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Inference Based On Record Values From Schabe Distribution*

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Abstract

In this paper, we discuss the lower record values arising from the Schabe distribution and derive the recurrence relations satisfied by single and product moments of these record values. With the help of these recurrence relations, we compute the means, variances and covariances of the lower record values. These values are then used to compute the best linear unbiased estimators (BLUEs) and the best linear invariant estimators (BLIEs) of the location and scale parameters. By using the BLUEs and BLIEs, we construct confidence intervals for the location and scale parameters through Monte Carlo simulations. Prediction for the future records is given in detail. Finally, simulation study is performed and prediction of future record, comparison between BLUE and BLIE are discussed.

1 Introduction

Let $\{X_1, X_2, ...\}$ be a sequence of independent and identically distributed (IID) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. An observation X_j is a lower record value of this sequence if it falls short of (in value) all preceding observations, i.e., if $X_j < X_i$, $i < j$. Upper records are analogously defined. Generally, if $L(n)$, $n \geq 1$, is defined by [Ahsanullah (1995)]

$$L(1) = 1, \quad L(n) = \min\left\{j : j > L(n-1),\ X_j < X_{L(n-1)}\right\},$$

then the sequence $\left\{X_{L(n)},\ n \geq 1\right\}$ provides a sequence of lower record statistics. The sequence $\{L(n),\ n \geq 1\}$ represents the lower record times. From the above definition, the sequence of record statistics can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of the observations. Note that from a sequence of n IID continuous random variables, only about $\log(n)$ records are expected, see Houchens (1984).

Chandler (1952) defined the model of record statistics as a model for successive extremes in a sequence of IID random variables. These statistics are of interest and important in many real life applications involving data relating to weather, economics, sports and life testing studies. For more details and applications regarding record values, see Ahsanullah (1995), Arnold et al. (1992, 1998) and Nevzorov (1987).

Schabe distribution is a lifetime distribution with bathtub shaped failure rate and is extremely useful in reliability. It is a reparameterization of Pareto distribution which has decreasing failure rate. Schabe (1994) gave a method to construct bathtub shaped failure rate distribution from a distribution with decreasing failure rate by the method of truncation as described below.

Let $F(x)$ be a twice differentiable function with decreasing failure rate $\lambda(x)$ and support on $[0, \infty)$. Let θ be a truncation point with $0 < \theta < \infty$. If

$$G(x) = \begin{cases} F(x)/F(\theta), & \text{for } x \leq \theta, \\ 1, & \text{otherwise,} \end{cases}$$

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then $G(x)$ has bathtub shaped failure rate if

$$\lambda^{'}(x)(F(\theta)-F(x))+(\lambda(x))^2(1-F(\theta))$$

has one and only one zero on $[0,\theta)$, (cf. Schabe (1994)).

Now, consider Pareto distribution defined by

$$F(x)=\frac{1}{(1+\frac{T}{x})^{\alpha}},\ \ x\geq 0,\ T\geq 0,\ \alpha\geq 0$$

which has decreasing failure rate. Here without loss of generality, we shall assume $\alpha=1$. Then

$$G(x)=\begin{cases}\frac{x(T+\theta)}{\theta(T+x)} & \text{for } x\leq\theta,\\ 1 & \text{otherwise.}\end{cases}$$

The model can be reparameterized setting $\gamma=\frac{T}{\theta}$. Hence, we get Schabe distribution with probability density function (pdf) of the form

$$f(x)=\frac{(1+\gamma)\gamma\theta}{(x+\theta\gamma)^2},\quad 0<x\leq\theta,\quad 0<\gamma<1,\ \theta>0, \tag{1}$$

and the cumulative distribution function (cdf) of the form

$$F(x)=\frac{(1+\gamma)x}{(x+\theta\gamma)}. \tag{2}$$

It may be noted that the characterizing differential equation of Schabe distribution (1) is given by

$$f(x)[x^2+x\theta\gamma]=\gamma\theta F(x). \tag{3}$$

More details on this distribution can be found in Schabe (1994). The cdf of the location-scale parameter Schabe distribution is given by

$$F(x)=\frac{(1+\gamma)\frac{(x-\mu)}{\sigma}}{(\frac{(x-\mu)}{\sigma}+\theta\gamma)},\ \mu<x\leq\mu+\theta\sigma,\ \mu\geq 0,\ 0<\gamma<1,\ \theta>0. \tag{4}$$

In this paper, we consider the lower record values from the Schabe distribution. We derive recurrence relations satisfied by single and product moments of lower record values from Schabe distribution. These recurrence relations help us to compute the means, variances and covariances of the lower record values. Then, we use these moments to calculate best linear unbiased estimators (BLUEs) and best linear invariant estimators (BLIEs) for the location and scale parameters of the location-scale Schabe distribution. Prediction for the future records is also discussed. Ahsanullah (1980) and Dunsmore (1983) discussed the BLUEs and prediction of future record values from a two-parameter exponential distribution. Some work in this direction has been done for the logistic distribution by Balakrishnan et al. (1995), for the normal distribution by Balakrishnan and Chan (1998), for the generalized exponential distribution by Raqab (2002), for the gamma distribution by Sultan et al.(2008), for the Nadarajah-Haghighi distribution by MirMostafaee et al.(2016), for Lindley distribution by Fallah et al. (2018), for generalized linear exponential distribution by Alam et al. (2021), for exponential distribution by Basiri et al. (2020), for additive Weibull distribution by Khan et al. (2017), for complementary beta distribution by Makouei et al. (2021) and for Weibull- power function distribution by Singh et al. (2020). The remaining paper is organized as below. In Section 2, we derive recurrenec relations satisfied by single and product moments of lower record values arising from Schabe distribution. Then we compute the means, variances and covariances of lower record values from Schabe distribution for arbitrarily chosen values of θ and γ, viz. $(\theta=6,\gamma=0.85)$, $(\theta=10,\gamma=1)$, $(\theta=20,\gamma=0.5)$, $(\theta=100,\gamma=0.05)$ and $(\theta=20,\gamma=0.85)$ and for sample sizes $n=1(1)6$. Next, in Section 3, we obtain the BLUEs and BLIEs of the location and scale parameters of the location-scale Schabe distribution. The BLUEs and BLIEs are then used to construct the confidence intervals (CIs) for the location and scale parameters. In Section 4, we discuss point and interval predictions for future records. In Section 5, two numerical examples are given to illustrate the estimation and prediction methods discussed in this paper.

2 Recurrence Relations for Single and Product Moments

Let $X_{L(1)}, X_{L(2)}, ..., X_{L(n)}$ be the first n lower record values from the Schabe distribution. Then the pdf of $X_{L(n)}$ is given by

$$f_n(x) = \frac{1}{\Gamma(n)} \left[-\log F(x)\right]^{n-1} f(x), \quad x > 0, \quad n = 1, 2, ..., \tag{5}$$

where $f(x)$ and $F(x)$ are given by (1) and (2), respectively. The joint pdf of $X_{L(m)}$ and $X_{L(n)}$ is given by

$$\begin{aligned} f_{m,n}(x, y) = &\frac{1}{\Gamma(m)\Gamma(n-m)} \left[-\log F(x)\right]^{m-1} \frac{f(x)}{F(x)} \left[-\log F(y) + \log F(x)\right]^{n-m-1} \\ &\times f(y), \quad 0 \leq y < x < \infty, \quad 1 \leq m < n, \; n \geq 2. \end{aligned} \tag{6}$$

Then, the k-th single moment of $X_{L(n)}$ denoted by $\mu_n^{(k)}$ $(k = 1, 2, 3, ...)$ is given by

$$\mu_n^{(k)} = \int_0^\infty x^k f_n(x) dx.$$

The double (k, s)th moment of $X_{L(m)}$ and $X_{L(n)}$, $(m < n)$ is given by

$$\mu_{m,n}^{(k,s)} = \int_0^\infty \int_0^x x^k y^s f_{m,n}(x, y) dy dx, \quad 1 \leq m < n, \; n \geq 2, \; k, \; s = 0, 1, 2, ...$$

Explicit expressions for single and double moments cannot be obtained in a closed form for Schabe distribution but we can compute these using the recurrence relations given in the following Theorems 1 to 2 along with the use of R software.

Theorem 1 *For the Schabe distribution given in* (1) *and for* $p, \; n = 1, 2, ...$,

$$\mu_n^{(p+2)} = \frac{\gamma\theta}{p+1} \left[\mu_{n-1}^{(p+1)} - (p+2)\mu_n^{(p+1)}\right]. \tag{7}$$

Proof. Using (5) and (3), we have

$$\begin{aligned} \mu_n^{(p+2)} + \theta\gamma\mu_n^{(p+1)} &= \frac{1}{\Gamma(n)} \int_0^\theta x^p \left(x^2 + \theta\gamma x\right) \left[-\log F(x)\right]^{n-1} f(x) dx \\ &= \frac{\theta\gamma}{\Gamma(n)} \int_0^\theta x^p F(x) \left[-\log F(x)\right]^{n-1} dx. \end{aligned}$$

Now, integrating by parts the integral on the R.H.S. of the above equation by taking $F(x) \left[-\log F(x)\right]^{n-1}$ for differentiation and the rest of the integrand for integration, and then after some simplification, it leads to the required result (7). ■

Theorem 2 *For the Schabe distribution given in* (1) *and for* $n, \; m = 1, 2, ..., \; m < n$ *and* $p, \; q = 0, 1, 2, ...$,

$$\mu_{m,n}^{(p,q+2)} = \frac{\theta\gamma}{(q+1)} \left[\mu_{m,n-1}^{(p,q+1)} - (q+2)\, \mu_{m,n}^{(p,q+1)}\right]. \tag{8}$$

Table 1: Means of record values

n	$\theta = 6$, $\gamma = 0.85$	$\theta = 10$, $\gamma = 1$	$\theta = 20$, $\gamma = 0.5$	$\theta = 100$, $\gamma = 0.05$	$\theta = 20$, $\gamma = 0.85$
1	2.237643	3.862944	6.479184	3.661272	7.458809
2	0.933552	1.644811	2.499078	0.604977	3.11184
3	0.417534	0.744264	1.070910	0.197852	1.391779
4	0.194797	0.349581	0.487904	0.081147	0.649325
5	0.093248	0.168012	0.230399	0.036480	0.310826
6	0.045349	0.081908	0.111152	0.017149	0.151165

Proof. Using (6) and (3), we have

$$\begin{aligned}
&\mu_{m,n}^{(p,q+2)} + \theta\gamma\mu_{m,n}^{(p,q+1)} \\
&= \frac{1}{(m-1)!(n-m-1)!}\int_0^\theta\int_0^x x^p y^q \left(y^2 + \theta\gamma y\right) f(y) \left[-\log F(x)\right]^{m-1} \frac{f(x)}{F(x)} \\
&\quad \times \left[-\log F(y) + \log F(x)\right]^{n-m-1} dydx \\
&= \frac{\gamma\theta}{(m-1)!(n-m-1)!}\int_0^\theta\int_0^x x^p y^q F(y) \left[-\log F(x)\right]^{m-1} \left[-\log F(y) + \log F(x)\right]^{n-m-1} \\
&\quad \times \frac{f(x)}{F(x)} dydx \\
&= \frac{\gamma\theta}{(m-1)!(n-m-1)!}\int_0^\theta x^p y^q \left[-\log F(x)\right]^{m-1} \frac{f(x)}{F(x)} I(x) dx, \qquad (9)
\end{aligned}$$

where

$$I(x) = \int_0^x y^q F(y) \left[-\log F(y) + \log F(x)\right]^{n-m-1} dy.$$

Now, integrating by parts the integral on the R.H.S. of the above equation by taking

$$F(y) \left[-\log F(y) + \log F(x)\right]^{n-m-1}$$

for differentiation and the rest of the integrand for integration, and putting the resultant into (9) and then making some simplification, it leads to the required result (8). ∎

By utilizing the recurrence relations given in Theorems 1 and 2, we have computed the means, variances and covariances of lower record values as given in Tables 1 and 2. R software (R Core Team, 2020) has been used to compute the means, variances and covariances.

3 Linear Estimators

Let $Y_{L(1)}, Y_{L(2)}, ..., Y_{L(n)}$ be the first n lower record values from the location-scale Schabe distribution with cdf as given in (4) is

$$F(y) = \frac{(1+\gamma)\frac{(y-\mu)}{\sigma}}{(\frac{(y-\mu)}{\sigma} + \theta\gamma)},\ \mu < y \le \mu + \theta\sigma,\ \mu \ge 0,\ 0 < \gamma < 1,\ \theta > 0. \qquad (10)$$

Let $X_{L(i)} = \frac{(Y_{L(i)} - \mu)}{\sigma}$, $i = 1, 2, ..., n$ be the corresponding lower record values from the Schabe distribution with pdf given in (1).

Let us denote $E(X_{L(i)})$ by μ_i, $Var(X_{L(i)})$ by $\sigma_{i,i:n}$ and $Cov(X_{L(i)}, X_{L(j)})$ by $\sigma_{i,j:n}$; furthermore, let

$$\mathbf{Y} = (Y_{L(1)}, Y_{L(2)}, ..., Y_{L(n)})^T,$$

Table 2: Variances and covariances of record values

m	n	$\theta=6,\ \gamma=0.85$	$\theta=10,\ \gamma=1$	$\theta=20,\ \gamma=0.5$	$\theta=100,\ \gamma=0.05$	$\theta=20,\ \gamma=0.85$
1	1	2.76900101	7.8187944	28.4364838	269.5199662	30.7666779
1	2	1.05068728	3.0620545	9.5373393	38.1476142	11.6743031
1	3	0.44095856	1.3099889	3.7214934	9.7210436	4.8995407
1	4	0.19744015	0.5933103	1.5976586	3.4196885	2.1937795
1	5	0.09206589	0.278569	0.7269651	1.410761	1.0229544
1	6	0.04403193	0.1337889	0.3426649	0.6314017	0.4892437
2	2	1.01822261	3.0278235	8.5648846	22.1128998	11.3135846
2	3	0.42089882	1.2783224	3.2651561	5.1048513	4.6766541
2	4	0.1866152	0.5740072	1.3811682	1.7059102	2.0735022
2	5	0.08647668	0.2680274	0.6227302	0.68578	0.960852
2	6	0.04119567	0.1282758	0.2918777	0.3027503	0.4577296
3	3	0.32793917	1.00889913	2.425742	2.9953249	3.643769
3	4	0.14387701	0.44882843	1.0105981	0.9616949	1.5986335
3	5	0.06622596	0.20832813	0.4513548	0.3788006	0.7358438
3	6	0.03141468	0.09932507	0.2103164	0.1654261	0.3490521
4	4	0.10454264	0.32880701	0.7129681	0.6004553	1.161585
4	5	0.04776589	0.15161118	0.3152002	0.2319280	0.5307321
4	6	0.02255177	0.07197848	0.1459463	0.1002131	0.2505751
5	5	0.03364485	0.10735274	0.2179666	0.15025464	0.3738317
5	6	0.01580093	0.05072328	0.1002155	0.06415331	0.1755658
6	6	0.01094251	0.03524779	0.06859598	0.04232419	0.1215835

$$\mu=(\mu_1,\mu_2,...,\mu_n)^T,$$
$$\mathbf{1}=(\underbrace{1,1,...,1})^T$$

and

$$\sum=(\sigma_{i,j:n})\,,\ 1\le i,\ j\le n.$$

Then, the BLUEs of μ and σ are obtained by minimizing the generalized variance $Q(\delta)=(\mathbf{Y}-A\delta)^T\sum^T(\mathbf{Y}-A\delta)$ with respect to δ, where $\delta=(\mu,\sigma)^T$, A is $n\times 2$ matrix $(\mathbf{1},\mu)$, $\mathbf{1}$ is $n\times 1$ vector with components all 1's, μ is the mean vector of $\mathbf{X}$, and $\sum$ is the variance-covariance matrix of $\mathbf{X}$. The minimization leads to the expressions for the BLUE's of μ and σ as (see Anrold et al. (1992) and Balakrishnan and Cohen (1991))

$$\hat{\mu}_{BLU}=\left\{\frac{\mu^T\sum^{-1}\mu\mathbf{1}^T\sum^{-1}-\mu^T\sum^{-1}\mathbf{1}\mu^T\sum^{-1}}{(\mu^T\sum^{-1}\mu)(\mathbf{1}^T\sum^{-1}\mathbf{1})-(\mu^T\sum^{-1}\mathbf{1})^2}\right\}\mathbf{Y}\ =\sum_{r=1}^{n}a_rY_{L(r)} \tag{11}$$

and

$$\hat{\sigma}_{BLU}=\left\{\frac{\mathbf{1}^T\sum^{-1}\mathbf{1}\mu^T\sum^{-1}-\mathbf{1}^T\sum^{-1}\mu\mathbf{1}^T\sum^{-1}}{(\mu^T\sum^{-1}\mu)(\mathbf{1}^T\sum^{-1}\mathbf{1})-(\mu^T\sum^{-1}\mathbf{1})^2}\right\}\mathbf{Y}\ =\sum_{r=1}^{n}b_rY_{L(r)}, \tag{12}$$

and the variances and covariance of these BLUEs are given by

$$Var(\hat{\mu}_{BLU})=\sigma^2\left\{\frac{\mu^T\sum^{-1}\mu}{(\mu^T\sum^{-1}\mu)(\mathbf{1}^T\sum^{-1}\mathbf{1})-(\mu^T\sum^{-1}\mathbf{1})^2}\right\}=\sigma^2V_1,$$

$$Var(\hat{\sigma}_{BLU})=\sigma^2\left\{\frac{\mathbf{1}^T\sum^{-1}\mathbf{1}}{(\mu^T\sum^{-1}\mu)(\mathbf{1}^T\sum^{-1}\mathbf{1})-(\mu^T\sum^{-1}\mathbf{1})^2}\right\}=\sigma^2V_2$$

and

$$Cov(\hat{\mu}_{BLU},\ \hat{\sigma}_{BLU}) = \sigma^2 \left\{ \frac{-\mu^T \sum^{-1} \mathbf{1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_3.$$

The coefficients of the BLUEs in (11) and (12) satisfy the conditions $\sum_{r=1}^{n} a_r = 1$ and $\sum_{r=1}^{n} b_r = 0$ respectively.

The coefficients $a_i^{'}s,\ b_i^{'}s,\ 1 \leq i \leq n$ and the values of V_1, V_2 and V_3 are computed and presented in Tables 3–5, respectively. From Table 5, we note that the variance of the BLUEs decreases as n increases.

Based on the BLUEs of the location and scale parameters, the confidence intervals (CIs) for μ and σ can be constructed through the pivotal quantities given by

$$R_1 = \frac{\hat{\mu}_{BLU} - \mu}{\hat{\sigma}_{BLU}\sqrt{V_1}} \quad \text{and} \quad R_2 = \frac{\hat{\sigma}_{BLU} - \sigma}{\sigma\sqrt{V_2}}.$$

Constructing such CIs requires the percentage points of R_1 and R_2 which can be computed by using BLUEs $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ via Monte Carlo method. In Table 6, we have determined the percentage points of R_1 and R_2 based on 10000 runs for arbitrarily chosen values of θ and γ, viz. $(\theta = 6, \gamma = 0.85)$, $(\theta = 10, \gamma = 1)$, $(\theta = 20, \gamma = 0.5)$, $(\theta = 100, \gamma = 0.05)$ and $(\theta = 20, \gamma = 0.85)$ and for sample sizes $n = 2(1)5$. Based on these simulated percentage points, we can determine a $100(1-\alpha)\%$ CI for μ through the pivotal quantity R_1 as follows:

$$P\left(\hat{\mu}_{BLU} - \hat{\sigma}_{BLU}\sqrt{V_1}R_1(1-\alpha/2) \leq \mu \leq \hat{\mu}_{BLU} - \hat{\sigma}_{BLU}\sqrt{V_1}R_1(\alpha/2)\right) = 1 - \alpha,$$

where $R_1(\gamma)$ is the left percentage point of R_1 at γ, i.e., $P(R_1 < R_1(\gamma)) = \gamma$.

Similarly, a $100(1-\alpha)\%$ CI for σ can be constructed through the pivotal quantity R_2 as follows:

$$P\left(\frac{\hat{\sigma}_{BLU}}{1 + \sqrt{V_2}R_2(1-\alpha/2)} \leq \sigma \leq \frac{\hat{\sigma}_{BLU}}{1 + \sqrt{V_2}R_2(\alpha/2)}\right) = 1 - \alpha.$$

Now, let us consider the BLIEs of μ and σ. Based on the results of Mann (1969), the BLIEs for μ and σ are (cf. Arnold et al. (1998), p.143)

$$\hat{\mu}_{BLI} = \hat{\mu}_{BLU} - \frac{V_3}{1 + V_2}\hat{\sigma}_{BLU} \quad \text{and} \quad \hat{\sigma}_{BLI} = \frac{\hat{\sigma}_{BLU}}{1 + V_2}.$$

Note: In Table 5, for each n $(n = 2, 3, 4, 5)$, the first, second, third and forth line represents $V_1 = \frac{1}{\sigma^2}Var(\hat{\mu}_{BLU})$, $V_2 = \frac{1}{\sigma^2}Var(\hat{\sigma}_{BLU})$, $V_3 = \frac{1}{\sigma^2}Cov(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU})$ and V_4, respectively, where V_4 will be defined in the next Setion 4.

Furthermore, the variances of these BLIEs are given by (see Arnold et al. (1998), p. 143)

$$Var(\hat{\mu}_{BLI}) = \sigma^2\left(V_1 - \frac{V_3^2(2 + V_2)}{(1 + V_2)^2}\right) \quad \text{and} \quad Var(\hat{\sigma}_{BLI}) = \frac{\sigma^2 V_2}{(1 + V_2)^2}.$$

Based on the BLIEs, we can again construct CIs for the location and scale parameters through pivotal quantites given by

$$R_3 = \frac{\hat{\mu}_{BLI} - \mu}{\hat{\sigma}_{BLI}\sqrt{V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}}} \quad \text{and} \quad R_4 = \frac{\hat{\sigma}_{BLI} - \sigma}{\sigma\frac{\sqrt{V_2}}{1+V_2}}.$$

Table 7 presents the percentage points of R_3 and R_4 based on 10000 runs and arbitrarily chosen values of θ and γ, viz. $(\theta = 6, \gamma = 0.85)$, $(\theta = 10, \gamma = 1)$, $(\theta = 20, \gamma = 0.5)$, $(\theta = 100, \gamma = 0.05)$ and $(\theta = 20, \gamma = 0.85)$ and for sample sizes $n = 2(1)5$. Using the BLIEs along with the use of Table 7, we can determine a $100(1-\alpha)\%$ CI for μ through the pivotal quantity R_3 as

$$P\left(\hat{\mu}_{BLI} - \hat{\sigma}_{BLI}\sqrt{V_1 - \frac{V_3^2(2 + V_2)}{(1 + V_2)^2}}R_3(1-\alpha/2) \leq \mu \leq \hat{\mu}_{BLI} - \hat{\sigma}_{BLI}\sqrt{V_1 - \frac{V_3^2(2 + V_2)}{(1 + V_2)^2}}R_3(\alpha/2)\right) = 1 - \alpha.$$

Table 3: Coefficients for the BLUEs of μ

n	$\theta = 6$, $\gamma = 0.85$	$\theta = 10$, $\gamma = 1$	$\theta = 20$, $\gamma = 0.5$	$\theta = 100$, $\gamma = 0.05$	$\theta = 20$, $\gamma = 0.85$
2	-0.715865	-0.741529	-0.627892	-0.310595	-0.715865
	1.715865	1.741529	1.627892	1.310595	1.715865
3	-0.166471	-0.173774	-0.141892	-0.060089	-0.166471
	-0.221965	-0.224661	-0.212523	-0.185666	-0.221965
	1.388436	1.398435	1.354415	1.245754	1.388436
4	-0.049966	-0.052252	-0.042354	-0.017041	-0.049966
	-0.062562	-0.063956	-0.057667	-0.041135	-0.062562
	-0.208800	-0.210705	-0.202686	-0.210179	-0.208800
	1.321328	1.326913	1.302708	1.268356	1.321328
5	-0.016429	-0.017147	-0.014036	-0.005572	-0.016429
	-0.019399	-0.019933	-0.017551	-0.011433	-0.019400
	-0.063513	-0.064559	-0.060003	-0.054621	-0.063513
	-0.207968	-0.209375	-0.203658	-0.215920	-0.207968
	1.307310	1.311014	1.295248	1.287546	1.307310

Table 4: Coefficients for the BLUEs of σ

n	$\theta = 6$, $\gamma = 0.85$	$\theta = 10$, $\gamma = 1$	$\theta = 20$, $\gamma = 0.5$	$\theta = 100$, $\gamma = 0.05$	$\theta = 20$, $\gamma = 0.85$
2	0.766818	0.450830	0.251250	0.119321	0.230045
	-0.766818	-0.450830	-0.251250	-0.119321	-0.230045
3	0.504930	0.295511	0.167881	0.079087	0.151479
	0.156919	0.087054	0.064455	0.120998	0.047076
	-0.661848	-0.382565	-0.232336	-0.200085	-0.198555
4	0.444040	0.259557	0.147992	0.065064	0.133212
	0.073609	0.039507	0.033512	0.073916	0.022083
	0.172924	0.093524	0.078794	0.274202	0.051877
	-0.690573	-0.392588	-0.260299	-0.413182	-0.207172
5	0.424604	0.248215	0.141290	0.058342	0.127381
	0.048595	0.025283	0.024018	0.056510	0.014579
	0.088726	0.046306	0.045025	0.183048	0.026618
	0.195698	0.103772	0.096216	0.456586	0.058710
	-0.757624	-0.423575	-0.306549	-0.754486	-0.227287

Table 5: Variances and covariances of the BLUEs of μ and σ and V_4

n	$\theta = 6$, $\gamma = 0.85$	$\theta = 10$, $\gamma = 1$	$\theta = 20$, $\gamma = 0.5$	$\theta = 100$, $\gamma = 0.05$	$\theta = 20$, $\gamma = 0.85$
2	1.835677	5.573803	14.411269	32.925767	20.396408
	0.991296	0.959837	1.131647	3.065888	0.991296
	-0.900533	-1.563318	-2.583745	-6.067265	-3.001775
	-0.502015	-0.863224	-1.486508	-3.804362	-1.673384
3	0.373466	1.161124	2.674333	3.418256	4.149623
	0.659038	0.629599	0.786278	2.304693	0.659038
	-0.203517	-0.356158	-0.570396	-1.327965	-0.678389
	-0.108890	-0.189655	-0.309209	-0.727228	-0.362966
4	0.098171	0.310547	0.658405	0.586788	1.090784
	0.583842	0.555143	0.705791	2.004216	0.583842
	-0.059638	-0.104502	-0.167586	-0.405581	-0.198792
	-0.030908	-0.054088	-0.086945	-0.207394	-0.103027
5	0.028447	0.091073	0.182577	0.131097	0.316081
	0.560425	0.532232	0.679138	1.847741	0.560425
	-0.019231	-0.033592	-0.054971	-0.138552	-0.064103
	-0.009743	-0.017033	-0.027700	-0.068457	-0.032478

Table 6: Simulated percentage points of R_1 and R_2

	n	R_1				R_2			
		2.50%	5%	95%	97.50%	2.50%	5%	95%	97.50%
$\theta = 6$, $\gamma = 0.85$	2	-0.6703	-0.6531	12.2586	25.3593	-0.9951	-0.9827	2.1178	2.5416
	3	-0.7587	-0.6936	4.0691	6.7874	-1.2049	-1.1733	1.9828	2.2419
	4	-0.8342	-0.7035	2.8296	4.2320	-1.2678	-1.2317	1.8919	2.1126
	5	-0.9096	-0.7414	2.0754	3.3612	-1.2976	-1.2525	1.9012	2.1053
$\theta = 10$, $\gamma = 1$	2	-0.6796	-0.6616	13.5482	28.4667	-1.0097	-0.9951	2.1609	2.5755
	3	-0.7686	-0.6983	4.1237	6.7840	-1.2287	-1.1931	1.8863	2.1748
	4	-0.8215	-0.7031	2.8569	4.5464	-1.3017	-1.2606	1.8511	2.0728
	5	-0.8805	-0.7305	2.1705	3.5971	-1.3267	-1.2804	1.8410	2.0089
$\theta = 20$, $\gamma = 0.5$	2	-0.6419	-0.6244	13.0466	26.4392	-0.9320	-0.9220	2.1852	2.6526
	3	-0.7340	-0.6640	4.3170	7.1874	-1.1052	-1.0811	2.0710	2.4035
	4	-0.7996	-0.6701	2.9057	4.4338	-1.1602	-1.1305	2.0057	2.2620
	5	-0.8523	-0.6865	2.2182	3.4857	-1.1797	-1.1430	1.9836	2.2098
$\theta = 100$, $\gamma = .05$	2	-0.4424	-0.4278	13.2952	30.1442	-0.5696	-0.5673	2.0861	3.2507
	3	-0.5418	-0.4867	5.3991	8.5430	-0.6526	-0.6456	2.1702	3.1362
	4	-0.5968	-0.5294	3.5410	5.6938	-0.6986	-0.6893	2.0934	3.0348
	5	-0.6505	-0.5490	2.4187	3.6710	-0.7242	-0.7135	2.2596	3.0087
$\theta = 20$, $\gamma = .85$	2	-0.6711	-0.6534	11.7201	26.7071	-0.9925	-0.9774	2.1594	2.5507
	3	-0.7659	-0.6928	4.3327	7.4763	-1.2036	-1.1702	1.9685	2.2521
	4	-0.8495	-0.7216	2.6654	4.2915	-1.2679	-1.2263	1.8962	2.1055
	5	-0.8731	-0.7149	2.0537	3.3664	-1.2970	-1.2555	1.8886	2.0743

Table 7: Simulated percentage points of R_3 and R_4

	n	R_3				R_4			
		2.50%	5%	95%	97.50%	2.50%	5%	95%	97.50%
$\theta = 6, \gamma = 0.85$	2	-0.8207	-0.7787	30.7092	62.6579	-1.9907	-1.9784	1.1222	1.5459
	3	-0.9797	-0.8654	7.4967	12.2696	-2.0167	-1.9852	1.1710	1.4301
	4	-1.1526	-0.9416	4.7617	7.0255	-2.0319	-1.9958	1.1278	1.3485
	5	-1.3143	-1.0500	3.3757	5.3959	-2.0462	-2.0012	1.1526	1.3567
$\theta = 10, \gamma = 1$	2	-0.8230	-0.7797	33.4451	69.3770	-1.9894	-1.9748	1.1812	1.5958
	3	-0.9763	-0.8550	7.4659	12.0565	-2.0222	-1.9866	1.0929	1.3814
	4	-1.1108	-0.9232	4.7189	7.3965	-2.0467	-2.0056	1.1060	1.3277
	5	-1.2462	-1.0148	3.4602	5.6608	-2.0563	-2.0099	1.1115	1.2794
$\theta = 20, \gamma = 0.5$	2	-0.8335	-0.7882	34.5321	69.1329	-1.9958	-1.9857	1.1214	1.5889
	3	-1.0180	-0.8857	8.5258	13.9492	-1.9919	-1.9678	1.1843	1.5168
	4	-1.1811	-0.9557	5.2686	7.9285	-2.0003	-1.9707	1.1656	1.4219
	5	-1.3129	-1.0322	3.8840	6.0293	-2.0038	-1.9671	1.1595	1.3857
$\theta = 100, \gamma = 0.05$	2	-0.9146	-0.8411	67.9749	152.4675	-2.3206	-2.3183	0.3351	1.4998
	3	-1.2014	-0.9973	20.7950	32.4353	-2.1708	-2.1637	0.6520	1.6180
	4	-1.3502	-1.1338	11.9342	18.8457	-2.1143	-2.1050	0.6777	1.6191
	5	-1.5235	-1.2239	7.5372	11.2340	-2.0835	-2.0728	0.9003	1.6494
$\theta = 20, \gamma = 0.85$	2	-0.8226	-0.7795	29.3959	65.9448	-1.9881	-1.9730	1.1638	1.5550
	3	-0.9923	-0.8639	7.9595	13.4791	-2.0154	-1.9820	1.1567	1.4402
	4	-1.1773	-0.9708	4.4966	7.1216	-2.0320	-1.9904	1.1321	1.3414
	5	-1.2570	-1.0085	3.3415	5.4040	-2.0456	-2.0041	1.1400	1.3257

Similarly, we can determine $100(1-\alpha)\%$ CI for σ, through the pivotal quantity R_4 as

$$P\left(\frac{\hat{\sigma}_{BLI}}{1+\frac{\sqrt{V_2}}{1+V_2}R_4(1-\alpha/2)} \le \sigma \le \frac{\hat{\sigma}_{BLI}}{1+\frac{\sqrt{V_2}}{1+V_2}R_4(\alpha/2)}\right) = 1-\alpha.$$

Now, let us compare the BLUEs and BLIEs using the relative efficiency criterion (REC). Since the mean squared errors (MSEs) of BLUEs are equal to their corresponding variances, we have

$$MSE\left(\hat{\mu}_{BLU}\right) = \sigma^2 V_1 \quad \text{and} \quad MSE\left(\hat{\sigma}_{BLU}\right) = \sigma^2 V_2.$$

On the other hand, the MSEs of BLIEs of μ and σ can be obtained as

$$MSE\left(\hat{\mu}_{BLI}\right) = \sigma^2\left(V_1 - \frac{V_3^2}{1+V_2}\right) \quad \text{and} \quad MSE\left(\hat{\sigma}_{BLI}\right) = \frac{\sigma^2 V_2}{1+V_2}.$$

Therefore, we can readily obtain the RECs of the BLIEs of μ and σ with respect to their corresponding BLUEs as follows:

$$REC\left(\hat{\mu}_{BLI}, \hat{\mu}_{BLU}\right) = \frac{MSE\left(\hat{\mu}_{BLU}\right)}{MSE\left(\hat{\mu}_{BLI}\right)} = \frac{V_1}{V_1 - \frac{V_3^2}{1+V_2}} \ge 1, \quad REC\left(\hat{\sigma}_{BLI}, \hat{\sigma}_{BLU}\right) = \frac{MSE\left(\hat{\sigma}_{BLU}\right)}{MSE\left(\hat{\sigma}_{BLI}\right)} = 1 + V_2 \ge 1.$$

Therefore, both of the BLIEs perform better than coressponding BLUEs in terms of MSEs.

4 Linear Predictors

Suppose we have observed the first n lower records $\mathbf{Y}' = \left\{Y_{L(1)}, Y_{L(2)}, ..., Y_{L(n)}\right\}$ from location-scale Schabe Distribution with cdf given in (10) and our aim is to predict the next lower record $Y = Y_{L(n+1)}$. The best linear unbiased predictor (BLUP) of Y is given by (Arnold et al. (1998), p. 150)

$$\hat{Y}_{BLUP} = \hat{\mu}_{BLU} + \hat{\sigma}_{BLU}\mu_{n+1} + \omega^{'}\Sigma^{-1}\left(\mathbf{Y} - \hat{\mu}_{BLU}\mathbf{1} - \hat{\sigma}_{BLU}\mu\right),$$

where

$$\omega^{'} = \left(Cov(X_{L(1)}, X_{L(n+1)}), ..., Cov(X_{L(n)}, X_{L(n+1)})\right),$$

in which $X_{L(i)} = \frac{Y_{L(i)}-\mu}{\sigma}$, $i = 1, 2, ..., n+1$. Moreover, the mean squared prediction error (MSPE) of $(\hat{Y}_{BLUP})$ is given by (Burkschat (2009))

$$\begin{aligned} MSPE\left(\hat{Y}_{BLUP}\right) &= E\left[\left(\hat{Y}_{BLUP} - Y_{L(n+1)}\right)^2\right] \\ &= \sigma^2\left[\left(1-\omega^{'}\Sigma^{-1}\mathbf{1}\right)^2 V_1 + \left(\mu_{n+1} - \omega^{'}\Sigma^{-1}\mu\right)^2 V_2 - \omega^{'}\Sigma^{-1}\omega \right. \\ &\left. +2\left(1-\omega^{'}\Sigma^{-1}\mathbf{1}\right)\left(\mu_{n+1} - \omega^{'}\Sigma^{-1}\mu\right)V_3 + Var\left(X_{L(n+1)}\right)\right]. \end{aligned}$$

Let us now consider the best linear invariant predictor (BLIP) of the next lower record value. From the results of Mann (1969), the BLIP of Y can be obtained based on the BLUP of Y as follows (see also Arnold et al. (1998), p.153)

$$\hat{Y}_{BLIP} = \hat{Y}_{BLUP} - \left(\frac{V_4}{1+V_2}\right)\hat{\sigma}_{BLU},$$

where $\hat{Y}_{BLUP}$ is the BLUP of $Y_{L(n+1)}$ and

$$V_4 = \left(1-\omega^{'}\Sigma^{-1}\mathbf{1}\right)V_3 + \left(\mu_{n+1} - \omega^{'}\Sigma^{-1}\mu\right)V_2.$$

In Table 5, we reported the values of V_4 for different values of n, θ and γ. The MSPE of $\hat{Y}_{BLIP}$ is given by (Brukschat (2009))

$$\begin{aligned} MSPE\left(\hat{Y}_{BLIP}\right) &= E\left[\left(\hat{Y}_{BLIP} - Y_{L(n+1)}\right)^2\right] \\ &= \sigma^2\left[\frac{\mu^{'}\Sigma^{-1}\mu+1}{\Delta}\left(1-\omega^{'}\Sigma^{-1}\mathbf{1}\right)^2 + \frac{\mathbf{1}^{'}\Sigma^{-1}\mathbf{1}}{\Delta}\left(\mu_{n+1} - \omega^{'}\Sigma^{-1}\mu\right)^2 - \omega^{'}\Sigma^{-1}\omega \right. \\ &\left. -2\frac{\mu^{'}\Sigma^{-1}\mathbf{1}}{\Delta}\left(1-\omega^{'}\Sigma^{-1}\mathbf{1}\right)\left(\mu_{n+1} - \omega^{'}\Sigma^{-1}\mu\right)V_3 + Var\left(X_{L(n+1)}\right)\right], \end{aligned}$$

where

$$\Delta = \left(\mu^{'}\Sigma^{-1}\mu + 1\right)\left(\mathbf{1}^{'}\Sigma^{-1}\mathbf{1}\right) - \left(\mu^{'}\Sigma^{-1}\mathbf{1}\right)^2.$$

Now we compare the BLUP and BLIP of Y using the REC. The REC of $\hat{Y}_{BLIP}$ relative to $\hat{Y}_{BLUP}$ is

$$REC\left(\hat{Y}_{BLIP}, \hat{Y}_{BLUP}\right) = \frac{MSPE\left(\hat{Y}_{BLUP}\right)}{MSPE\left(\hat{Y}_{BLIP}\right)}.$$

In Table 8, we presented the REC of $\hat{Y}_{BLIP}$ relative to $\hat{Y}_{BLUP}$ for different choices of n, θ and γ. From Table 8, we observe that the BLIP works better than BLUP in terms of MSPE. Suppose we are now interested in prediction intervals (PIs) for $Y_{L(n+1)}$. The PIs can be constructed using the pivotal quantities (cf. Balakrishnan and Chan (1998))

$$T_1 = \frac{Y_{L(n)} - Y_{L(n-1)}}{\hat{\sigma}_{BLU}}$$

Table 8: The REC of $\hat{Y}_{BLIP}$ with respect to $\hat{Y}_{BLUP}$.

n	$\theta = 6, \gamma = 0.85$	$\theta = 10, \gamma = 1$	$\theta = 20, \gamma = 0.5$	$\theta = 100, \gamma = 0.05$	$\theta = 20, \gamma = 0.85$
2	1.657798	1.633792	1.763426	3.146248	1.657798
3	1.414268	1.394886	1.497118	2.392495	1.414268
4	1.354725	1.337456	1.426776	2.083502	1.354725
5	1.332975	1.317046	1.398328	1.926255	1.332975

and

$$T_2 = \frac{Y_{L(n)} - Y_{L(n-1)}}{\hat{\sigma}_{BLI}}.$$

Constructing such PIs require the percentage points of T_1 and T_2. In Tables 9 and 10, we presented the simulated percentage points of T_1 and T_2 respectively using Monte Carlo method based on 10000 runs and for arbitrarily chosen values of θ and γ, viz. ($\theta = 6, \gamma = 0.85$), ($\theta = 10, \gamma = 1$), ($\theta = 20, \gamma = 0.5$), ($\theta = 100, \gamma = 0.05$) and ($\theta = 20, \gamma = 0.85$) and for sample sizes n=3(1)5. Using the pivotal quantity T_1, a $100(1-\alpha)\%$ PI for $Y = Y_{L(n+1)}$ is given by

$$P\left(Y_{L(n)} - \hat{\sigma}_{BLU}\ T_1(1-\alpha) \leq Y \leq Y_{L(n)} - \hat{\sigma}_{BLU}\ T_1(\alpha/2)\right) = 1-\alpha.$$

Similarly, using the pivotal quantity T_2, a $100(1-\alpha)\%$ PI for Y is given by

$$P\left(Y_{L(n)} - \hat{\sigma}_{BLI}\ T_2(1-\alpha) \leq Y \leq Y_{L(n)} - \hat{\sigma}_{BLI}\ T_2(\alpha/2)\right) = 1-\alpha.$$

5 Illustrative Examples

In this section, we present two numerical examples for illustrative purposes.

5.1 Example 1 (Simulated Data)

We generated $n = 4$ lower record values from location-scale Schabe distribution with parameters $\theta = 100,\ \gamma = 0.05,\ \mu = 0,\ \sigma = 1$ as follows:

$$3.4835475, 1.3453739, 0.8840804, 0.4189143$$

The BLUEs of μ and σ are computed to be

$$\hat{\mu}_{BLU} = 0.2308101 \text{ and } \hat{\sigma}_{BLU} = 0.3954255$$

and the corresponding variances and covariances of $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ are computed to be:

$$Var\left(\hat{\mu}_{BLU}\right) = 0.5867875\sigma^2, \quad Var\left(\hat{\sigma}_{BLU}\right) = 2.0042163\sigma^2, \quad Cov\left(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU}\right) = -0.4055810\sigma^2.$$

The BLIEs of the location and scale parameters are given by

$$\hat{\mu}_{BLI} = 0.2841941 \text{ and } \hat{\sigma}_{BLI} = 0.1316235.$$

The variances of $\hat{\mu}_{BLI}$ and $\hat{\sigma}_{BLI}$ are

$$Var\left(\hat{\mu}_{BLI}\right) = 0.5138064\sigma^2 \text{ and } Var\left(\hat{\sigma}_{BLI}\right) = 0.2220661\sigma^2.$$

The 95% CIs for μ based on R_1 and R_3 are $(-1.493865, 0.4115832)$ and $(-1.493862, 0.4115829)$, respectively. Also, the 95% CIs for σ based on R_2 and R_4 are $(0.07465962, 35.98191)$ and $(0.07465955, 35.96588)$, respectively.

Let us consider now the BLUP and BLIP of the next record $Y_{L(5)}$. The BLUP of $Y_{L(5)}$ is $\hat{Y}_{BLUP} = 0.1316235$ and the BLIP is $\hat{Y}_{BLIP} = 0.2869653$. In addition, the 95% PIs for the next lower record $Y_{L(5)}$ based on the pivotal quantities T_1 and T_2 are given by

$$\left(L_1(Y_{L(5)}), U_1(Y_{L(5)})\right) = (-3.615338, 0.417787)$$

and

$$\left(L_2(Y_{L(5)}), U_2(Y_{L(5)})\right) = (-4.018835, 0.417674)$$

respectively.

5.2 Example 2 (Simulated Data)

We generated $n = 5$ lower record values from location-scale Schabe distribution with parameters $\theta = 20,\ \gamma = 0.85,\ \mu = 0,\ \sigma = 1$ as follows:

$$14.794021,\ 6.672330,\ 4.620351,\ 4.327030,\ 3.488474$$

The BLUEs of μ and σ are computed to be

$$\hat{\mu}_{BLU} = 2.994686 \text{ and } \hat{\sigma}_{BLU} = 1.565891$$

and the corresponding variances and covariances of $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ are computed to be:

$$Var\left(\hat{\mu}_{BLU}\right) = 0.31608140\sigma^2, \quad Var\left(\hat{\sigma}_{BLU}\right) = 0.56042514\sigma^2, \quad Cov\left(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU}\right) = -0.06410301\sigma^2.$$

The BLIEs of the location and scale parameters are given by

$$\hat{\mu}_{BLI} = 3.059013 \text{ and } \hat{\sigma}_{BLI} = 1.003503.$$

The variances of $\hat{\mu}_{BLI}$ and $\hat{\sigma}_{BLI}$ are

$$Var\left(\hat{\mu}_{BLI}\right) = 0.3117604\sigma^2 \text{ and } Var\left(\hat{\sigma}_{BLI}\right) = 0.230161\sigma^2.$$

The 95% CIs for μ based on R_1 and R_3 are $(0.03103703, 3.763329)$ and $(0.03109311, 3.763324)$, respectively. Also, the 95% CIs for σ based on R_2 and R_4 are $(0.6133886, 53.91125)$ and $(0.6133858, 53.88974)$, respectively.

Let us consider now the BLUP and BLIP of the next record $Y_{L(6)}$. The BLUP of $Y_{L(6)}$ is $\hat{Y}_{BLUP} = 3.234521$ and the BLIP is $\hat{Y}_{BLIP} = 3.267113$. In addition, the 95% PIs for the next lower record $Y_{L(6)}$ based on the pivotal quantities T_1 and T_2 are given by

$$\left(L_1(Y_{L(6)}), U_1(Y_{L(6)})\right) = (-1.370155, 3.487232)$$

and

$$\left(L_2(Y_{L(6)}), U_2(Y_{L(6)})\right) = (-1.443068, 3.487216)$$

respectively.

6 Conclusion

In this paper, we have established recurrence relations for the single and product moments of lower record values arising from the Schabe distribution. With the help of recurrence relations along with use of R software, we have computed all the means, variances and covariances of lower record values. These moments have then been used to obtain the best linear unbiased estimators (BLUEs) and best linear invariant estimators (BLIEs) of location and scale parameters of location-scale Schabe distribution (10). These BLUEs and BLIEs are then used to construct confidence intervals for the location and scale parameters through

Monte Carlo simulations. We have compared the BLUEs and BLIEs for location and scale parameters using the relative efficiency criterion (REC) and observed that the BLIEs work better than BLUEs in terms of mean squared errors (MSEs). We have also discussed Predictions and Prediction Intervals for the future records. We have compared the best linear unbiased predictors (BLUPs) and best linear invariant predictors (BLIPs) for next record value using the REC and concluded that the BLIPs work better than BLUPs in terms of mean squared prediction errors (MSPEs). Finally, two numerical examples are given to illustrate the estimation and prediction methods discussed here.

Table 9: Simulated percentage points of T_1

	n	T_1			
		2.50%	5%	95%	97.50%
$\theta = 6$, $\gamma = 0.85$	3	0.004453	0.010162	9.677063	20.577763
	4	0.000930	0.002550	1.692273	2.716815
	5	0.000115	0.002873	0.844331	1.020733
$\theta = 10$, $\gamma = 1$	3	0.006943	0.017819	16.524751	33.228657
	4	0.001999	0.005109	2.776881	4.438588
	5	0.000421	0.001117	1.058293	1.641202
$\theta = 20$, $\gamma = 0.5$	3	0.012169	0.029111	28.564340	58.065615
	4	0.002860	0.006715	4.628703	7.566805
	5	0.006072	0.001794	1.645213	2.502674
$\theta = 100$, $\gamma = .05$	3	0.015644	0.038101	47.611590	94.484262
	4	0.002851	0.007088	5.918252	10.202272
	5	0.000822	0.001924	1.760334	2.705235
$\theta = 20$, $\gamma = .85$	3	0.015168	0.038906	33.05122	71.444618
	4	0.003217	0.007828	5.387537	8.516205
	5	0.000793	0.002224	2.041887	3.102789

Table 10: Simulated percentage points of T_2

	n	T_2			
		2.50%	5%	95%	97.50%
$\theta = 6$, $\gamma = 0.85$	3	0.008868	0.020350	19.269898	40.976423
	4	0.001542	0.004231	2.807546	4.507301
	5	0.000928	0.003550	1.337287	1.616680
$\theta = 10$, $\gamma = 1$	3	0.013610	0.034927	32.385824	65.122797
	4	0.003258	0.008325	4.525203	7.233111
	5	0.000654	0.001737	1.645797	2.552303
$\theta = 20$, $\gamma = 0.5$	3	0.025939	0.062055	60.889079	123.775374
	4	0.005109	0.011994	10.268148	13.516413
	5	0.001036	0.003060	2.806390	4.269039
$\theta = 100$, $\gamma = 0.05$	3	0.063607	0.154915	193.58340	384.162441
	4	0.009424	0.023423	19.55810	33.715383
	5	0.002467	0.005780	5.288485	8.127111
$\theta = 20$, $\gamma = 0.85$	3	0.302040	0.077472	65.814777	142.267401
	4	0.005337	0.012985	8.928130	14.536574
	5	0.001257	0.003523	3.234025	4.914327

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