

A Note On The Boundedness Of Solutions Of Generalized Functional Differential Equations*

Juan Eduardo Napoles Valdes^{1,2†}, Sayooj Aby Jose^{3,4‡}

Received 7 January 2021

Abstract

This work presents some asymptotic properties of the solutions, mainly related to the boundedness, of certain functional differential equations in the framework of the generalized local derivative.

1 Introduction

The development of differential operator theory has taken two very clear directions in recent years: both global and local operators. In the case of the first they are defined by integral global transformations, that is, their nature is not local, in other words they have “memory”, in the case of local operators, they are defined by means of the limit of a certain incremental quotient, that is, they are defined locally. Classical global operators are associated with the emergence of Fractional Calculus itself, that is, in the original works of the founders of this area, even the establishment of the classical definitions of fractional calculus, and the local are directly linked to the classical notion of derivative.

Generalized differential equations and fractional differential equations have recently been shown to be a useful resource in modelling many phenomena in different fields of science and engineering. An excellent account in the study and analysis of fractional differential equations and generalized differential equation can be found in [3, 6, 9, 20, 31, 32, 33, 34]. In [19] Khalil et al. (also cf. [1]) defined a local derivative, no fractional, that he called conformable, which is based on the classical definition of the derivative, that is, using the limit of a certain incremental quotient. Thus, define the conformable derivative of order α of the function h is given by the following limit, if it exists

$$T_\alpha(h)(\tau) = \lim_{\epsilon \rightarrow 0} \frac{h(\tau + \epsilon\tau^{1-\alpha}) - h(\tau)}{\epsilon}, \quad \alpha \in (0, 1), \tau > 0.$$

Later, in 2018, a local derivative of a new type, called non-conformable, is defined with different behaviors than the previous one when α tends to 1.

Definition 1 ([14]) *Given a function $f : [0, +\infty) \rightarrow \mathbb{R}$. The N -derivative of f of order α is defined by $N_1^\alpha f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t+\epsilon e^{t-\alpha}) - f(t)}{\epsilon}$ for all $t > 0$, $\alpha \in (0, 1)$. If f is α -differentiable in some $(0, a)$, and $\lim_{t \rightarrow 0^+} N_1^{(\alpha)} f(t)$ exists, then define $N_1^{(\alpha)} f(0) = \lim_{t \rightarrow 0^+} N_1^{(\alpha)} f(t)$.*

The following definition presents a generalized derivative defined in [26] (see also [35]):

Definition 2 *Let $h : [0, +\infty) \rightarrow \mathbb{R}$, $\alpha \in (0, 1)$ and $F(\cdot, \alpha)$ be some function. We define the derivative N_F^α of order α of a function h by the following limit, if it exists*

$$N_F^\alpha h(\tau) = \lim_{\epsilon \rightarrow 0} \left[\frac{h(\tau + \epsilon F(\tau, \alpha)) - h(\tau)}{\epsilon} \right], \quad t > 0.$$

*Mathematics Subject Classifications: 34A08; 34K12; 34K25.

[†]Department of Mathematics, UNNE-FACENA, Corrientes 3400, Argentina ²Department of Mathematics, UNNE-FACENA, Corrientes 3400, Argentina

[‡]Ramanujan Centre for Higher Mathematics, Alagappa University, Karaikudi-630 004, India ⁴Department of Mathematics, Alagappa University, Karaikudi-630 004, India

We can define $N_F^\alpha h(0) = \lim_{\tau \rightarrow 0^+} [N_F^\alpha h(\tau)]$, in the case that h is N -differentiable in some $0 < \alpha \leq 1$, and $\lim_{\tau \rightarrow 0^+} N_F^\alpha h(\tau)$ exists.

Remark 1 It is clear that this definition encompasses both conformable and non-conformable derivatives, which have appeared in recent years (see also [25]).

Remark 2 We will cite a group of works, where the generalized derivative $N_F^\alpha f(t)$ has been used and that demonstrate its usefulness in the modeling of different processes and phenomena, due to the double dependency of the kernel and the order (see the results obtained in [2, 11, 12, 17, 24, 27, 28, 29]) all of which shows the strength of this new tool in applications.

Remark 3 From the above definition, it is not difficult to extend the order of the derivative for $0 \leq n - 1 < \alpha \leq n$ by putting

$$N_F^\alpha h(\tau) = \lim_{\varepsilon \rightarrow 0} \left[\frac{h^{(n-1)}(\tau + \varepsilon F(\tau, \alpha)) - h^{(n-1)}(\tau)}{\varepsilon} \right].$$

If $h^{(n)}$ exists on some interval $I \subseteq \mathbb{R}$, then we have $N_F^\alpha h(\tau) = F(\tau, \alpha)h^{(n)}(\tau)$, with $0 \leq n - 1 < \alpha \leq n$.

Next we define a generalized integral operator, studied in detail in [18], see also [35]. In this definition, we take the kernel F as an absolutely continuous function.

Definition 3 Consider the real interval $I := [a, b]$, with $a < b$, $\tau \in I$ and $\alpha \in \mathbb{R}$. We define the integral operators, right side and left side, for a function h , locally integrable by the following expressions

$$J_{F,a+}^\alpha h(\tau) = \int_a^\tau \frac{h(s)}{F(\tau - s, \alpha)} ds, \quad a < \tau,$$

and

$$J_{F,b-}^\alpha h(\tau) = \int_\tau^b \frac{h(s)}{F(s - \tau, \alpha)} ds, \quad b > \tau.$$

Remark 4 We will also use the “central” integral operator defined by (see [15] and [35])

$$J_{F,a}^\alpha h(b) = \int_a^b \frac{h(\tau)}{F(\tau, \alpha)} dt, \quad a < b.$$

Remark 5 One of the strengths of this operator J_F^α is that it contains, for adequate choices of the kernel F , well-known integral operators, both fractional and generalized, reported in many works, both generated by conformable kernels or not. In particular, if $F(t - s, \alpha) = (t - s)^{1-\alpha}$ of the Definition 6, we obtain the classic Riemann-Liouville Fractional Integral.

The following result is similar to a known result from classical calculus (see [15]).

Theorem 1 If $\tau > \tau_0$, and h is an N -differentiable function on (τ_0, ∞) with $\alpha \in (0, 1]$, then it is true that

a) $J_{F,\tau_0}^\alpha (N_F^\alpha h(\tau)) = h(\tau) - h(\tau_0)$ with h being differentiable;

b) $N_F^\alpha (J_{F,\tau_0}^\alpha h(\tau)) = h(\tau)$.

Proof.

a) Using the definitions of the integral and differential operator, we have

$$J_{F,\tau_0}^\alpha (N_F^\alpha h(\tau)) = \int_{\tau_0}^\tau \frac{N_F^\alpha h(s)}{F(s, \alpha)} ds = \int_{\tau_0}^\tau \frac{h'(s)F(s, \alpha)ds}{F(s, \alpha)} = h(\tau) - h(\tau_0).$$

b) Analogously, we have

$$N_F^\alpha (J_{F,\tau_0}^\alpha h(\tau)) = F(s, \alpha) \frac{d}{d\tau} \left[\int_{\tau_0}^t \frac{h(s)}{F(s, \alpha)} ds \right] = h(\tau).$$

■

Remark 6 From the definition of the integral operator J , it is easy to deduce that if h is differentiable, then $J_{F,a}^\alpha h(b) = J_{F,a}^\alpha h(\tau) - J_{F,b}^\alpha h(\tau)$.

It is clear that many “classical” properties of integration theory can be proved without much difficulty. For example, we can prove the well-known mean value theorems for integral calculus (see [18]).

Theorem 2 For a continuous function h defined on $[a, b]$, with $0 < a < b$, then there exists a value c in the interval (a, b) satisfying $J_{F,a}^\alpha h(b) = h(c)(b - a)$.

Theorem 3 For a continuous function h defined on $[a, b]$, g is an integrable function that does not change sign on $[a, b]$, then there exists a value c in the interval (a, b) satisfying $J_{F,a}^\alpha (hg)(b) = h(c)J_{F,a}^\alpha g(b)$.

Remark 7 Using the definition of the integral operator J , and the iterated use of Theorem 1, if $f^{(n-1)}$ exists on some interval $I \subseteq \mathbb{R}$, then we can obtain

$$J_{F,a}^\alpha (N_F^\alpha h(\tau)) = h(\tau) - \left[\sum_{i=0}^{n-1} \frac{h^{(i)}(a)}{i!} (\tau - a)^i \right],$$

with $\tau \in I$, and $0 \leq n - 1 < \alpha \leq n$.

In this paper, we presented a functional generalized integral inequalities using the above generalized derivative, and we will use them in the study of the qualitative behavior of generalized functional equations, mainly referred to the boundedness of solutions. For all the above, we realize that the subject of this work is recent, and tries to extend known results to the case of fractional differential equations and functional equations, to the case of generalized functional differential equations. While there are some attempts in the aforementioned areas (see survey [30], [5] and the references cited there), these are referred to the classical (global) fractional case and the ordinary functional equations; hence our work establishes a work area unknown until now, to the knowledge of the authors.

2 Properties

In 1919, Gronwall proved a remarkable differential inequality. The integral form of this inequality was proven by Richard Bellman in 1943, and a non-linear version was demonstrated, independently, by LaSalle in 1949 and Bihari in 1956. By that, this inequality is sometimes called, the inequality of Gronwall-Bellman-Bihari, this inequality has been the object of much attention in the last 50 years, because it has been used, and is used, in the qualitative study of the solutions of differential and integral equations in various contexts. The first use of the Gronwall inequality to establish boundedness and stability is due to Bellman, and today there are innumerable applications to the study of various qualitative properties in differential, integral and integro-differential equations.

Next, we present a generalized version of Gronwall’s Inequality, which will show its usefulness in studying the stability, according to Lyapunov, of solutions of systems of generalized differential equations.

Theorem 4 Let r, a, φ be a continuous, nonnegative functions defined on $[0, b]$, with $0 < b$, $\varphi(\tau) \leq t$ and c nonnegative constant such that $(0 < \alpha \leq 1)$:

$$r(\tau) \leq c + J_{F,a}^\alpha (a(s)r(s))(\varphi(\tau)).$$

Then we have

$$r(\tau) \leq c + cJ_{F,0}^\alpha a(z)e^{J_{F,z(s)}^\alpha a(u)(z(\tau))} dz,$$

for all $t \in [a, b]$.

Proof. Define $R(\tau) = J_{F,0}^\alpha (a(s)r(s))(\varphi(\tau))$. Then $R(0) = 0$ since $\varphi(0) = 0$, $R(\tau) \geq r(\tau)$, and

$$N_F^\alpha R(\tau) = a(\varphi(\tau))r(\varphi(\tau))N_F^\alpha \varphi(\tau) \leq a(\varphi(\tau))r(\tau)N_F^\alpha \varphi(\tau).$$

From here we have

$$N_F^\alpha R(\tau) - a(\varphi(\tau))N_F^\alpha \varphi(\tau)R(\tau) \leq ca(\varphi(\tau))N_F^\alpha \varphi(\tau).$$

Multiplying the last inequality by $e^{-J_{F,0}^\alpha a(s)(\varphi(\tau))}$, we see that

$$N_F^\alpha \left[e^{-J_{F,0}^\alpha a(s)(\varphi(\tau))} R(\tau) \right] \leq ma(\varphi(\tau))N_F^\alpha \varphi(\tau)e^{-J_{F,0}^\alpha a(s)(\varphi(\tau))}.$$

Since $e^{J_{F,0}^\alpha a(s)(\varphi(\tau))} R(\tau)$ is differentiable on $(0, b)$ we have from Theorem 3 that:

$$R(\tau) \leq mJ_{F,0}^\alpha a(\varphi(\tau))N_F^\alpha \varphi(\tau)e^{-J_{\varphi(s)}^\alpha a(s)(\varphi(\tau))}(\tau).$$

Making a change of variables we obtain

$$R(\tau) \leq mJ_{F,0}^\alpha a(z)e^{-J_{\varphi(s)}^\alpha a(u)(\varphi(\tau))}(\varphi(\tau)).$$

From this, and the fact that $R(\tau) \geq r(\tau)$, we obtain the desired result. ■

Similarly, the following result can be proved.

Theorem 5 *Considering that $r(\tau)$, $c(\tau)$ and $d(\tau)$ are continuous functions defined on $[0, b]$, with $0 < b$ and φ as before. Suppose that on $[0, b]$ we have the inequality (with $d(\tau) > 0$ and $0 < \alpha \leq 1$):*

$$r(\tau) \leq c(\tau) + J_{F,a}^\alpha (d(s)r(s))(\varphi(\tau)).$$

Then, we have

$$r(\tau) \leq c(\tau) + J_{F,0}^\alpha (d(s)c(s))e^{-J_{F,0}^\alpha d(u)(\varphi(s))}(\varphi(\tau)),$$

for all $t \in [0, b]$.

More general is the following result, an inequality of the Bihari type.

Theorem 6 *Let $r(\tau)$ and $e(\tau)$ be real continuous functions defined on an interval $[0, b]$, with $0 < b, c > 0$. On $[0, b]$ we suppose φ as before, $e(\tau)$ nonnegative on $[0, b]$, be b a continuous function non decreasing on $[0, b]$ and $0 < \alpha \leq 1$, then have the inequality:*

$$r(\tau) \leq c + \leq B^{-1} [B(M) + J_{F,0}^\alpha a(z) (\varphi(\tau))], \quad B(u) = J_{F,0}^\alpha b^{-1}(z)(u).$$

Then we have

$$r(\tau) \leq c + J_{F,0}^\alpha (c(s)e(s))e^{-J_{F,0}^\alpha d(u)e(u)(s)}(\tau),$$

for all $t \in [0, b]$.

Proof. This proof is typically as those Gronwall's inequalities type. We consider the function $R(\tau) = J_{F,0}^\alpha (e(s)b(r(s)))(\varphi(\tau))$. Then, we have $R(0) = 0$ and

$$N_F^\alpha R(\tau) \leq a(\varphi(\tau))b(\varphi(\tau))N_F^\alpha \varphi(\tau) \leq a(\varphi(\tau))N_F^\alpha \varphi(\tau)b(M + R(\tau)).$$

From this we have:

$$\frac{N_F^\alpha R(\tau)}{b(M + R(\tau))} \leq a(\varphi(\tau))N_F^\alpha \varphi(\tau).$$

Separating variables, and integrating, we obtain

$$M + R(\tau) \leq B^{-1} [B(M) + J_{F,0}^\alpha a(z) (\varphi(\tau))], \quad B(u) = J_{F,0}^\alpha b^{-1}(z)(u).$$

From this and knowing that $r(\tau) \leq c + R(\tau)$, we get the proof of the theorem. ■

3 On the Boundedness of Solutions

Consider a paradigmatic case in the nonlinear analysis, the Liénard N-generalized functional system:

$$\left. \begin{aligned} N_F^\alpha x(\tau) &= y, \\ N_F^\alpha y(\tau) &= -g(x(\varphi(\tau))) - f(x)y. \end{aligned} \right\} \tag{1}$$

Under assumptions on the continuous functions involved:

1. $f(x) > 0$ for all $x \in \mathbb{R}$,
2. $xg(x) > 0$ for all $x \neq 0$,
3. $\varphi(\tau) \leq \tau$.

Consider the following equation equivalent to the system (1)

$$\frac{dy}{dx} = \frac{-g(x(\varphi(\tau))) - f(x)y}{y}. \tag{2}$$

We have the following result:

Lemma 1 *Under above assumptions on f, g, φ , the solutions of system (1), and, therefore, the solutions of its equivalent equation (2), do not admit vertical asymptotes.*

Proof. To prove this, it is sufficient to show that the solutions of the equation (2) they do not escape to infinity in a finite time, i.e., solutions do not admit vertical asymptotes. By the purpose of contradiction, let us suppose the existence of a solution to the equation (2)

$$y = \phi(x), \quad c \leq x < d$$

such that

$$\lim_{x \rightarrow d^-} \phi(x) = +\infty. \tag{3}$$

We can consider, without losing generality, that this solution satisfies $0 < \phi(a) \leq \phi(x)$ for $a \leq x < d$. Putting

$$F \geq \max_{c \leq x < d} |f(x)|, \quad G \geq \max_{c \leq x < d} |g(x)|,$$

and using the mean value theorem, we obtain

$$\phi(x) - \phi(a) \leq \left[\frac{F + G}{\phi(a)} \right] (d - c)$$

for $c < x < d$. But this contradicts (3). The other cases can be analyzed in a similar way. Thus, the proof is completed. ■

Remark 8 *The result obtained is equivalent to the continuation of the solutions of equation (2), and accordingly, of system (1). Similar results on the differential Liénard equation in other frameworks, can be consulted in [18] and [15].*

4 Conclusions

In this work, we have presented some results from a fairly new area: local functional differential equations, in this case generalized since we use the derivative of [14], hence the results obtained are not reported in the literature.

If in our work we consider $\varphi(\tau) = \tau$ and $F \equiv 1$, the results obtained are consistent with those of [22], where the existence of non-continuable solutions of a Liénard type non-linear equation is studied. Under the same previous conditions, our results coincide with those of [21], obtained using the Second Lyapunov Method, on the other hand, if we consider $F(t, \alpha) = e^{t^{-\alpha}}$ then our results are similar to those of [29] for a Liénard-type Equation, with the non-conformable derivative of [14].

Given the development that fractional and generalized calculus have undergone in recent years (we recommend [4, 7, 8, 13], the readers can consult [23] where a historical development related to the study of the qualitative properties of ordinary differential equations is presented, mainly the boundedness of the solutions), a development that leads to the intersection of global and local operators in the multiple definitions have been presented.

From the results obtained, it is clear that we can point out some possible directions of work:

1. Study of the stability of solutions of generalized functional equations, either from the point of view of the Second Method of Lyapunov or from the notion of Hyers-Ulam stability.
2. Study of qualitative properties of the solutions of new classes of generalized functional equations.
3. Applying the results obtained to the modeling of processes and phenomena, for example, the system (1) referred to a generalized functional Liénard Equation, can be the starting point for the investigation of the existence of periodic solutions or limit cycles in such systems.

Acknowledgment. The authors would like to thank the anonymous referee for his valuable comments.

References

- [1] T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.*, 279(2015), 57–66.
- [2] R. Abreu-Blaya, A. Fleitas, J. E. Nápoles Valdés, R. Reyes, J. M. Rodríguez and J. M. Sigarreta, On the conformable fractional logistic models, *Math. Methods Appl. Sci.*, 43(2020), 4156–4167.
- [3] M. J. Alves, Singular functional differential equation of the second order: Diss. kand. phys.-matem. nauk, Perm. University, Perm, (1999), 114 (Russian).
- [4] A. Atangana, Derivative with a New Parameter Theory, *Methods and Applications*, Academic Press, 2016.
- [5] N. V. Azbelev, V. P. Maksimov and L. F. Rakhmatullina, *Introduction to the Theory of Functional Differential Equations*, Nauka, Moscow, 1991.
- [6] N. V. Azbelev, V. P. Maksimov and L. F. Rakhmatullina, *Introduction to the Theory of Functional Differential Equations: Methods and Applications*, Hindawi Publishing Corporation, 2007.
- [7] D. Baleanu, Comments on: "The failure of certain fractional calculus operators in two physical models" by M. Ortigueira, V. Martynyuk, M. Fedula and J.A.T. Machado in: *Fract. Calc. Appl. Anal.* 22, No 2 (2019), 255–270, *Fract. Calc. Appl. Anal.*, 23(2020), 292–297.
- [8] D. Baleanu and A. Fernández, On fractional operators and their classifications, *Mathematics*, 7(2019), 830.

- [9] B. Khaminsou, C. Thaiprayoon, W. Sudsutad and S. A. Jose, Qualitative analysis of a proportional Caputo fractional Pantograph differential equation with mixed nonlocal conditions, *Nonlinear Funct. Anal. Appl.*, 26(2021), 197–223.
- [10] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, 1985.
- [11] A. Fleitas, J. F. Gómez-Aguilar, Juan E. Nápoles Valdés, José M. Rodríguez, José M. Sigarreta, Analysis of the local Drude model involving the generalized fractional derivative, *Optik - International Journal for Light and Electron Optics*, 193(2019) 163–008
- [12] A. Fleitas, J. A. Méndez, J. E. Nápoles and J. M. Sigarreta, On fractional Liénard-type systems, *Revista mexicana de fisica*, 65(2019), 618–625.
- [13] A. Fleitas, J. E. Nápoles Valdés, J. M. Rodríguez and J. M. Sigarreta, Note on the generalized conformable derivative. *Rev. Un. Mat. Argentina*, 62(2021), 443–457.
- [14] P. M. Guzmán, G. Langton, L. M. Lugo, J. Medina and J. E. Nápoles Valdés, A new definition of a fractional derivative of local type, *J. Math. Anal. Appl.*, 9(2018), 88–98.
- [15] P. M. Guzmán, L. M. Lugo and J. E. Nápoles Valdés, A Note on Stability of Certain Lienard Fractional Equation, *Int. J. Math. Comput. Sci.*, 14(2019), 301–315.
- [16] P. M. Guzmán, L. M. Lugo, J. E. Nápoles Valdés and M. Vivas-Cortez, On a new generalized integral operator and certain operating properties, *Axioms*, 9(2020), 69.
- [17] P. M. Guzmán and J. E. Nápoles Valdés, A note on the oscillatory character of some non conformable generalized lienard system, *Advanced Mathematical Models & Applications*, 4(2019), 127–133
- [18] P. M. Guzmán, L. Lugo Motta and J. E. Nápoles Valdés, On the stability of solutions of fractional non conformable differential equations, *Studia Universitatis Babes-Bolyai Mathematica*, 65(2020), 495–502.
- [19] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.*, 264(2014), 65–70.
- [20] A. A. Kirillov and A. D. Gvishiani, *Theorems and Problems of Functional Analysis*, Nauka, Moscow, (1979).
- [21] J. E. Nápoles Valdés, A continuation result for a bidimensional system of differential equations, *Rev. Integr. Temas Mat.*, 13(1995), 49–54.
- [22] J. E. Nápoles Valdés, On the continuability of solutions of bidimensional systems, *Extracta Math.*, 11(1996), 366–368.
- [23] J. E. Nápoles Valdés, The historical legacy of Ordinary Differential Equations, *Bol. Mat.*, 5(1998), 53–79.
- [24] J. E. Nápoles Valdés, Y. S. Gasimov and A. R. Aliyeva, On the oscillatory behavior of some generalized differential equation, *Punjab Univ. J. Math.*, 53(2021), 73–82.
- [25] J. E. Nápoles Valdés, P. M. Guzmán and L. M. Lugo, Some new results on nonconformable fractional calculus, *Adv. Dyn. Syst. Appl.*, 13(2018), 167–175.
- [26] J. E. Nápoles Valdés, P. M. Guzmán, L. M. Lugo and A. Kashuri, The local non-conformable derivative and Mittag Leffler function, *Sigma Journal of Engineering and Natural Sciences*, 38(2020), 1007–1017.
- [27] J. E. Nápoles Valdés and M. N. Quevedo Cubillos, On the oscillatory nature of some generalized Emden-Fowler equation, *Punjab Univ. J. Math.*, 52(2020), 97–106.
- [28] J. E. Nápoles Valdés, M. N. Quevedo Cubillos and A. R. Gómez Plata, On the asymptotic behavior of a generalized nonlinear equation, *Sigma J. Eng. & Nat Sci*, 38(2020), 2109–2121.

- [29] J. E. Nápoles Valdés and C. Tunc, On the boundedness and oscillation of non-conformable Lienard equation, *J. Fract. Calc. Appl.*, 11(2020), 92–101.
- [30] R. R. Nigmatullin and D. Baleanu, The derivation of the generalized functional equations describing self-similar processes, *Fract. Calc. Appl. Anal.*, 15(2012), 718–740.
- [31] S. A. Jose, W. Yukunthorn, J. E. Napoles and H. Leiva, Some existence, uniqueness and stability results of nonlocal random impulsive integro-differential equations, *Appl. Math. E-Notes*, 20(2020), 481–492.
- [32] S. A. Jose, A. Tom, S. Abinaya and W. Yukunthorn, Some characterization results of nonlocal special random impulsive differential evolution equation, *J. Appl. Nonlinear Dyn.*, 10(2021), 695–707.
- [33] S. A. Jose, A. Tom, M. S. Ali, S. Abinaya and W. Sudsutad, Existence, uniqueness and stability results of semilinear functional special random impulsive differential equations, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 28(2021), 269–293.
- [34] S. Pleumpreedaporn, W. Sudsutad, C. Thaiprayoon and S. A. Jose, Qualitative analysis of generalized proportional fractional functional integro-differential Langevin equation with variable coefficient and nonlocal integral conditions, *Mem. Differ. Equ. Math. Phys.*, 83(2021), 99–120.
- [35] D. Zhao and M. Luo, General conformable fractional derivative and its physical interpretation, *Calcolo*, 54(2017), 903–917.