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On Powers And Roots Of Triangular Toeplitz Matrices*

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Abstract

In the present work, a general expression for the positive integer power of triangular Toeplitz matrices is suggested. Furthermore, it is given the structure of the fractional power of this matrix. As an application, an algorithm for computing the square root of the triangular Toeplitz matrix is considered.

1 Introduction

The powers and the roots of a matrix have been studied by many mathematicians. (see [1, 6, 8]). Newtons binomial theorem has been widely used to calculate the positive integer and fractional powers of square matrices as reported in the references [1, 2, 3, 4, 5].

The objective of this article is to study a class of matrices called Toeplitz matrices. The aim is to prove that the positive integer or fractional power of a triangular Toeplitz matrix always remains a triangular Toeplitz matrix. The square root of a triangular Toeplitz matrix has been considered here as an application, an algorithm is established to calculate it, performed by Matlab software. Some examples are given to better illustrate the work.

The paper includes three sections. In section 2, it is presented the result obtained from Theorem 1, in which it is demonstrated that the positive integer power of a triangular Toeplitz matrix is still the same structure. In section 3, the same result is found for the fractional powers (or the roots) of a triangular Toeplitz matrix. Finally, a formula to calculate the square root of a triangular Toeplitz matrix is suggested in section 4. An algorithm is performed by Matlab software and some concrete examples can be applied to illustrate this study. The paper is ended by a conclusion and some future works.

2 Positive Integer Power of Triangular Toeplitz Matrices

In this section, it is shown that the positive integer power of the triangular Toeplitz matrix stays triangular Toeplitz. We start with a result of the power of triangular matrices in the following Lemma.

Lemma 1 *If $A = (a_{ij})_{n\times n}$ is a triangular matrix, then $A^p = (\alpha_{ij})_{n\times n}$ is of the same type for all positive integer p. Furthermore,*

(1) If A is an upper triangular matrix, we have

$$\alpha_{ij} = \begin{cases} 0 & \text{if } i > j, \\ a_{ii}^p & \text{if } i = j, \\ \sum\limits_{i \le k_1 \le \cdots \le k_{p-1} \le j} a_{ik_1} a_{k_1k_2} \cdots a_{k_{p-1}j} & \text{if } i < j. \end{cases} \tag{1}$$

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(2) If A is a lower triangular matrix, we have

$$\alpha_{ij}=\begin{cases}0 & \text{if } i<j,\\ a_{ii}^{p} & \text{if } i=j,\\ \sum\limits_{j\leq k_1\leq\cdots\leq k_{p-1}\leq i} a_{ik_1}a_{k_1k_2}\cdots a_{k_{p-1}j} & \text{if } i>j.\end{cases} \tag{2}$$

For more details see [6]. In this paper, an $n\times n$ Toeplitz matrix T is denoted by

$$T=(t_{ij})=(t_{i-j}),\ \text{ for } 1\leq i,j\leq n. \tag{3}$$

Theorem 1 *Let A be an $n\times n$ triangular Toeplitz matrix. Then $A^p=(\alpha_{ij})_{n\times n}$ is still a triangular Toeplitz matrix for all positive integer p. Furthermore:*

1. *If A is an upper triangular Toeplitz matrix, we have*

$$\alpha_{ij}=\begin{cases}0 & \text{if } i>j,\\ a_{ii}^{p} & \text{if } i=j,\\ \sum\limits_{i\leq k_1\leq\cdots\leq k_{p-1}\leq j} a_{ik_1}a_{k_1k_2}\cdots a_{k_{p-1}j} & \text{if } i<j.\end{cases} \tag{4}$$

2. *If A is a lower triangular Toeplitz matrix, we have*

$$\alpha_{ij}=\begin{cases}0 & \text{if } i<j,\\ a_{ii}^{p} & \text{if } i=j,\\ \sum\limits_{j\leq k_1\leq\cdots\leq k_{p-1}\leq i} a_{ik_1}a_{k_1k_2}\cdots a_{k_{p-1}j} & \text{if } i>j.\end{cases} \tag{5}$$

Proof. Let A be an $n\times n$ triangular Toeplitz matrix, and $p\in\mathbb{N}$. From Lemma 1, it is clear that A^p is a triangular matrix. By subtitution in relation (1) and (2), we obtain the expressions (4) and (5) of A^p. To prove that A^p is a Toeplitz matrix, we verify that $\alpha_{ij}=\alpha_{i+1j+1}$, for each $i,j\in\{1,\cdots,n-1\}$. So we have two cases:

1. **case 1**: If A is an upper triangular matrix, then we have
   (a) if $i>j$, then $\alpha_{ij}=\alpha_{i+1j+1}=0$.
   (b) if $i=j$, then

$$\begin{aligned}\alpha_{i+1j+1} &= a_{i+1j+1}^{p}\\ &= a_{ij}^{p}\ \ \text{(because } A \text{ is Toeplitz)}\\ &= \alpha_{ij}.\end{aligned}$$

   (c) if $i<j$, then

$$\begin{aligned}\alpha_{ij} &= \sum_{i\leq k_1\leq\cdots\leq k_{p-1}\leq j} a_{ik_1}a_{k_1k_2}\cdots a_{k_{p-1}j}\\ &= \sum_{i+1\leq k_1+1\leq\cdots\leq k_{p-1}+1\leq j+1} a_{i+1k_1+1}a_{k_1+1k_2+1}\cdots a_{k_{p-1}+1j+1}\ \ \text{(because } A \text{ is Toeplitz)}\\ &= \sum_{i+1\leq h_1\leq\cdots\leq h_{p-1}\leq j+1} a_{i+1h_1}a_{h_1h_2}\cdots a_{h_{p-1}j+1}\\ &= \alpha_{i+1j+1},\end{aligned}$$

   where $h_s=k_s+1$, with $s\in\{1,\cdots,p-1\}$.

Consequently, A^p is Toeplitz matrix.

2. **case 2** : If A is a lower triangular matrix, we have:

   (a) if $i < j$, then $\alpha_{ij} = \alpha_{i+1j+1} = 0$.

   (b) if $i = j$, then

$$\begin{aligned}\alpha_{i+1j+1} &= a^p_{i+1j+1} \\ &= a^p_{ij} \quad \text{(because } A \text{ is Toeplitz)} \\ &= \alpha_{ij}.\end{aligned}$$

   (c) if $i > j$, then

$$\begin{aligned}\alpha_{ij} &= \sum_{j\le k_1\le\cdots\le k_{p-1}\le i} a_{ik_1}a_{k_1k_2}\cdots a_{k_{p-1}j} \\ &= \sum_{j+1\le k_1+1\le\cdots\le k_{p-1}+1\le i+1} a_{i+1k_1+1}a_{k_1+1k_2+1}\cdots a_{k_{p-1}+1j+1} \quad \text{(because } A \text{ is Toeplitz)} \\ &= \sum_{j+1\le h_1\le\cdots\le h_{p-1}\le i+1} a_{i+1h_1}a_{h_1h_2}\cdots a_{h_{p-1}j+1} \\ &= \alpha_{i+1j+1},\end{aligned}$$

   where $h_s = k_s + 1$, with $s \in \{1, \cdots, p-1\}$.

Consequently, A^p is Toeplitz matrix.

By using the notation in (3), the expressions (4) and (5) of A^p are written as follows:

1. If $A = (a_{i-j})_{n\times n}$ is an upper triangular Toeplitz matrix, we have

$$\alpha_{ij} = \alpha_{i-j} = \begin{cases} 0 & \text{if } i > j, \\ a_0^p & \text{if } i = j, \\ \sum\limits_{i\le k_1\le\cdots\le k_{p-1}\le j} a_{i-k_1}a_{k_1-k_2}\cdots a_{k_{p-1}-j} & \text{if } i < j. \end{cases} \tag{6}$$

2. If $A = (a_{i-j})_{n\times n}$ is a lower triangular Toeplitz matrix, we have

$$\alpha_{ij} = \alpha_{i-j} = \begin{cases} 0 & \text{if } i < j, \\ a_0^p & \text{if } i = j, \\ \sum\limits_{j\le k_1\le\cdots\le k_{p-1}\le i} a_{i-k_1}a_{k_1-k_2}\cdots a_{k_{p-1}-j} & \text{if } i > j. \end{cases} \tag{7}$$

■

Example 1 *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

To compute A^3 we put $A^3 = (\alpha_{ij})_{1\le i,j\le 4}$. Using (7) we have

$$\alpha_{ij} = \begin{cases} 0 & \textit{if } i < j, \\ a_0^3 & \textit{if } i = j, \\ \sum\limits_{j\le k_1\le k_2\le i} a_{i-k_1}a_{k_1-k_2}a_{k_2-j} & \textit{if } i > j. \end{cases}$$

Then

$$A^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ 21 & 6 & 1 & 0 \\ 56 & 21 & 6 & 1 \end{pmatrix}.$$

It is clear that A^3 is a lower triangular Toeplitz matrix.

Remark 1 *In [7], if an upper (respectively: a lower) triangular Toeplitz matrix $A = (a_{ij})_{1\le i,j\le n}$ is invertible, then its inverse $A^{-1} = (a_{ij}^{-1})_{1\le i,j\le n}$ is Toeplitz, because the product of two upper (respectively: a lower) triangular Toeplitz matrices is again an upper (respectively: a lower) triangular Toeplitz matrix. Furthermore the diagonal elements of A^{-1} are inverses of that of A, i-e: $a_{i,i}^{-1} = \dfrac{1}{a_{ii}}$ for each $1 \le i, j \le n$.*

Remark 2 *In [7], since an upper (respectively: a lower) unitriangular matrix is always invertible and its inverse is an upper (respectively: a lower) unitriangular matrix, the inverse of any upper (respectively: a lower) unitriangular Toeplitz matrix is also an upper (respectively: a lower) unitriangular Toeplitz matrix.*

Example 2 *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}.$$

Then, we have $spectre(A) = \{1\}$ and $\det(A) = 1 \neq 0$ so A is invertible. The polynomial characteristic of A is $P(X) = (X-1)^3$. By Cayley Hamilton Theorem A is a root of P then

$$\begin{aligned} P(A) = 0 &\Longrightarrow (A - I)^3 = 0 \\ &\Longrightarrow A^3 - 3A^2 + 3A = I \\ &\Longrightarrow A(A^2 - 3A + 3I) = I \\ &\Longrightarrow A^{-1} = A^2 - 3A + 3I \\ &\Longrightarrow A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}. \end{aligned}$$

Observe then that A^{-1} is a lower triangular Toeplitz matrix.

3 Roots of Triangular Toeplitz Matrices

The square roots of triangular matrices are not necessarily triangular matrices. For example,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}.$$

In this section, based on the work of Arias et al. in [1], we use the Newton's Binomial Theorem to prove that every triangular Toeplitz matrix has a p^{th} ($p \in \mathbb{N}^*$) root with the same structure (Triangular Toeplitz matrix).

The Newton's Binomial Theorem can be expressed by the next expression

$$(1 + X)^r = \sum_{k=0}^{\infty} \binom{r}{k} X^k \tag{8}$$

for any real number r that is not a non-negative integer, when $|X| < 1$ and

$$\binom{r}{k} = \frac{r\,(r-1)\,(r-2)\cdots(r-k+1)}{k!}.$$

Let A be an $n \times n$ triangular Toeplitz matrix. Then we can write A as the following

$$A = I + B.$$

So B is a triangular Toeplitz matrix. By using (8) we have

$$\begin{aligned} A^{\frac{1}{p}} &= (I+B)^{\frac{1}{p}} \\ &= \sum_{k=0}^{\infty} \binom{\frac{1}{p}}{k} B^k \\ &= I + \frac{1}{p}B + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)}{2!}B^2 + \cdots . \end{aligned} \tag{9}$$

The series will converge if: $\|B\| < 1$.

From theorem 1, B^k is a triangular Toeplitz matrix, Furthermore, for all positive integer k: B^k is a strictly triangular Toeplitz matrix wich is nilpotent then its sum is finite and remains a triangular Toeplitz matrix.

Example 3 *Let A be a triangular Toeplitz matrix:*

$$A = \begin{pmatrix} 1.25 & 0.10 & 0.15 \\ 0 & 1.25 & 0.10 \\ 0 & 0 & 1.25 \end{pmatrix}.$$

Take $A = I + B$ with

$$B = \begin{pmatrix} 0.25 & 0.10 & 0.15 \\ & 0.25 & 0.10 \\ & & 0.25 \end{pmatrix},$$

then we have $\|B\| < 1$. We suppose that $p = 3$. The matrix 3^{rd} root R of A is given by the following (we will take the series (9) in the term of 4th degree).

$$\begin{aligned} R = A^{\frac{1}{3}} &= (I+B)^{\frac{1}{3}} \\ &= I + \frac{1}{3}B - \frac{1}{9}B^2 + \frac{5}{81}B^3 - \frac{10}{243}B^4 \\ &= \begin{pmatrix} 1.0772 & 0.0287 & 0.0422 \\ 0 & 1.0772 & 0.0287 \\ 0 & 0 & 1.0772 \end{pmatrix}. \end{aligned}$$

To evaluate the approximation, we have

$$R^3 = \begin{pmatrix} 1.2499 & 0.0998 & 0.1496 \\ 0 & 1.2499 & 0.0998 \\ 0 & 0 & 1.2499 \end{pmatrix} \simeq A.$$

4 Square Roots of Triangular Toeplitz Matrix

In this section, it is presented an algorithm to find the triangular Toeplitz square root of a triangular Toeplitz matrix.

Let $A \in M_{n\times n}(\mathbb{R})$ be a lower triangular Toeplitz matrix with even order n ($n = 2k$) and strictly positive elements ($a_i > 0$ for all $i = 1, \cdots, n$)

$$A = \begin{pmatrix} a_1 & 0 & \cdots & \cdots & 0 \\ a_2 & a_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ a_{n-1} & & \ddots & a_1 & 0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{pmatrix}$$

and R its square root, so R is a lower triangular Toeplitz matrix and is written by

$$R = \begin{pmatrix} r_1 & 0 & \cdots & \cdots & 0 \\ r_2 & r_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r_{n-1} & & \ddots & r_1 & 0 \\ r_n & r_{n-1} & \cdots & r_2 & r_1 \end{pmatrix}.$$

Then we have

$$R^2 = A \Longrightarrow \begin{cases} r_1^2 = a_1 \\ 2r_1r_2 = a_2 \\ 2r_1r_3 + r_2^2 = a_3 \\ 2r_1r_4 + 2r_2r_3 = a_4 \\ \vdots \\ 2r_1r_{n-1} + 2r_2r_{n-2} + 2r_3r_{n-3} + \cdots + 2r_{\frac{n}{2}-1}r_{\frac{n}{2}+3} + 2r_{\frac{n}{2}}r_{\frac{n}{2}+2} + r^2_{\frac{n}{2}+1} = a_{n-1} \\ 2r_1r_{n+2} + 2r_2r_{n-1} + 2r_3r_{n-2} + \cdots + 2r_{\frac{n}{2}-1}r_{\frac{n}{2}+2} + 2r_{\frac{n}{2}}r_{\frac{n}{2}+1} = a_n \end{cases}$$

$$\Longrightarrow \begin{cases} r_1 = \pm\sqrt{a_1} \\ r_2 = a_2/2r_1 \\ r_3 = \left(a_3 - r_2^2\right)/2r_1 \\ r_4 = \left(a_4 - 2r_2r_3\right)/2r_1 \\ \vdots \\ r_{2k} = \left(a_{2k} - 2r_2r_{2k-1} - 2r_3r_{2k-2} - \cdots - 2r_{k-1}r_{k+2} - 2r_kr_{k+1}\right)/2r_1 \\ r_{2k+1} = \left(a_{2k+1} - 2r_2r_{2k-2}r_3r_{2k-1} - \cdots - 2r_{k-1}r_{k+3} - 2r_kr_{k+2} - r^2_{k+1}\right)/2r_1 \\ \vdots \\ r_{n-1} = \left(a_{n-1} - 2r_2r_{n-2} - 2r_3r_{n-3} - \cdots - 2r_{\left[\frac{n-1}{2}\right]-1}r_{\left[\frac{n-1}{2}\right]+3} - 2r_{\left[\frac{n-1}{2}\right]}r_{\left[\frac{n-1}{2}\right]+2} - r^2_{\left[\frac{n-1}{2}\right]+1}\right)/2r_1 \\ r_n = \left(a_{n-2}r2r_{n-1} - 2r_3r_{n-2} - \cdots - 2r_{\frac{n}{2}-1}r_{\frac{n}{2}+2} - 2r_{\frac{n}{2}}r_{\frac{n}{2}+1}\right)/2r_1. \end{cases}$$

The solutions of this system of n equations are the entries of the matrix R, and it's given by the following recurrence formula:

$$\begin{cases} r_1 = \pm\sqrt{a_1}, & r_2 = a_2 \diagup 2r_1, \\ r_2 = a_2 \diagup 2r_1, & r_3 = \left(a_3 - r_2^2\right) \diagup 2r_1, \\ r_3 = \left(a_3 - r_2^2\right) \diagup 2r_1, & r_i = \left(a_i - 2\sum\limits_{k=1}^{\frac{i}{2}-1} r_{k+1} r_{i-k}\right) \diagup 2r_1, \\ r_i = \left(a_i - 2\sum\limits_{k=1}^{\frac{i}{2}-1} r_{k+1} r_{i-k}\right) \diagup 2r_1, & \text{if } i \text{ is even}, \\ r_i = \left(a_i - 2\sum\limits_{k=1}^{\left[\frac{i}{2}\right]-1} r_{k+1} r_{i-k} - r^2_{\left[\frac{i}{2}\right]+1}\right) \diagup 2r_1, & \text{if } i \text{ is odd}. \end{cases}$$

Application in Matlab : The following algorithm can compute the square root of a lower triangular Toeplitz matrix.

```
clear all
clc
n=input('donner la taille de la matrice ');
i=0;j=1; m=n;
A=zeros(n,n);
for kk=1:n
fprintf('donner la valeur de A%d=',kk);
x=input(' ');
for k=1:m
i=i+1;
A(i,k)=x;
end
m=m-1;
i=n-m;
end
A
R=zeros(n,n);
TR=zeros(1,n);%table des racines
TR(1)=sqrt(A(1,1));
TR(2)=A(2,1)/(2*TR(1));
TR(3)=(A(3,1)-TR(2)^2)/2*TR(1);
spair=0; simpair=0;
for i=4:n
if mod(i,2)==0 %pour i pair
for k=1:i/2-1
spair=spair+TR(k+1)*TR(i-k);
end
TR(i)=(A(i,1)-2*spair)/2*TR(1);

else %i impair
for k=1:fix(i/2)-1
```

```
simpair=simpair+TR(k+1)*TR(i-k);
end
TR(i)=(A(i,1)-2*simpair-TR(fix(i/2)+1)^2)/2*TR(1);
end
end
TR   %la matrice des racines
i=0;j=1; m=n;
for kk=1:n
x=TR(kk);
for k=1:m
i=i+1;
R(i,k)=x;
end
m=m-1;
i=n-m;
end
R
```

Example 4 *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 2 & 1 & 0 & \ddots & & \vdots \\ 3 & 2 & 1 & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & & \\ n-1 & & & \ddots & 1 & 0 \\ n & n-1 & \cdots & & 2 & 1 \end{pmatrix}.$$

It's clear that A is non singular matrix and it has a square root R given by

$$R = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & \ddots & & \vdots \\ 1 & 1 & 1 & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & & \\ 1 & & & \ddots & 1 & 0 \\ 1 & 1 & \cdots & & 1 & 1 \end{pmatrix}.$$

Note that $R^2 = A$ and A and R has the same structure.

Conclusion and Perspectives

In this work, it is studied the powers and roots of triangular Toeplitz matrices. A new formula to compute the positive integer power of a triangular Toeplitz matrix is suggested and an algorithm computing its square root is established. Furthermore, a description of the structure of the p^{th} root of this type of matrices is given. One can easily find the formulas enabling us to give the entries of the p^{th} root or to study other properties.

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