then equation (1) is oscillatory.

Proof. Let $u(t)$ be a nonoscillatory solution of equation (1), say $u(t) > 0$, $u(\theta(t)) > 0$, and $u(t-\eta_i) > 0$ for $i = 1, 2, ..., m$ for all $t \geq t_1$ for some $t_1 \geq t_0$. Applying Lemmas 3 and 4, we conclude that Lemma 3(iii) and (9) hold for $t \geq t_2$, respectively. Using (9) in (1) gives

$$(b(t)(w'(t))^{\delta})' + g^{\beta} f(t) w^{\beta}(\theta(t)) \leq 0, \quad t \geq t_2. \tag{14}$$

From Lemma 3(iii), we have

$$w^{\beta}(\theta(t)) \geq B^{\beta}(\theta(t)) \left[b(\theta(t))(w'(\theta(t)))^{\delta} \right]^{\beta/\delta}.$$

Using this in (14) and letting $X(t) = b(t)(w'(t))^{\delta}$, we see that $X(t)$ is a positive solution of the differential inequality

$$X'(t) + g^{\beta} f(t) B^{\beta}(\theta(t)) X(\theta(t)) \leq 0, \tag{15}$$

but this contradicts Theorem 2.1.1 in [16], according to which condition (13) ensures that (15) has no positive solution. The proof is complete. ■

Theorem 2 *Let $\delta > 1$, (B_1)–(B_5) and (4) hold. Then equation (1) is oscillatory provided that*

$$(b^{\frac{1}{\delta}}(t)w'(t))' + \frac{g^{\beta}}{\delta} f(t)B^{\delta-1}(\theta(t))w^{\beta+1-\delta}(\theta(t)) = 0 \tag{16}$$

is oscillatory.

Proof. Let $u(t)$ be a nonoscillatory solution of equation (1), say $u(t) > 0$, $u(\theta(t)) > 0$, and $u(t-\eta_i) > 0$ for $i = 1, 2, ..., m$ for all $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in Theorem 1, we see that Lemma 3(iii) and (14) hold for $t \geq t_2$. It is easy to see that

$$(b(t)(w'(t))^{\delta})' = \left((b^{\frac{1}{\delta}}(t)w'(t))^{\delta}\right)' = \delta \left(b^{\frac{1}{\delta}}(t)w'(t)\right)^{\delta-1} \left(b^{\frac{1}{\delta}}(t)w'(t)\right)'.$$

Using the above relation in (14), we obtain

$$(b^{\frac{1}{\delta}}(t)w'(t))' + \frac{g^{\beta}}{\delta}(b^{\frac{1}{\delta}}(t)w'(t))^{1-\delta} f(t)w^{\beta}(\theta(t)) \leq 0. \tag{17}$$

For $\delta > 1$, Lemma 3(iii) yields

$$w^{1-\delta}(t) \leq \left(b^{\frac{1}{\delta}}(t)w'(t)\right)^{1-\delta} B^{1-\delta}(t),$$

hence

$$\left(b^{\frac{1}{\delta}}(t)w'(t)\right)^{1-\delta} \geq \left(\frac{w(t)}{B(t)}\right)^{1-\delta}. \tag{18}$$

Since $w(t)/B(t)$ is decreasing and $\theta(t) \leq t$, we get

$$\left(\frac{w(t)}{B(t)}\right)^{1-\delta} \geq \left(\frac{w(\theta(t))}{B(\theta(t))}\right)^{1-\delta}. \tag{19}$$

Substituting (19) into (18) yields

$$\left(b^{\frac{1}{\delta}}(t)w'(t)\right)^{1-\delta} \geq \left(\frac{w(\theta(t))}{B(\theta(t))}\right)^{1-\delta}.$$

Using this in (17), we see that $w(t)$ is a positive solution of the differential inequality

$$(b^{\frac{1}{\delta}}(t)w'(t))' + \frac{g^{\beta}}{\delta} f(t)B^{\delta-1}(\theta(t))w^{\beta+1-\delta}(\theta(t)) \leq 0, \quad t \geq t_2. \tag{20}$$

It follows from [15, Corollary 1] that the delay differential equation (16) corresponding to (20) has also a positive solution, but this contradicts our assumption on Eq. (16). This completes the proof. ■

We denote

$$F(t) = \left[\int_t^{\infty} f(s)ds\right]^{(1-\delta)/\delta}.$$

Theorem 3 *Let $0 < \delta < 1$, (B_1)–(B_5), (4) and (10) hold and let $\theta'(t) \geq 0$. Then equation (1) is oscillatory provided that*

$$(b^{\frac{1}{\delta}}(t)w'(t))' + \frac{g^{\frac{\beta}{\delta}}}{\delta} F(t)f(t)w^{\frac{\beta}{\delta}}(\theta(t)) = 0 \tag{21}$$

is oscillatory.

Proof. Assume on the contrary that $u(t)$ is a positive solution of (1). From Lemmas 4 and 5, we see that (9) and (11) hold, respectively. Now note that (11) implies

$$\lim_{t\to\infty} b^{\frac{1}{\delta}}(t)w'(t) = 0.$$

Therefore an integration of (1) yields

$$b^{\frac{1}{\delta}}(t)w'(t) = \left[\int_t^\infty f(s)u^\beta(\theta(s))ds\right]^{1/\delta}. \tag{22}$$

Differentiating (22) leads to the equation

$$(b^{\frac{1}{\delta}}(t)w'(t))' + \frac{1}{\delta}\left[\int_t^\infty f(s)u^\beta(\theta(s))ds\right]^{\frac{1-\delta}{\delta}} f(t)u^\beta(\theta(t)) = 0.$$

Using (9) in the above equation yields

$$(b^{\frac{1}{\delta}}(t)w'(t))' + \frac{g^\beta}{\delta}\left[g^\beta\int_t^\infty f(s)w^\beta(\theta(s))ds\right]^{\frac{1-\delta}{\delta}} f(t)w^\beta(\theta(t)) \le 0.$$

Employing $w(t)$ is an increasing function, we see that $w(t)$ is a positive solution of the differential inequality

$$(b^{\frac{1}{\delta}}(t)w'(t))' + \frac{g^{\frac{\beta}{\delta}}}{\delta}F(t)f(t)w^{\frac{\beta}{\delta}}(\theta(t)) \le 0. \tag{23}$$

It follows from [15, Corollary 1] that the delay differential equation (21) corresponding to (23) has also a positive solution, but this contradicts our assumption on Eq. (21). This completes the proof. ■

In the following, we obtain explicit criteria for the oscillation of (1) for different values of δ and β.

Theorem 4 *Let $\beta = \delta > 1$, (B_1)–(B_5) and (4) hold and let $\theta'(t) \ge 0$. If*

$$\begin{aligned}\limsup_{t\to\infty}\Bigg\{\frac{1}{B(\theta(t))}\int_{t_0}^{\theta(t)} f(s)B(s)B^\delta(\theta(s))ds + \int_{\theta(t)}^t f(s)B^\delta(\theta(s))ds \\ +B(\theta(t))\int_t^\infty f(s)B^{\delta-1}(\theta(s))ds\Bigg\} > \frac{\delta}{g^\beta},\end{aligned} \tag{24}$$

then equation (1) is oscillatory.

Proof. Let $u(t)$ be a nonoscillatory solution of equation (1), say $u(t) > 0$, $u(\theta(t)) > 0$, and $u(t-\eta_i) > 0$ for $i = 1, 2, ..., m$ for all $t \ge t_1$ for some $t_1 \ge t_0$. Proceeding as in Theorem 2, we see that (20) holds for $t \ge t_2$. Integrating (20) from t to ∞ yields

$$w'(t) \ge \frac{g^\beta}{\delta b^{\frac{1}{\delta}}(t)}\int_t^\infty f(s)B^{\delta-1}(\theta(s))w(\theta(s))ds.$$

Integrating this inequality from t_2 to t yields

$$\begin{aligned} w(t) &\ge \frac{g^\beta}{\delta}\int_{t_2}^t \frac{1}{b^{\frac{1}{\delta}}(s)}\int_s^\infty f(j)B^{\delta-1}(\theta(j))w(\theta(j))djds \\ &= \frac{g^\beta}{\delta}\Bigg[\int_{t_2}^t \frac{1}{b^{\frac{1}{\delta}}(s)}\int_s^t f(j)B^{\delta-1}(\theta(j))w(\theta(j))djds \\ &\quad + \int_{t_2}^t \frac{1}{b^{\frac{1}{\delta}}(s)}\int_t^\infty f(j)B^{\delta-1}(\theta(j))w(\theta(j))djds\Bigg].\end{aligned}$$

Changing the order of integration, we obtain

$$w(t) \geq \frac{g^\beta}{\delta}\left[\int_{t_2}^{t} f(s)B(s)B^{\delta-1}(\theta(s))w(\theta(s))ds + B(t)\int_t^\infty f(s)B^{\delta-1}(\theta(s))w(\theta(s))ds\right].$$

Thus,

$$\begin{aligned} w(\theta(t)) \geq \frac{g^\beta}{\delta}\Bigg[&\int_{t_2}^{\theta(t)} f(s)B(s)B^{\delta-1}(\theta(s))w(\theta(s))ds + B(\theta(t))\int_{\theta(t)}^{t} f(s)B^{\delta-1}(\theta(s))w(\theta(s))ds \\ &+B(\theta(t))\int_t^\infty f(s)B^{\delta-1}(\theta(s))w(\theta(s))ds\Bigg]. \end{aligned}$$

Applying the fact that $w(t)/B(t)$ is decreasing and $w(t)$ is increasing, the previous inequality implies

$$\begin{aligned} w(\theta(t)) \geq{}& \frac{g^\beta}{\delta}\frac{w(\theta(t))}{B(\theta(t))}\int_{t_2}^{\theta(t)} f(s)B(s)B^{\delta}(\theta(s))ds + \frac{g^\beta}{\delta}w(\theta(t))\int_{\theta(t)}^{t} f(s)B^{\delta}(\theta(s))ds \\ &+\frac{g^\beta}{\delta}B(\theta(t))w(\theta(t))\int_t^\infty f(s)B^{\delta-1}(\theta(s))ds. \end{aligned}$$

After simplification, one can see that

$$\begin{aligned} \Bigg\{&\frac{1}{B(\theta(t))}\int_{t_2}^{\theta(t)} f(s)B(s)B^{\delta}(\theta(s))ds + \int_{\theta(t)}^{t} f(s)B^{\delta}(\theta(s))ds \\ &+B(\theta(t))\int_t^\infty f(s)B^{\delta-1}(\theta(s))ds\Bigg\} \leq \frac{\delta}{g^\beta}, \end{aligned}$$

which is a contradiction. This completes the proof. ■

Theorem 5 *Let $\beta = \delta < 1$, (B_1)–(B_5), (4) and (10) hold and let $\theta'(t) \geq 0$. If*

$$\begin{aligned} \limsup_{t\to\infty}\Bigg\{&\frac{1}{B(\theta(t))}\int_{t_0}^{\theta(t)} F(s)f(s)B(s)B(\theta(s))ds + B(\theta(t))\int_{\theta(t)}^{t} F(s)f(s)\frac{B(\theta(s))}{B(s)}ds \\ &+B(\theta(t))\int_t^\infty F(s)f(s)ds\Bigg\} > \frac{\delta}{g}, \end{aligned} \tag{25}$$

then equation (1) is oscillatory.

Proof. Let $u(t)$ be a nonoscillatory solution of equation (1), say $u(t) > 0$, $u(\theta(t)) > 0$, and $u(t-\eta_i) > 0$ for $i = 1, 2, ..., m$ for all $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in Theorem 3, we see that (23) holds for $t \geq t_2$. Integrating (23) from t to ∞ yields,

$$w'(t) \geq \frac{g}{\delta}\frac{1}{b^{\frac{1}{\delta}}(t)}\int_t^\infty F(s)f(s)w(\theta(s))ds.$$

Integrating this inequality from t_2 to t yields

$$\begin{aligned} w(t) &\geq \frac{g}{\delta}\int_{t_2}^{t}\frac{1}{b^{\frac{1}{\delta}}(s)}\int_s^\infty F(j)f(j)w(\theta(j))djds \\ &= \frac{g}{\delta}\int_{t_2}^{t}\frac{1}{b^{\frac{1}{\delta}}(s)}\int_s^t F(j)f(j)w(\theta(j))djds \\ &\quad+\frac{g}{\delta}\int_{t_2}^{t}\frac{1}{b^{\frac{1}{\delta}}(s)}\int_t^\infty F(j)f(j)w(\theta(j))djds. \end{aligned}$$

Hence

$$w(t) \geq \frac{g}{\delta} \int_{t_2}^{t} B(s)F(s)f(s)w(\theta(s))ds + \frac{g}{\delta} B(t) \int_{t}^{\infty} F(s)f(s)w(\theta(s))ds,$$

and so

$$\begin{aligned} w(\theta(t)) \quad &\geq \quad \frac{g}{\delta} \int_{t_2}^{\theta(t)} B(s)F(s)f(s)w(\theta(s))ds + \frac{g}{\delta} B(\theta(t)) \int_{\theta(t)}^{t} F(s)f(s)w(\theta(s))ds \\ &\quad + \frac{g}{\delta} B(\theta(t)) \int_{t}^{\infty} F(s)f(s)w(\theta(s))ds. \end{aligned}$$

Since $w(t)/B(t)$ is decreasing and $w(t)$ is increasing, the last inequality implies

$$\begin{aligned} &\left\{ \frac{1}{B(\theta(t))} \int_{t_2}^{\theta(t)} F(s)f(s)B(s)B(\theta(s))ds + B(\theta(t)) \int_{\theta(t)}^{t} F(s)f(s) \frac{B(\theta(s))}{B(s)} ds \right. \\ &\left. + B(\theta(t)) \int_{t}^{\infty} F(s)f(s)ds \right\} \leq \frac{\delta}{g}. \end{aligned}$$

This is a contradiction and the proof is completed. ■

Theorem 6 *Let $\delta > 1$ and $\beta > \delta$ be hold. Moreover, assume that (B_1)–(B_5) and (4) are satisfied. If*

$$\int_{t_0}^{\infty} f(t)B^{\delta-1}(\theta(t))dt = \infty, \tag{26}$$

then (1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2, we again arrive at (20) for $t \geq t_2$. Since $w(t)$ is increasing, there exists a constant $M > 0$ such that $w(t) \geq M$, and so $w^{\beta-\delta+1}(t) \geq M^{\beta-\delta+1}$ for $t \geq t_2$. Using this in (20) gives

$$(b^{\frac{1}{\delta}}(t)w'(t))' + \frac{g^{\beta}}{\delta} M^{\beta-\delta+1} f(t)B^{\delta-1}(\theta(t)) \leq 0 \quad \text{for} \quad t \geq t_2.$$

Integrating the last inequality from t_2 to t yields

$$\frac{g^{\beta}}{\delta} M^{\beta-\delta+1} \int_{t_2}^{t} f(s)B^{\delta-1}(\theta(s))ds \leq b^{\frac{1}{\delta}}(t_2)w'(t_2) < \infty \quad \text{as } t \to \infty,$$

which contradicts (26). This completes the proof. ■

Theorem 7 *Let $\delta > 1$ and $\beta < \delta$ be hold. Moreover, assume that (B_1)–(B_5) and (4) are satisfied. If*

$$\int_{t_0}^{\infty} f(t)B^{\beta-1}(\theta(t))dt = \infty, \tag{27}$$

then (1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2, we again arrive at (20) for $t \geq t_2$. Since $w(t)/B(t)$ is decreasing, there exists a constant $K > 0$ such that $w^{\beta-\delta}(t) \geq K^{\beta-\delta}B^{\beta-\delta}(t)$ for $t \geq t_2$. Using this in (20), we obtain

$$(b^{\frac{1}{\delta}}(t)w'(t))' + \frac{g^{\beta}}{\delta} K^{\beta-\delta} f(t)B^{\beta-1}(\theta(t))w(\theta(t)) \leq 0, \quad t \geq t_2. \tag{28}$$

Since $w(t)$ is increasing there exists a constant $M > 0$ such that $w(t) \geq M$ for $t \geq t_2$. Using this in (28) and then integrating from t_2 to t, we see that

$$\frac{g^{\beta}}{\delta} K^{\beta-\delta} M \int_{t_2}^{t} f(s)B^{\beta-1}(\theta(s))ds \leq b^{\frac{1}{\delta}}(t_2)w'(t_2) < \infty \quad \text{as } t \to \infty,$$

which contradicts (27). This completes the proof of the theorem. ■

3 Examples

In this section, we present three examples to show effectiveness and feasibility of the main results.

Example 1 *Consider the following differential equation with a couple of sublinear neutral terms*

$$\left(t^{\frac{1}{2}}\left(u(t)+\frac{1}{4}u^{\frac{1}{3}}(t-1)+\frac{1}{16}u^{\frac{1}{5}}\left(t-\frac{1}{2}\right)\right)'\right)'+\frac{a}{t^{\frac{3}{2}}}u\left(\frac{t}{2}\right)=0, \quad t\geq 1. \tag{29}$$

Here $b(t)=t^{\frac{1}{2}}$, $f(t)=a/t^{3/2}$ *with* $a>0$, $g_1=1/4$, $g_2=1/16$, $\alpha_1=1/3$, $\alpha_2=1/5$, $\eta_1=1$, $\eta_2=1/2$, $\theta(t)=t/2$, *and* $\beta=\delta=1$. *A simple calculation shows that* $g=43/120$, *and* $B(t)=2(\sqrt{t}-1)$. *It is easy to see that* (B_1)*–*(B_5) *are satisfied. Condition (4) becomes*

$$\int_1^\infty\left(\frac{1}{\sqrt{t}}\int_t^\infty\frac{a}{s^{\frac{3}{2}}}ds\right)dt=2a\int_1^\infty\frac{1}{t}dt=\infty,$$

i.e., condition (4) is satisfied. Also condition (13) becomes

$$\liminf_{t\to\infty}\int_{t/2}^{t}\frac{2a}{s^{\frac{3}{2}}}\left(\sqrt{\frac{s}{2}}-1\right)ds=\liminf_{t\to\infty}\int_{t/2}^{t}\left(\frac{\sqrt{2}a}{s}-\frac{2a}{s^{\frac{3}{2}}}\right)ds=\sqrt{2}a\log 2,$$

and so condition (13) is satisfied if $a>1.0473$. *Hence, by Theorem 1, equation (29) is oscillatory if* $a>1.0473$.

Example 2 *Consider the following second-order differential equation with a couple of sublinear neutral terms*

$$\left(t\left(w'(t)\right)^3\right)'+\frac{a}{t^2}u^5\left(\frac{t}{2}\right)=0, \quad t\geq 1, \tag{30}$$

where $w(t)=u(t)+\frac{1}{4}u^{1/3}(t-1)+\frac{1}{16}u^{1/5}(t-2)$ *and* $a>0$ *is a constant.*

Here $b(t)=t$, $f(t)=a/t^2$ *with* $a>0$, $g_1=1/4$, $g_2=1/16$, $\alpha_1=1/3$, $\alpha_2=1/5$, $\delta=3$, $\beta=5$, $\eta_1=1$, $\eta_2=2$, *and* $\theta(t)=t/2$. *It is easy to see that* (B_1)*–*(B_5) *are satisfied. Condition (4) becomes*

$$\int_1^\infty\left(\frac{1}{t}\int_t^\infty\frac{a}{s^2}ds\right)^{\frac{1}{3}}dt=\int_1^\infty\frac{a^{\frac{1}{3}}}{t^{\frac{2}{3}}}dt=\infty,$$

i.e., condition (4) is satisfied. Since $B(t)=\frac{3}{2}(t^{2/3}-1)$, *condition (26) becomes*

$$\frac{9a}{4}\int_1^\infty\frac{1}{t^2}\left(\frac{t^{2/3}}{2^{2/3}}-1\right)^2dt=\infty,$$

i.e., condition (26) is satisfied. Therefore, by Theorem 6, equation (30) is oscillatory.

Example 3 *Consider the second-order differential equation with two sublinear neutral terms*

$$\left(\frac{1}{t}\left(w'(t)\right)^{\frac{1}{3}}\right)'+\frac{1}{t^2}u^{\frac{1}{3}}\left(\frac{t}{2}\right)=0, \quad t\geq 1, \tag{31}$$

where $w(t)=u(t)+\frac{1}{2^{8/5}}u^{1/5}(t-1)+\frac{1}{2^{12/7}}u^{1/7}(t-2)$.

Here $b(t)=1/t$, $f(t)=1/t^2$, $\beta=\delta=1/3$, $\eta_1=1$, $\eta_2=2$, $\alpha_1=1/5$, $\alpha_2=1/7$, $\theta(t)=t/2$, $g_1=1/2^{8/5}$, *and* $g_2=1/2^{12/7}$. *As in Examples 1 and 2, it is easy to show that all conditions of Theorem 5 hold, and so equation (31) is oscillatory.*

4 Conclusion

In this paper, we have obtained some new criteria for the oscillation of (1). Our criteria are new in the sense that almost all the results already established for (1) required that $g_i(t) \to 0$ as $t \to \infty$ for $i = 1, 2, ..., m$, but we assume that $g_i(t) = g_i \in (0, 1)$ for $i = 1, 2, ..., m$. Therefore the oscillation criteria already known in the literature cannot be applied to our examples.

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