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Extended Newton's Method With Applications To Interior Point Algorithms Of Mathematical Programming[*]

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Abstract

We use a weaker Newton–Kantorovich theorem for solving equations, introduced in [3] to analyze interior point methods. This way our approach extends earlier works in [6] on Newton's method and interior point algorithms.

1 Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1}$$

where, F is a differentiable operator defined on a convex domain $\mathcal{D}$ of $\mathbb{R}^i$ (i an integer) with values in $\mathbb{R}^i$.

The famous Newton-Kantorovich theorem [4] has been used extensively to solve equation (1). A survey of such results can be found in [1] and the references therein. Recently, in [1]–[3], we improved the Newton-Kantorovich theorem. We use this development to show that the results obtained in the elegant work in [6] in connection to interior point methods can be extended, if our convergence conditions simply replace the stronger ones given there.

Finally a numerical example is provided to show that fewer iterations than the ones suggested in [6] are needed to achieve the same error tolerance.

2 An Improved Newton–Kantorovich Theorem

Let $\|\cdot\|$ be a given norm on $\mathbb{R}^i$, and x_0 be a point of $\mathcal{D}$ such that the closed ball of radius r centered at x_0, $\overline{U}(x_0, r) = \{x \in \mathbb{R}^i \colon \|x - x_0\| \le r\}$ is included in $\mathcal{D}$, i.e., $\overline{U}(x_0, r) \subseteq \mathcal{D}$. We assume that the Jacobian $F'(x_0)$ is nonsingular and that the following affine-invariant Lipschitz condition is satisfied:

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \le w\,\|x - y\| \quad \text{for some } w \ge 0, \text{ and for all } x, y \in \overline{U}(x_0, r). \tag{2}$$

Our technique extends other methods using inverses along the same lines [1, 2, 3].

The famous Newton–Kantorovich Theorem [4] states that if the quantity

$$\alpha := \|F'(x_0)^{-1} F(x_0)\| \tag{3}$$

together with w satisfy

$$k = \alpha w \le \frac{1}{2}, \tag{4}$$

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then there exists $x^\star \in \overline{U}(x_0, r)$ with $F(x^\star) = 0$. Moreover the sequences produced by Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad (n \geq 0), \tag{5}$$

and by the modified Newton's method

$$y_{n+1} = y_n - F'(y_0)^{-1} F(y_n), \quad y_0 = x_0 \quad (n \geq 0) \tag{6}$$

are well defined and converge to $x^\star$. Notice that we assume that $F'(x_0)$ is invertible, so it does not become zero on $\overline{U}(x_0, r)$.

In [1]–[3] we introduced the center–Lipschitz condition

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq w_0 \, \|x - x_0\|, \quad \text{for some } w_0 \geq 0, \text{ and for all } x \in \overline{U}(x_0, r). \tag{7}$$

This way, we provided a finer local and semilocal convergence analysis of method (5) by using the combination of conditions (2) and (7) given by

$$\ell^0 = \alpha \overline{w} \leq \frac{1}{2}, \tag{8}$$

where,

$$\overline{w} = \frac{1}{8} \, (w + 4 \, w_0 + \sqrt{w^2 + 8 \, w_0 \, w}) \quad [3]. \tag{9}$$

In general

$$w_0 \leq \overline{w} \leq w \tag{10}$$

holds, and $\frac{w}{w_0}$, $\frac{\overline{w}}{w_0}$, $\frac{w}{\overline{w}}$ can be arbitrarily large [1]. Note also that

$$k \leq \frac{1}{2} \Rightarrow \ell^0 \leq \frac{1}{2} \tag{11}$$

but not vice versa unless if $w_0 = w$. Examples where weaker condition (8) holds but (4) fails have been also given in [1]–[3].

We can do even better as follows:
Set

$$U_0 := \bar{U}(x_0, r) \cap U(x_0, \frac{1}{w_0} - \|F'(x_0)^{-1} F(x_0)\|).$$

Suppose there exists $w_1 > 0$ such that for each $x, y \in U_0$:

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq w_1 \|x - y\|, \tag{12}$$

and there exist $\beta > 0$, $\gamma > 0$ such that for each $x \in U_0$ and $\theta \in [0, 1]$,

$$\|F'(x_0)^{-1}(F'(x_0 + \theta(x_1 - x_0)) - F'(x_0))\| \leq \gamma\theta \|x_1 - x_0\|, \tag{13}$$

$$\|F'(x_0)^{-1}(F'(x_1) - F'(x_0))\| \leq \beta \|x_1 - x_0\|,$$

where

$$x_1 = x_0 - F'(x_0)^{-1} F(x_0). \tag{14}$$

Notice that

$$\beta \leq \gamma \leq w_0 \leq w \text{ and } w_1 \leq w. \tag{15}$$

Define

$$\delta = \frac{2w_1}{w_1 + \sqrt{w_1^2 + 8w_0 w_1}} \tag{16}$$

and

$$\delta^1 = \begin{cases} \frac{1}{w_0+\beta}, & \text{if } q = w_1\gamma + \delta w_0(\gamma - 2\beta) = 0, \\ 2\frac{-\delta(w_0+\beta)+\sqrt{\delta^2(w_0+\beta)^2+\delta(w_1\gamma+2\delta w_0(\gamma-2\beta))}}{q}, & \text{if } q > 0, \\ -\frac{2\delta(w_0+\beta)+\sqrt{\delta^2(w_0+\beta)^2+\delta(w_1\gamma+2\delta L_0(\gamma-2\beta))}}{q}, & \text{if } q < 0. \end{cases} \tag{17}$$

In [3], we presented a semi-local convergence analysis using the conditions below (11) and the condition

$$k^0 = \alpha\delta \le \frac{1}{2}, \tag{18}$$

replacing (8). In view of (15)–(18), we have that

$$w_0 \le \delta \le \bar{w}, \tag{19}$$

so

$$\ell^0 \le \frac{1}{2} \Longrightarrow k^0 \le \frac{1}{2}. \tag{20}$$

Notice that by (18) $\alpha w_0 < 1$, so the set U_0 is defined.

Similarly by simply replacing w with w_0 (since (7) instead of (2) is actually needed in the proof) and condition (4) by the weaker

$$k^1 = \alpha w_0 \le \frac{1}{2} \tag{21}$$

in the proof of Theorem 1 in [6] we show that method (6) also converges to $x^\star$ and the improved bounds

$$\|y_n - x^\star\| \le \frac{2\,\beta_0\,\lambda_0^2}{1-\lambda_0^2}\;\xi_0^{n-1} \quad (n \ge 1) \tag{22}$$

where

$$\beta_0 = \frac{\sqrt{1-2k^1}}{k^1}, \quad \lambda_0 = \frac{1-\sqrt{1-2k^1}-h^1}{k^1} \quad \text{and} \quad \xi_0 = 1-\sqrt{1-2k^1}$$

hold. In case $w_0 = w$, (22) reduces to (10) in [6]. Otherwise our error bounds are finer. Note also that

$$k \le \frac{1}{2} \Rightarrow k^1 \le \frac{1}{2}$$

but not vice versa unless if $w_0 = w$. Let us provide an example to show that (8) or (18) or (21) hold but (4) fails.

Example 1 *Let $i = 1$, $x_0 = 1$, $\mathcal{D} = [p, 2-p]$, $p \in \left[0, \frac{1}{2}\right)$, and define functions F on D by*

$$F(x) = x^3 - p. \tag{23}$$

Using (2), (3), (7) and (23), we obtain

$$\alpha = \frac{1}{3}(1-p), \quad w = 2(2-p) \quad \textit{and} \quad w_0 = 3-p, \tag{24}$$

which imply that

$$k = \frac{2}{3}(1-p)(2-p) > \frac{1}{2} \quad \textit{for all } p \in \left[0, \frac{1}{2}\right). \tag{25}$$

That is there is no guarantee that Newton's method (5) converges to $x^\star = \sqrt[3]{p}$, since the Newton-Kantorovich hypothesis (4) is violated.

Moreover, condition (21) holds for all $p \in \left[\frac{4-\sqrt{10}}{2}, \frac{1}{2}\right)$. Furthermore, (8) holds for $p \in \left[0.450339002, \frac{1}{2}\right)$. We also have that $\beta = \frac{5+p}{3}$, $\gamma = 2$, and $w_1 = \frac{2}{2(3-p)}(-2p^2+5p+6)$. Finally, (18) is satisfied for $p \in [0.0984119, 0.5)$ which is the largest interval. The above suggest that all results on interior point methods obtained in [6] for Newton's method using (4) can now be rewritten using only (18). The same holds true for the modified Newton's method, where (21) also replaces (4).

3 Applications to Interior Point Algorithm

It has already been shown in [5] that the Newton-Kantorovich theorem can be used to construct and analyze optimal-complexity path following algorithms for linear complementary problems. Potra has chosen to apply this theorem to linear complementary problems because such problems provide a convenient framework for analyzing primal-dual interior point algorithms. Theoretical and experimental work conducted over the past decade has shown that primal-dual path following algorithms are among the best solution methods for (LP), quadratic programming (QP), and linear complementary problems (LCP) (see for example [7], [11]). Primal-dual path following algorithms are the basis of the best general–purpose practical methods, and they have important theoretical properties [10], [11], [12].

Potra, using (4), in particular showed how to construct path–following algorithms for LCP that have $\mathrm{O}(\sqrt{n}L)$ iteration complexity [6].

Given a point x that approximates a point $x(\tau)$ on the central path of the LCP with complementarity gap τ, the algorithms compute a parameter $\theta \in (0,1)$ so that x satisfies the Newton–Kantorovich hypothesis (4) for the equation defining $x\left((1-\theta)\,\tau\right)$. It is proven that θ is bounded below by a multiple of $n^{-1/2}$. Since (4) is satisfied, the sequence generated by Newton's method or by the modified Newton method (take $F'(x_n) = F'(x_0)$, $n \geq 0$) with starting x, will converge to $x\left((1-\theta)\,\tau\right)$. He showed that the number of steps required to obtain an acceptable approximation of $x\left((1-\theta)\,\tau\right)$ is bounded above by a number independent of n. Therefore, a point with complementarity less than ε can be obtained in at most $O\left(\sqrt{n}\log\left(\frac{\varepsilon}{\varepsilon_0}\right)\right)$ steps (for both methods), where ε_0 is the complementary gap of the starting point. For linear complementarity problems with rational input data of bit length L, this implies that an exact solution can be obtained in at most $O\left(\sqrt{n}L\right)$ iterations plus a rounding procedure including $O\left(n^3\right)$ arithmetic operations [11] (see also [8, 9]).

We also refer the reader to the excellent monograph of Nesterov and Nemirovskii [5] for an analysis of the construction of interior point methods for a larger class of problems than that considered in [6] (see also [9]).

We can now describe the linear complementarity problem as follows: Given two matrices $Q, R \in \mathbb{R}^{n\times n}$ ($n \geq 2$) and a vector $b \in \mathbb{R}^n$, the horizontal linear complementarity problem (HLCP) consists of approximating a pair of vectors (w, s) such that

$$\begin{aligned} ws &= 0 \\ Q(w) + R(s) &= b \\ w, s &\geq 0. \end{aligned} \tag{26}$$

The monotone linear complementarity problem (LCP) is obtained by taking $R = -I$ and Q positive semidefinite.

Moreover the linear programming problem (LP) and the quadratic programming problem (QP) can be formulated as HLCPs. That is, HLCP is a suitable way for studying interior point methods.

We assume HLCP (26) is monotone in the sense that:

$$Q(u) + R(v) = 0 \text{ implies } u^t v \geq 0, \text{ for all } u, v \in \mathbb{R}^n. \tag{27}$$

Condition (27) holds if the HLCP is a reformulation of a QP. If the HLCP is a reformuation of a LP then the following stronger condition holds:

$$Q(u) + R(v) = 0 \text{ implies } u^t v = 0, \text{ for all } u, v \in \mathbb{R}^n. \tag{28}$$

Then we say in this case that the HLCP is skew–symmetric.

If the HLCP has an interior point, i.e. there is $(w, s) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$ satisfying $Q(w) + R(s) = b$, then for any parameter $\tau > 0$ the nonlinear system

$$\begin{aligned} ws &= \tau e \\ Q(w) + R(s) &= b \\ w, s &\geq 0 \end{aligned} \tag{29}$$

has a unique positive solution $x(\tau) = [w(\tau)^t, s(\tau)^t]^t$.

The set of all such solutions defines the central path C of the HLCP. It can be proved that $(w(\tau), s(\tau))$ converges to a solution of the HLCP as $\tau \to 0$. Such an approach for solving the HLCP is called the path following algorithm.

At a basic step of a path following algorithm, an approximation (w, s) of $(w(\tau),\, s(\tau))$ has already been computed for some $\tau > 0$. The algorithm determines the smaller value of the central path parameter $\tau_+ = (1-\theta)\,\tau$, where the value $\theta \in (0,1)$ is computed in some unspecified way. The approximation (w^t, s^t) of $(w(\tau_+), s(\tau_+))$ is computed. The procedure is then repeated with $(w^+, s^+,\, \tau^+)$ in place of $(w, s.\tau)$.

In order for us to relate the path following algorithm and the the Newton-Kantorovich theorem we introduce the notations

$$x = \begin{bmatrix} w \\ s \end{bmatrix}, \quad x\,(\tau) = \begin{bmatrix} w\,(\tau) \\ s\,(\tau) \end{bmatrix}, \quad x^+ = \begin{bmatrix} w^+ \\ s^+ \end{bmatrix}, \quad x\,(\tau_+) = \begin{bmatrix} w\,(\tau_+) \\ s\,(\tau_+) \end{bmatrix}, \text{ etc.}$$

Then for any $\theta > 0$ we define the nonlinear operator

$$F_\sigma(x) = \begin{bmatrix} ws - \sigma e \\ Q(w) + R(s) - b \end{bmatrix}. \tag{30}$$

Then system (29) defining $x(\tau)$ becomes

$$F_\sigma(x) = 0, \tag{31}$$

whereas the system defining $x(\tau_+)$ is given by

$$F_{(1-\theta)\tau}(x) = 0. \tag{32}$$

We assume that the initial guess x belongs in the interior of the feasible set of the HLCP

$$F^0 = \left\{ x = (w^t, s^t)^t \in \mathbb{R}^{2n}_{++} : Q(w) + R(s) = b \right\}. \tag{33}$$

In order to verify the Newton–Kantorovich hypothesis for equation (31) we introduce the quantity

$$\eta = \eta(x,\tau) = \left\| F'(x)^{-1} F_\tau(x) \right\|, \tag{34}$$

the measure of proximity

$$\begin{aligned} k &= k(x,\tau) = \eta\ell,\ w = w(x), \\ k^0 &= k^0(x,\tau) = \eta\bar{\ell},\ \delta = \delta(x), \\ k^1 &= k^1(x,\tau) = \eta w_0,\ w_0 = w_0(x), \end{aligned} \tag{35}$$

and the normalized primal–dual gap

$$\mu = \mu(x) = \frac{w^t s}{\eta}. \tag{36}$$

If for a given interior point x and a given parameter τ we have $k^0(x,\tau) \le .5$ for the Newton–Kantorovich method or $k^1(x,\tau) \le .5$ for the modified Newton–Kantorovich method then corresponding sequences starting from x will converge to the point $x\,(\tau)$ on the central path. We can now describe our algorithm which is a weaker version of the one given in [6]:

Algorithm 1 *(using Newton–Kantorovich method).*
Given $0 < k^0_1 < k^0_2 < .5$, $\varepsilon > 0$, *and* $x_0 \in F^0$ *satisfying* $k^0(x_0, \mu\,(x_0)) \le k^0_1$;
Set $k^0 \leftarrow 0$ *and* $\tau_0 \leftarrow \mu\,(x_0)$;
repeat (outer iteration)
Set $(x,\tau) \leftarrow (x_k, \tau_k)$, $\bar{x} \leftarrow x_k$;
Determine the largest $\theta \in (0,1)$ *such that* $k^0(x, (1-\theta)\tau) \le k^0_2$;

Set $\tau \leftarrow (1-\theta)\tau$;
repeat (inner iteration)

$$\text{Set } x \leftarrow x - F'(x)^{-1}F_\tau(x) \tag{37}$$

until $k^0(x,\mu) \le k_1^0$;
Set $(x_{k+1}, \tau_{k+1}) \leftarrow (x,\tau)$;
Set $k \leftarrow k+1$;
until $\left(w^k\right)^t s^k \le \varepsilon$.

For the modified Newton–Kantorovich algorithm k_1^0, k_2^0, k^0 should be replaced by k_1^1, k_2^1, k^1, and (37) by

$$\text{Set } x \leftarrow x - F'(\bar{x})^{-1}F_\tau(x)$$

respectively.

In order to obtain Algorithm 1 in [6] we need to replace k_1^0, k_2^0, k^0 by k_1, k_2, k respectively.

The above suggests that all results on interior point methods obtained in [6] using (4) can now be rewritten using only the weaker (8) (or (21)).

We only state those results for which we will provide applications.

Let us introduce the notation

$$\Psi_i^a = \begin{cases} 1+\theta_i^a + \sqrt{2\theta_i^a + r_i^a} & \text{if HLCP is monotone,} \\ 1+q_{ia} + \sqrt{2q_{ia} + q_{ia}^2} & \text{if HLCP is skew-symmetric,} \end{cases} \tag{38}$$

where $\sqrt{r_i^a} = \theta_i^a$, $\sqrt{t_i^a} = k_i^a$, $a = 0,1$,

$$\theta_i^a = t_i\left[1 + \frac{t_i^a}{1-t_i^a}\right], \quad q_{ia} = \frac{t_i^a}{2}, \quad i = 1,2. \tag{39}$$

Then by simply replacing k, k_1, k_2 by k^0, k_1^0, k_2^0 respectively in the corresponding results in [6] we obtain the following improvements:

Theorem 1 *The parameter θ determined at each outer iteration of algorithm 1 satisfies*

$$\theta \ge \frac{\chi^a}{\sqrt{n}} = \lambda^a$$

where

$$\chi^a = \begin{cases} \frac{\sqrt{2}(k_2^a - k_1^a)}{\sqrt{2+p^2 t_i}\sqrt{\psi_1^a}} & \textit{if HLCP is skew-symmetric or if no simplified Newton-Kanorovich steps are performed,} \\ \frac{\sqrt{2}(k_2^a - k_1^a)}{\left(\sqrt{2}+pk_1^a\right)\sqrt{\psi_1^a}} & \textit{otherwise,} \end{cases} \tag{40}$$

where

$$p = \begin{cases} \sqrt{2} & \textit{if HLCP is monotone} \\ 1 & \textit{if HLCP is skew-symmetric,} \end{cases} \tag{41}$$

Clearly, the lower bound on λ^a on θ is an improvement over the corresponding one in [6, Corollary 4].

In the next result a bound on the number of steps of the inner iteration that depends only on k_1^0 and k_2^0 is provided.

Theorem 2 *If Newton-Kantorovich method is used in Algorithm 1 then each inner iteration terminates in at most $N^0\left(k_1^0, k_2^0\right)$ steps, where*

$$N^0\left(k_1^0,k_2^0\right) = \text{integer}\,part\left[\log_2\left(\frac{\log_2\left(x_{N^0}\right)}{\log_2\left[\left(1-\sqrt{1-2k_2^0} - k_2^0\right)/k_2^0\right]}\right)\right] \tag{42}$$

and

$$x_{N^0}=\frac{\left(1-pk_2^0/\sqrt{2}\right)\left[t_2^0-\left(1-\sqrt{1-2k_2^0}\right)^2\right]k_1^0}{2\sqrt{2}t_2^0\sqrt{1-2k_2^0}\left[\sqrt{\psi_2^0}+1-\sqrt{1-2k_2^0}\right]\left(1+k_1^0\right)}. \tag{43}$$

If the modified the Newton-Kantorovich method is used in Algorithm 1, then each iteration terminates in at most $S^0\left(k_1,k_2\right)$ steps, where

$$S^1\left(k_1^1,k_2^1\right)=\text{integer}\,part\left[\frac{\log_2\left(x_{S^1}\right)}{\log_2\left(1-\sqrt{1-\sqrt{1-2k_2^1}}\right)}\right]+1 \tag{44}$$

and

$$x_{S^1}=\frac{\left(1-pk_2^1/\sqrt{2}\right)\left[t_2^1-\left(1-\sqrt{1-2k_2^1}-k_2^1\right)^2\right]k_1^0}{2\sqrt{2}\sqrt{1-2k_2^1}\left(1-\sqrt{1-2k_2^1}-k_2^1\right)^2\left(\sqrt{\psi_2^1}+1-\sqrt{1-2k_2^1}\right)\left(1+k_1^1\right)}.$$

Remark 1 *Clearly, if $k_1^1=k_1^0=k_1$, $k_2^1=k_2^0=k_2$, $k^1=k^0=k$, Theorem 1 reduces to the corresponding Theorem 2 in [6]. Otherwise the following improvement hold:*

$$N^0\left(k_1^0,k_2^0\right)<N\left(k_1,k_2\right),\ N^0<N,\ S^1\left(k_1^1,k_2^1\right)<S\left(k_1,k_2\right)\ \textit{and}\ S^1<S.$$

Since $\frac{k_1}{k_1^0}$, $\frac{k_2}{k_2^0}$, $\frac{k_1}{k_1^1}$ and $\frac{k_2}{k_2^1}$ can be arbitrarily large [1]–[3] for given triplet β,γ,η, w_1 and w_0, the choices

$$k_1^0=k_1^1=.12,\quad k_2^0=k_2^1=.24,\quad \textit{when}\quad k_1=.21,\ \textit{and}\ k_2=.42$$

and

$$k_1^0=k_1^1=.24,\quad k_2^0=k_2^1=.48,\quad \textit{when}\quad k_1=.245,\ \textit{and}\ k_2=.49$$

are possible.

Then using formulas (41), (42) and (44), we obtain the following tables:

(a) If the HLCP is monotone and only Newton directions are performed, then:

Potra (40)	Argyros (40)
$\chi(.21,.42)>.17$	$\chi(.12,.24)>.09$
$\chi(.245,.49)>.199$	$\chi(.24,.48)>.190$

Potra (42)	Argyros (42)
$N(.21,.42)=2$	$N(.12,.24)=1$
$N(.245,.49)=4$	$N(.24,.48)=3$

(b) If the HLCP is monotone and Modified Newton directions are performed:

Potra (40)	Argyros (40)
$\chi(.21,.42)>.149$	$\chi(.12,.24)>.097$
$\chi(.245,.49)>.164$	$\chi(.24,.48)>.160$

Potra (44)	Argyros (44)
$S(.21,.42)=5$	$S(.12,.24)=1$
$S(.245,.49)=18$	$S(.24,.48)=12$

All the above improvements are obtained under weaker hypotheses and the same computational cost (in the case of Newton's method) or less computational cost (in the case of the modified Newton method) since in practice the computation of w requires that of w_0 and w_1. In general, the computation of w_0 is less expensive than that of w.

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